Algebra Homework Set 3

James Cummings (jcumming@andrew.cmu.edu)

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1. Let p and q be distinct primes. Prove that if G has order pq then it has either a normal subgroup of order p or one of order q. Conclude that G is solvable.

2. Find the derived subgroup of the dihedral group with 2n elements.

3. Let G be a nonabelian group of order p^3 where p is prime. Show that

   (a) The centre of G has order p.

   (b) The derived subgroup of G equals the centre of G.

   (c) $G/Z(G)$ is isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

4. Let G be a simple group of order 168. What can you say about the Sylow subgroups of G and their normalisers?

5. Let F be a finite field with q elements (we see later that such fields exist exactly when $q = p^n$ for p prime and $n > 0$, and for each q there is exactly one such field up to isomorphism).

   (a) Let $GL_2(F)$ be the group of 2×2 matrices with entries in F and non-zero determinant, with the group operation being matrix multiplication. Find the order of $GL_2(F)$. Hint: a matrix is in the group when its columns are non-zero and linearly independent.

(b) Let $SL_2(F)$ be the subgroup of matrices with determinant 1. Find the order of $SL_2(F)$.

(c) Prove that $SL_2(F)$ is generated by the set of *coordinate transvections*, that is to say the matrices with ones on the diagonal and a single nonzero entry off the diagonal (actually you don't need F finite for this bit).

6. Let G have order pq^2 where p and q are distinct primes. Prove that G is not simple.

7. * Let F be the field with 5 elements, and let $PSL_2(F)$ be the quotient of $SL_2(F)$ by the normal subgroup $\{I, -I\}$. Prove that $PSL_2(F)$ is isomorphic to A_5.

8. Let G be a nonabelian group of order 8. Fill in the following outline of a (rather laborious) analysis of G.

   (a) G has an element of order 4, which we call a.

   (b) $Z(G) = \{e, a^2\}$.

   (c) If $H = \langle a \rangle$ then $H \lhd G$.

   (d) Fix $b \notin H$. Every element can be written in a unique way as $a^i b^j$ where $0 \leq i < 4$ and $0 \leq j < 2$. Hint: coset decomposition.

   (e) $ab \neq ba$.

   (f) $a^b = a^3 = a^{-1}$.

(g) b has order 2 or order 4.

(h) Both possibilities can occur.

9. Let G be finite, let $\{p_1, \ldots p_n\}$ be the primes dividing the order of G and for each i let P_i be some Sylow p_i-subgroup. Prove that G is generated by $\bigcup_i P_i$.