

ALGEBRA HOMEWORK SET V

JAMES CUMMINGS

You may collaborate on this homework set, but must write up your solutions by yourself. Please contact me by email if you are puzzled by something, would like a hint or believe that you have found a typo.

Due March 31st.

- (1) Let R be a ring. The *formal power series ring* $R[[x]]$ consists of infinite expressions $a_0 + a_1x + a_2x^2 + \dots$ with the natural operations. Prove that if R is Noetherian then $R[[x]]$ is Noetherian.
- (2) Let $n \geq 1$ and let A be an $n \times n$ integer matrix. Let G be the subgroup of $\mathbb{Z}^n$ generated by the rows of A . Prove that G has rank n (as a $\mathbb{Z}$ -module) iff A is non-singular, and that in this case the order of the quotient group $\mathbb{Z}^n/G$ is $|\det(A)|$.
- (3) Let E be a field and let F be algebraic over E . Prove that F is algebraically closed iff every polynomial in $E[x]$ splits over F .
- (4) Let E be a field of characteristic p and consider the map $\phi : a \mapsto a^p$. Show that ϕ is an injective HM from E to E . Prove that if E is either finite or algebraically closed then ϕ is an automorphism of E . Prove that if $|E| = p^n$ then ϕ is an element of order n in the automorphism group of E .
- (5) Let E be the splitting field for $x^4 + 2$ over $\mathbb{Q}$ (that is to say the subfield of the complex numbers generated by the roots). Compute $[E : \mathbb{Q}]$ and find an element α such that $E = \mathbb{Q}(\alpha)$. Repeat the exercise for $x^4 + 3$.
- (6) Prove that every algebraically closed field is infinite.
- (7) Find an example of a field which is isomorphic to a proper subfield of itself.