Algebra Midterm

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Due midnight on Saturday March 6. Electronic submission (in PostScript or PDF only) is encouraged. No collaboration is allowed. You may consult any books or papers you wish, but please make a note that you have done so. You may also consult with me.

Hint: later problems sometimes build on earlier ones. When working on a problem you may freely use results from any earlier problem (or earlier part of the current problem), even if you have not yet solved that problem/part.

1. (10 pts) Let n be an integer with $n > 0$ and consider the ring $\mathbb{Z}/n\mathbb{Z}$. Prove that $m + n\mathbb{Z}$ is a unit in this ring if and only if m and n are coprime.

2. (25 pts) Let $n > 1$, let G be a cyclic group with n elements, and let $g \in G$ be an element of order n. For each m let $\psi_m : G \longrightarrow G$ be given by $\psi_m : g^i \mapsto g^{im}$.

   (a) Prove that $\{\psi_m : 0 \leq m < n\}$ is the set of homomorphisms from G to G.

   (b) For which m is ψ_m an automorphism of G?

(c) Determine the automorphism group of a cyclic group with 15 elements.

(d) Find all the homomorphisms from a cyclic group with two elements to the automorphism group of a cyclic group with 15 elements.

3. (15 pts) Prove that if G is a group of order 15, G is cyclic.

4. (40 pts) Let G be a group of order 30.

   (a) Let P be a Sylow 3-subgroup and let Q be a Sylow 5-subgroup. Prove that at least one of P and Q is normal. Hint: Sylow + counting.

   (b) Prove that PQ is a normal subgroup of G.

   (c) Let R be a Sylow 2-subgroup. Prove that $(PQ)R = G$ and $PQ \cap R = \{e\}$.

   (d) Let PQ be generated by some element a of order 15 and let R be generated by some element b of order 2. Prove that a^b is one of a, a^4, a^{11}, a^{14}.

   (e) Prove that up to isomorphism there are exactly four possibilities for G, and describe them.

5. (20 pts) Prove that the following are equivalent for an element r of a ring R:

   (a) $1 + rs$ is a unit for all $s \in R$.

   (b) r is in every maximal ideal of R.

You may use the fact that $x \in R$ is not a unit if and only if $x \in M$ for some maximal ideal M of R.

6. (15 pts) Let p be prime and let $\mathbb{Z}_{(}p)$ be the set of rationals that can be written as a/b where $b \neq 0$ and p does not divide b. Prove that $\mathbb{Z}_{(}p)$ is a PID, and describe the ideals.

7. (35 pts) For one of the non-abelian non-dihedral groups of order 30 determine the set of all the subgroups, and draw a picture of the inclusion relations between these subgroups.

8. (25 pts) Let $R = \mathbb{Z}[i\sqrt{5}]$, that is the least subring of $\mathbb{C}$ containing $\mathbb{Z} \cup \{i\sqrt{5}\}$. This ring is known not to be a UFD, in fact each of the elements $2, 3, 1+i\sqrt{5}, 1-\sqrt{5}$ is irreducible but $2\times 3 = (1+i\sqrt{5})(1-\sqrt{5})$.

   Show that

   (a) R is a Noetherian ring.

   (b) Every nonzero prime ideal gives a finite quotient.

   (c) Every nonzero prime ideal is maximal.

9. True or false: if R is Noetherian, every submodule of a free R-module of finite rank is free? Either prove it or give a counterexample.