Algebra Final

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This final is due at midnight on May 10. You may work on it during any continuous 24 hour period of your choosing. No collaboration is allowed. You may consult any books or papers you wish, but please make a note that you have done so. You may also consult with me.

You should attempt questions 1-11 and at least 5 of questions 12-25 (you will be graded on your best five attempts at these questions, partial answers will receive much less credit than complete answers).

1. Let G be a group. Let $H < G$ be a subgroup whose index in G is finite. Let $N \lhd G$ be a finite normal subgroup of G. Assume that the greatest common divisor of $|N|$ and $[G : H]$ is 1. Prove that $N \subseteq H$. CAUTION: WE ARE NOT ASSUMING G IS FINITE HERE.

2. Let G be a finite group, let H be a subgroup of G, and let X be the set of left cosets of H. Define an action ρ of H on X by $\rho(h)(xH) = hxH$. Prove that

   (a) xH is a fixed point for this action iff $x \in N_G(H)$.

   (b) If H is a p-group for some prime p, then $[G : H]$ is congruent to $[N_G(H) : H]$ modulo p.

3. Let G be the subgroup of the free abelian group $\mathbb{Z}^3$ generated by $(1,0,1)$, $(0,1,1)$ and $(1,1,0)$. Find a basis $\{y_1, y_2, y_3\}$ for $\mathbb{Z}^3$ and integers $m_1, \ldots m_k$ such that $m_j | m_{j+1}$ for $j < k$ and $G = \langle m_1 y_1, \ldots m_k y_k \rangle$. Describe the group $\mathbb{Z}^3/G$ as a product of cyclic groups.

4. Let G be a group. Let H_1, H_2 be distinct proper subgroups of G. Prove that $G \neq H_1 \cup H_2$. Is it possible for a group to be the union of 3 distinct proper subgroups?

5. Let p be prime and let P be a Sylow p-subgroup of $G = S_p$. Find the order of $N_G(P)$.

6. Prove that groups of the following orders are never simple: 56, 200, 312, 351.

7. Classify the abelian groups of order 9801 up to isomorphism.

8. Consider the ideals (x, y) and $(2, x, y)$ in $\mathbb{Z}[x, y]$. Which of them is prime? Which of them is maximal?

9. Give an example of a ring R and a finitely generated R-module M such that not every R-submodule of M is finitely generated.

10. Let $\zeta = e^{\pi i/4}$ and let $E = \mathbb{Q}(\zeta)$.

    (a) Prove that $E = \mathbb{Q}(\sqrt{2}, i)$.

    (b) Prove that $[E : \mathbb{Q}] = 4$ and find a basis for E as a VS over $\mathbb{Q}$.

    (c) Prove that E is a Galois extension of $\mathbb{Q}$.

    (d) Describe the Galois group $\Gamma(E/\mathbb{Q})$.

(e) Find the minimal polynomial of ζ over $\mathbb{Q}$.

11. Let F be a splitting field for $x^4 + 3$ over $\mathbb{Q}$.

    (a) Find $[F : \mathbb{Q}]$.

    (b) Describe $\Gamma(F/\mathbb{Q})$.

    (c) Find all the fields intermediate between $\mathbb{Q}$ and F, and identify the Galois extensions of $\mathbb{Q}$ among them.

    Hint: the preceding question may prove helpful here.

12. Let F be a field. Let $f = \sum_{i=0}^{n} c_i x^i \in F[x]$ be a monic irreducible polynomial of degree n. Let g be any polynomial in $F[x]$ and let $h \in F[x]$ be the polynomial $\sum_{i=0}^{n} c_i (g(x))^i$. Prove that if k is an irreducible factor of h then $deg(k)$ is divisible by n.

13. A subgroup M of a finite group G is said to be a *maximal subgroup* if $M \neq G$ and there is no subgroup N such that $M \subsetneq N \subsetneq G$. The "Frattini subgroup" of G is defined to be the intersection of all the maximal subgroups of G. We denote the Frattini subgroup by $\Phi(G)$.

    (a) Prove that $\Phi(G) \lhd G$.

    (b) Prove that if H is a subgroup of G whose index in G is prime then H is maximal.

    (c) Prove that if G is a finite abelian p-group then $\Phi(G) = \{g^p : g \in G\}$. (You may assume that G is a product of cyclic p-groups).

14. Let G be the group of matrices with integer entries and determinant 1. Prove that G is generated by the matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, -I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

15. Let G be a group, and let $\phi \in Aut(G)$ be such that $\forall g \in G\ g \neq e \rightarrow \phi(g) \neq g$.

    (a) Prove that the function from G to G defined by $F(g) = g^{-1}\phi(g)$ is injective.

    (b) Prove that if G is finite and $\phi \circ \phi = id_G$, then G is abelian and the order of G is odd. Hint: prove first that $\phi(x) = x^{-1}$ for all $x \in G$.

16. Let G be a finite group. Prove that if a and b are elements of order 2 and ab has odd order then a and b are conjugate in G. Hint: what can you say about $(ab)^l b (ab)^l$ where $l > 0$?

17. Let G be a finite group, p a prime dividing the order of G and P a Sylow p-subgroup of G. Let x and y be elements of $Z(P)$ which are conjugate in G. Show that x and y are conjugate in $N_G(P)$. Hint: look at $C_G(x)$.

18. Prove that the group of $n \times n$ real upper triangular matrices with ones on the diagonal is nilpotent.

19. Let σ be the element of $\Gamma(\mathbb{Q}(x)/\mathbb{Q})$ given by $\sigma(x) = 1/(1-x)$. Show that $\sigma^3 = id$. Find the fixed field of the group generated by σ.

20. Classify all the groups of order 75.

21. Find (with justification) the minimal polynomial of $1 + \sqrt{2} + \sqrt[3]{3}$ over $\mathbb{Q}$.

22. Show that $\mathbb{Z}[\sqrt{10}]$ is not a UFD. Find all the prime ideals P of $\mathbb{Z}[\sqrt{10}]$ such that $(3) \subseteq P$.

23. Find the Galois group for the polynomial $(x^3 - 2)(x^4 + 1)$ over $\mathbb{Q}$. Determine all the subgroups of the Galois group and all the intermediate fields. Which of the intermediate fields are normal extensions of $\mathbb{Q}$?

24. Let p and q be distinct primes. Prove that all groups of order p^2q^2 are solvable.

25. Let G be a finite group which acts transitively on X. Let $x \in X$, let p be a prime dividing the order of $Stab(x)$ and let P be a Sylow p-subgroup of $Stab(x)$. Prove that $N_G(P)$ acts transitively on $Fix(P)$.