Algebra Homework Set 2

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Due in class on Friday.

1. Find all the Sylow p-subgroups of S_3 for the relevant primes p. Confirm the predictions made by Sylow's theorem.

2. Prove that if H, K are normal in G and $HK = G$, $H \cap K = \{e\}$ then $(h, k) \mapsto hk$ is an isomorphism from $H \times K$ to G. Hint: what is $hkh^{-1}k^{-1}$?

3. Prove every group of order 15 is isomorphic to $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$. Is every group with order the product of two odd primes abelian?

4. $Aut(G)$ is the set of automorphisms of G, which becomes a group if we define the group operation to be composition. Prove that the map which takes g to the automorphism $h \mapsto h^g$ is a homomorphism from G to $Aut(G)$. What is the kernel? Prove that if $Z(G) = \{e\}$ then

   (a) $Aut(G)$ has a subgroup isomorphic to G.

   (b) $Z(Aut(G)) = \{e\}$.

5. Find the centre of the dihedral group of order 8 (symmetries of the square). In general what is the centre of the dihedral group of order 2n?

6. Prove that two permutations in S_n are conjugate in S_n if and only if they have the same "cycle type" (that is when they are written as products of disjoint cycles, the cycles have the same sizes and each size appears the same number of times: taking S_8 as an example, (12)(345)(678) has the same cycle type as (123)(456)(78)) but a different type from (12)(34)(567)). Prove that every conjugacy class of S_n either consists of even permutations or of odd ones. Find the conjugacy classes of S_4, and for each class c choose a representative element g and describe $C_{S_4}(g)$.

7. * Prove that if c is a conjugacy class of S_n consisting of even permutations, then either c is a conjugacy class of A_n or it splits into two classes of equal size. Hint: use an exercise from last week. Find the conjugacy classes of A_4, and for each class c choose a representative element g and describe $C_{A_4}(g)$.

8. * Repeat exercise 1 for A_4 and S_4.

9. ** Let π be a set of primes and let G be a finite group such that every $p \in \pi$ divides $|G|$, with n_p maximal such that p^{n_p} divides $|G|$. A *Hall π-subgroup* of G is a subgroup of order $\prod_{p\in\pi} p^{n_p}$. Find an example of G, π where G has no Hall π-subgroups.