ALGEBRA HOMEWORK SET II SOLUTIONS

JAMES CUMMINGS

You may collaborate on this homework set, but must write up your solutions by yourself. Please contact me by email if you are puzzled by something, would like a hint or believe that you have found a typo.

NOTE: This is one is shorter because it's late. Apologies for the delay.

C_n denotes a cyclic group of order n.

(1) Use Sylow's theorem to show that there is no simple group of any of the following orders: 20, 26, 34, 48.

Convention: n_p is the number of Sylow p-subgroups.

20: $n_5 = 1$ so the unique subgroup of order 5 is normal.

26: $n_{13} = 1$.

34: $n_{17} = 1$.

$48 = 2^4 3$. This one is tricky. Clearly $n_3 = 1$ or $n_3 = 4$. Suppose the latter, and note that in this case there is a subgroup of index 4, namely $H = N_G(P)$ for P a Sylow 3-subgroup.

Now consider the action of G on the left cosets of H by left multiplication, that is $g(aH) = (ga)H$. The kernel of this action is a normal subgroup contained in H, so is trivial by simplicity. But the action can't be injective because $4! = 24 \leq 48$.

Note: There are other ways but this is the slickest one I know.

(2) Describe the automorphism group $Aut(C_5)$.

Let a generate C_5. Then any AM must map a to another generator, and is determined by the image of a. So the possible images are a^i for $i = 1, 2, 3, 4$. A little thought shows that $Aut(C_5)$ is isomorphic to the set $\{1, 2, 3, 4\}$ with the group operation being multiplication mod 5. This is a cyclic group of order 4, note that 2 is a generator (so the AM which maps a to a^2 generates the AM group). Let ψ_i be the AM which maps a to a^i.

Find all the HMs from G to $Aut(C_5)$ for

(a) $G = C_4$.

(b) $G = C_2 \times C_2$.

Let $G = C_4$ with generator b. If ϕ is a HM then $\phi(b)$ determines ϕ. We must have $\phi(b)^4 = \phi(b^4) = e$ but this is true for any element of the AM group. So we get 4 HMs.

Now let $G = C_2 \times C_2$ where C_2 is generated by c. A HM is determined by the images of (c, e) and (e, c); these images can be chosen independently (since the AM group is abelian) but must be elements which square to the identity, IE must be either $\psi_1 = id$ or ψ_4 (which is actually inversion). So again there are 4 HMs.

(3) Suppose that $G = HN$ where $H \leq G$, $N \lhd G$ and $H \cap N = \{e\}$.

(a) Show that every element of G can be written as hn for unique elements $h \in H$ and $n \in N$.

Since $G = HN$ every element has at least one representation of this kind. If $h_1 n_1 = h_2 n_2$ for $h_i \in H$, $n_i \in N$ then $h_2^{-1} h_1 = n_2 n_1^{-1} \in H \cap N = \{e\}$, so $h_1 = h_2$ and $n_1 = n_2$.

(b) Express $h_1 n_1 h_2 n_2$ where $h_i \in H$, $n_i \in N$ in the form hn with $h \in H$, $n \in N$.

$h_1 h_2 [(h_2^{-1} n_1 h_2) n_2]$.

(4) Suppose that H, N are groups and $\psi : H \to Aut(N)$ is a HM. Define a binary operation $*$ on the set of ordered pairs $H \times N$ as follows:

$$(h_1, n_1) * (h_2, n_2) = (h_1 \times_H h_2, \psi(h_2^{-1})(n_1) \times_N n_2).$$

Prove that this binary operation makes $H \times N$ into a group.

Agonisingly tedious, so I skip it. Note that $\psi(h^{-1}) = \psi(h)^{-1}$ because ψ is a HM, also $\psi(h)(n^{-1}) = (\psi(h)(n))^{-1}$ because $\psi(h)$ is an AM. Also (e, e) is the identity BUT the inverse of (h, n) is not (h^{-1}, n^{-1}), it is $(h^{-1}, \psi(h)(n^{-1})$.

Cultural note: this is the "semidirect product" construction.

(5) Explain the relationship between the previous two questions.

If $G = HN$ as in the first question and we define $\psi(h)(n) = n^h$, then the group structure on G is given by the formula from the second question. Conversely if in the second question we let $\bar{H} = H \times \{e\}$ and $\bar{N} = \{e\} \times N$, then we are in the situation of the first question.

Use Sylow's theorem and the ideas of the previous two questions to classify groups of order 20 up to isomorphism.

By Sylow the unique Sylow 5-subgroup N is normal. Let it have a generator a. Let H be a Sylow 2-subgroup and note that by Lagrange $H \cap N = \{e\}$. We already know the possibilities for H and the possible HMs from H to $Aut(N)$, so we just work through them.

NOTE: We have to watch out for overcounting here. C_4 and C_2^2 must yield non-isomorphic semi-direct products (think about this) but is possible that distinct HMs from the same group to $Aut(C_5)$ yield isomorphic semi-direct products. See discussion below.

Case I: Semi-direct products of C_4 and C_5. Let b generate C_4 and a generate C_5. Then as we saw above $Aut(C_5)$ is cyclic of order 4, and is generated by the AM which maps a to a^2. There are four HMs from C_4 to $Aut(C_5)$ but it's not hard to see that the the HMs which map b to the generators $a \mapsto a^2$ and $a \mapsto a^3$ will give isomorphic semi-direct products. (The point is that there is an AM of $Aut(C_5)$ which exchanges its two generators).

So we get three (possibly) distinct groups: in generator relation language they are generated by $a^4 = b^5 = e$ and one of the relations $bab^{-1} = a$, $bab^{-1} = a^2$, $bab^{-1} = a^4$.

Routinely the relations $a^4 = b^5 = e$ and $bab^{-1} = a$ give $C_4 \times C_5 \simeq C_{20}$.

The other relations can be written as $ba = a^2 b$ and $ba = a^4 b$. So in each case every group element can be uniquely written as $a^i b^j$ for $0 \leq i < 4, 0 \leq j < 5$ and we can use the relations to figure out the multiplication table. Once can check that the results are distinct non-abelian groups (look at orders of elements, conjugacy classes, centers etc till you see a difference).

Case II: Semi-direct products of C_2^2 and C_5. Let C_2 be generated by c so that a HM from C_2^2 to $Aut(C_5)$ is determined by the images of (c, e) and (e, c), which each must be either the identity or $a \mapsto a^4$. One possibility is that everything maps to the identity. The other is that two of the elements (c, e), (e, c) and (c, c) map to the $a \mapsto a^4$ and the third maps to the identity; these are all equivalent (the point this time is that C_2^2 is very "homogeneous", any permutation of its non-identity elements gives an AM).

So we actually get two groups this time. they trun out to be $C_2^2 \times C_5 \simeq C_2 \times C_{10}$ and the dihedral group of order 20.