ALGEBRA HOMEWORK SET IV

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You may collaborate on this homework set, but must write up your solutions by yourself. Please contact me by email if you are puzzled by something, would like a hint or believe that you have found a typo.

NB: This is due a week from Monday, that is to say on Monday Feb 25.

We write 0 for the zero module $\{0\}$, and $M \leq N$ when M is a submodule of N.

(1) (a) (internal sum) Prove that if M_1 and M_2 are submodules of an R-module M, then the submodule generated by $M_1 \cup M_2$ is
$$M_1 + M_2 = \{m_1 + m_2 : m_i \in M_i\}.$$
(b) (external direct sum) Prove that if M_1 and M_2 are R-modules and we define operations on
$$M_1 \oplus M_2 = \{(m_1, m_2) : m_i \in M_I\}$$
by $r(m_1, m_2) = (rm_1, rm_2)$, $(m_1, m_2) + (n_1, n_2) = (m_1 + n_1, m_2 + n_2)$ then the resulting structure is an R-module.
(c) (internal direct sum) Prove that if $M_1, M_2 \leq M$ and $M_1 \cap M_2 = 0$ then $M_1 + M_2 \simeq M_1 \oplus M_2$. What can you say about the structure of $M_1 + M_2$ in general?

(2) Let M be an R-module and I an ideal of R. Prove that
(a) If we define IM to be the set of all finite sums $\sum_i r_i m_i$ with $r_i \in I, m_i \in M$ then $IM \leq M$.
(b) M/IM has the structure of an R/I module if we define $(r + I)(m + IM) = rm + IM$.

(3) Consider a sequence of R-modules M_i where the index i runs through some interval of integers, together with R-module HMs $\alpha_i : M_i \to M_{i+1}$. The sequence is said to be *exact at* M_i if $im(\alpha_{i-1}) = ker(\alpha_i)$.
(a) Show that $0 \to M_1 \to M_2$ is exact at M_1 iff α_1 is injective.
(b) Show that $M_1 \to M_2 \to 0$ is exact at M_2 iff α_1 is surjective.
(c) When is the sequence $0 \to M_1 \to M_2 \to 0$ exact at both of M_1, M_2?
(d) Suppose that $0 \to M_1 \to M_2 \to M_3 \to 0$ is exact at all of M_1, M_2, M_3. What does this tell you about the relation between the modules M_i?

(4) Recall that when R is an ID we defined the *field of fractions* by considering the set $X = \{(r, s) : r \in R, s \in R \setminus \{0\}\}$ and introducing an equivalence relation $(r_1, s_1) \simeq (r_2, s_2)$ iff $r_1 s_2 \simeq r_2 s_1$.
(a) Show by example that if R is not an ID then the binary relation defined in this way may not be an equivalence relation.
(b) Let R be an arbitrary ring. A subset $S \subseteq R$ is called *multiplicatively closed* iff $1 \in S$ and S is closed under multiplication.
Prove that if we define a relation on $R \times S$ by $(r_1, s_1) \simeq^* (r_2, s_2)$ iff there is $s_3 \in S$ such that $s_3(r_1 s_2 - r_2 s_1) = 0$ then $\simeq^*$ is an equivalence relation. What is this in the case when R is an ID and $S = R \setminus \{0\}$?

(c) With the same assumptions as the last part, we write r/s for the $\simeq^*$-class of the pair (r,s) and RS^{-1} for the set of such classes. Prove that if we attempt to define

$$r_1/s_1 \times r_2/s_2 = (r_1r_2)/(s_1s_2), r_1/s_1 + r_2/s_2 = (r_1s_2 + r_2s_1)/(s_1s_2),$$

then we get well-defined operation which make RS^{-1} into a ring.

(d) Prove that the map $r \mapsto r/1$ is a HM from R to RS^{-1}, and that every element of S is mapped to a unit in RS^{-1}. When is the map injective? When is it surjective?

(5) Let R be a ring and let $Spec(R)$ be the set of prime ideals of R. For each ring element a, let $O_a = \{P \in Spec(R) : a \notin P\}$. Say that a set X of prime ideals is *open* if for every $P \in X$ there exists a such that $P \in O_a \subseteq X$.
  (a) Prove that $O_a \cap O_b = O_{ab}$, $O_0 = \emptyset$, $O_1 = Spec(R)$.
  (b) Prove that the collection of open sets form a topology for $Spec(R)$, and describe it when $R = \mathbb{Z}$.
  (c) (Trickier) Prove that this topology is compact.

(6) Let p be prime. Let $R_n = \mathbb{Z}/p^n\mathbb{Z}$, and let $\pi_n : R_{n+1} \mapsto R_n$ be the surjective HM which maps $a + p^{n+1}\mathbb{Z}$ to $a + p^n\mathbb{Z}$. Define a ring $\mathbb{Z}_p$ as follows: the elements are infinite sequences $(r_0, r_1, \ldots)$ such that $r_i \in R_i$ and $\pi_i(r_{i+1}) = r_i$ for all i. Addition are multiplication are defined coordinatewise

  Prove that
  (a) $\mathbb{Z}_p$ is uncountable.
  (b) $\mathbb{Z}_p$ is an ID which contains an isomorphic copy of $\mathbb{Z}$.
  (c) The sequence $(2, 2, 2, \ldots)$ has a square root in $\mathbb{Z}_7$.

(7) The R-module M is said to be *Artinian* iff there is no infinite strictly decreasing sequence of submodules. R is *Artinian* iff it is Artinian as an R-module, that is there is no infinite strictly decreasing sequence of ideals.
  (a) Give an example of an infinite Artinian ring.
  (b) Prove that if R is a field, the classes of Artinian and Noetherian modules coincide.
  (c) Give an example of a Noetherian module which is not Artinian.
  (d) Give an example of a Artinian module which is not Noetherian.

  Note: all Artinian rings are Noetherian, but it is not completely easy to see this.