ALGEBRA HOMEWORK SET I

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You may collaborate on this homework set, but must write up your solutions by yourself. Please contact me by email if you are puzzled by something, would like a hint or believe that you have found a typo.

(1) Let G be a group and let $H, K \leq G$. The *double cosets of H and K* are the subsets of G of the form HaK where $a \in G$. Prove that the double cosets form a partition of G. Is it true in general that all the double cosets have the same cardinality? (Either prove it or give a counterexample)

(2) Let G be a group and let $H, K \leq G$. Prove that if $h_1, h_2 \in H$ then $h_1 K = h_2 K \iff h_1(H \cap K) = h_2(H \cap K)$. Use this to prove that if H, K are both finite then the set of products HK has cardinality $|H| \times |K| / |H \cap K|$.

(3) Recall from class that for any group G
  (a) An *automorphism (AM)* of G is an isomorphism from G to G.
  (b) For any $g \in G$, the map $h \mapsto h^g$ is an AM of G, where as usual $h^g = ghg^{-1}$.
  (c) $N \lhd G$ iff $N^g = N$ for all $g \in G$.

  A subgroup H of G is said to be *characteristic* iff $\alpha(H) = H$ for every AM H of G (so that clearly any characteristic ssubgroup is normal). In this case we write "$H \, \mathit{char} \, G$".
  (a) Prove that $Z(G)$ is a characteristic subgroup of G.
  (b) Give an example of a group G and a normal subgroup of G which is not characteristic. Hint: every subgroup of an abelian group is normal.
  (c) Let G be a group. Prove that if $A \, \mathit{char} \, B$ and $B \, \mathit{char} \, G$ then $A \, \mathit{char} \, G$, and that if $A \, \mathit{char} \, B$ and $B \lhd G$ then $A \lhd G$.
  (d) Give an example of a group G and subgroups $B \lhd G$ and $A \lhd B$ such that A is not normal in G.

(4) Let G be a group and let $H \leq G$ with $[G : H] = 2$.
  (a) Prove that $H \lhd G$, and that if $h \in H$ then the G-conjugacy class of g is a subset of H.
  (b) Prove that if $h \in H$ then *either* the G-conjugacy class of h equals the H-conjugacy class of h *or* the G-conjugacy class of h is the union of two disjoint H-conjugacy classes.

(5) Show that if G is a group such that $a^2 = e$ for all $a \in G$, then G is abelian. (Trickier) Produce an example of a non-abelian group G such that $a^3 = e$ for all $a \in G$.

(6) Prove that if $G/Z(G)$ is cyclic then G is abelian (so in fact $Z(G) = G$). Give an example of a non-abelian group such that $G/Z(G)$ is abelian.

(7) Find $Z(G)$ when G is the group of non-singular $n \times n$ real matrices under matrix multiplication (and $n \geq 2$).