Algebra Homework Set 1

James Cummings (jcumming@andrew.cmu.edu)

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You may collaborate on these exercises but must write up your solutions yourself. Solutions should be either typed or written clearly in black link on alternate lines, illegible homework will be returned ungraded. Starred exercises are harder.

1. For a fixed positive integer n, let G be the set of all $n \times n$ matrices with real entries and non-zero determinant. Consider the operation of matrix multiplication.
   (a) Prove that G is a group.
   (b) Prove that G is abelian only for $n = 1$.
   (c) What is the set of $x \in G$ such that $\forall y \in G\ xy = yx$?
2. Prove if G is abelian then the product of two elements of finite order has finite order. What can you say about the order of the product in terms of the orders of the factors?
3. Prove that if H and K are two finite subgroups of a group G and HK is the set of products $\{hk : h \in H, k \in K\}$ then HK has $\frac{|H| \times |K|}{|H \cap K|}$ elements.

4. Let X be an infinite set and let G be the subset of Σ_X consisting of permutations which fix all but finitely many elements of X. Prove that G is a subgroup. Is it normal?

5. * Find a countable group G which is not finitely generated (that is to say $G \neq \langle X \rangle$ for any finite $X \subseteq G$, where $\langle X \rangle$ is the subgroup generated by X).

6. * True or false: if G is a group such that $g^3 = e$ for all $g \in G$ then G is abelian? (It is well known to be true if $g^2 = e$ for all g).

7. * Let $H \leq G$ with $[G : H] = 2$. Let $x \in H$ and let C_0, C_1 be the conjugacy classes of x in G, H respectively. Prove that $C_0 \subseteq H$, and that if C_0 is finite then either $C_0 = C_1$ or $|C_0| = 2 \times |C_1|$.

8. ** A group G is said to be simple if the only normal subgroups of G are $\{e\}$ and G. Let G be simple and let H be a subgroup with $[G : H] = n$ for some integer $n > 1$. Prove that G is a finite group with at most $n!$ elements.

9. ** (For those who know a little bit of topology). A topological group is a group equipped with a Hausdorff topology which makes the maps $x \mapsto x^{-1}$ and $(x, y) \mapsto xy$ continuous. Prove that the kernel of a continuous homomorphism between topological groups is closed, and that in general not all normal subgroups of such a group are closed.