An Introduction to Itreated Ultrapowers

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Introduction

In these notes, we develop the basic theory of iterated ultrapowers of models of set theory. The notes are intended for a student who has taken one or two semesters of graduate-level set theory, but may have little or no prior exposure to ultrapowers and iteration.

We shall develop the pure theory, which centers on the question of well-foundedness for the models produced in various iteration processes. In addition, we consider two sorts of application:

(1) Large cardinal hypotheses yield regularity properties for definable sets of reals. Large cardinal hypotheses yield that logical simple sentences are absolute between V and its generic extensions.
(2) Large cardinal hypotheses admit canonical inner models having wellorders of $\mathbb{R}$ which are simply definable

Roughly, applications of type (1) involves using the large cardinal hypotheses to construct complicated iterations. Applications of type (2) involves bounding the complexity of the iterations one can produce under a given large cardinal hypothesis.

The notes are organized as follows. In lecture 1, we develop the basic theory of ultrapower $\mathrm{Ult}(M, U)$, where M is a transitive model of **ZFC** and U is an ultrafilter over M. In lecture 2, we develop the pure theory of iterations of such ultrapowers, and present some applications of type (1). Lecture 3 concerns applications of type (2). In lecture 4 we develop the basic theory of ultrapower $\mathrm{Ult}(M, E)$, where M is as before, but E is now a system of ultrafilters over M known as an "extender". Lecture 5 concerns linear iteration of such extender-ultrapowers, and its applications of type (1) and (2).

In lecture 6, we move from linear iteration to iteration trees, and develop the pure theory of this more general iteration process. This theory is far from complete; indeed, it contains one of the most important question in pure large cardinal theory, the Unique Branches Hypothesis (UBH) for countably closed iteration tree on V. We shall discuss UBH, and its role.

In lecture 7, we present some applications of iteration trees in proofs of generic absoluteness and Lebesgue measurability. Those involves Woodin's "extender algebra", and the corresponding "genericity iterations".

Finally, we outline in lecture 8 how iteration trees contribute to the theory of canonical inner models with Woodin cardinals.

LECTURE 1

Measures and Embeddings

For this standard material, see for example [[**2**], Chapter 17].

Let $M \models \mathbf{ZFC}^-$ be transitive, and

$$j : M \to N$$

be elementary. Let $\kappa = \mathrm{crit}(j) =$ least α such that $j(\alpha) \neq \alpha$. We suppose $\kappa \in \mathrm{wfp}(N)$, the well-founded part of N. (Here and later, we assume all well-founded parts have been transitivized.) For $A \subset \kappa$, $A \in M$, put

$$A \in U_j \quad \text{iff} \quad \kappa \in j(A).$$

Then

(1) U_j is a nonprincipal ultrafilter on $\mathcal{P}(\kappa)^M$;
(2) U_j is M κ-complete: if $\langle A_\alpha | \alpha < \beta\rangle \in M$, and $\beta < \kappa$, and each $A_\alpha \in U_j$, then $\bigcap_{\alpha<\beta} A_\alpha \in U_j$;
(3) U_j is M normal: if $f : \kappa \to \kappa$ and $f \in M$, and $f(\alpha) \in \alpha$ for U_j almost every α, then $\exists \beta < \kappa (f(\alpha) = \beta)$ for U_j a.e. α;
(4) $\mathcal{P}(\kappa)^M \subseteq \mathcal{P}(\kappa)^N$. Moreover, $\mathcal{P}(\kappa)^M = \mathcal{P}(\kappa)^N$ iff U_j is M-amenable: whenever $f : \kappa \to \mathcal{P}(\kappa)$ and $f \in M$, then $\{\alpha < \kappa \mid f(\alpha) \in U_j\} \in M$.

Remark. $M = V$ is an interesting special case. Then $U_j \in M$, so it is certainly amenable. κ is a measurable cardinal in this case.

Exercise 1. Prove (1)-(4) above.

Definition 1.1. An M**-ultrafilter** on κ is an M-amenable nonprinciple ultrafilter on $\mathcal{P}(\kappa)^M$.
An M-normal, M-ultrafilter on κ is called an M**-nuf** on κ.

Conversely, suppose $M \models \mathbf{ZFC}^-$ is transitive, and U is a non-principal ultrafilter on $\mathcal{P}(\kappa)^M$. We can, using functions in M, form an ultrapower:

$$f \sim g \quad \text{iff} \quad \{\alpha \mid f(\alpha) = g(\alpha)\} \in U_j$$

let $[f] = \{g \mid g \sim f\}$, then

$$[f] \widetilde{\in} [g] \quad \text{iff} \quad \{\alpha \mid f(\alpha) \in g(\alpha)\} \in U.$$

We set

$$\mathrm{Ult}(M, U) = (\{[f] \mid f \in M\}, \in),$$

where once again, we assume the well-founded part is transitivized. We then have:

(1) Łoś Theorem: for any $f_0, \cdots, f_n$ and $\varphi(v_0, \cdots, v_n)$,
$$\mathrm{Ult}(M, U) \models \varphi[[f_0], \cdots, [f_n]] \text{ iff for } U \text{ a.e. } \alpha,\ M \models \varphi[f_0(\alpha), \cdots, f_n(\alpha)].$$
(2) $i^M_U : M \to \mathrm{Ult}(M, U)$ is elementary, where
$$i^M_U(x) = [\lambda\alpha.x]$$
the equivalent class of constantly x function.
(3) If U is M-κ-complete, then $\kappa = \mathrm{crit}(i^M_U)$, (In general, the critical point is the M-completeness of U.)
(4) Letting $id : \kappa \to \kappa$ be the identity function, then
$$[f] = i^M_U(f)([id])$$
for any f. (Apply Los to see this.) Also for $A \in \mathcal{P}(\kappa)^M$:
$$A \in U \quad \text{iff} \quad [id] \in i^M_U(A).$$

Thus $\mathrm{Ult}(M,U) = \left\{ i_U^M(f)([id]) \mid f \in M \right\}$ =Skolem closure of $\mathrm{ran}(i_U^M) \cup \{[id]\}$ inside $\mathrm{Ult}(M,U)$.

(5) If U is M-normal, then
$$[id] = \kappa.$$
Thus
$$[f] = i_U^M(f)(\kappa),$$
and
$$A \in U \quad \text{iff} \quad \kappa \in i_U^M(a),$$
and
$$\begin{aligned} \mathrm{Ult}(M,U) &= \left\{ i_U^M(f)(\kappa) \mid f \in M \right\} \\ &= \text{Skolem closure of } \mathrm{ran}(i_U^M) \cup \{\kappa\} \text{ in } \mathrm{Ult}(M,U). \end{aligned}$$
From (4) above, we easily get

(6) Let U be an M-ultrafilter on κ, and $\langle f_\alpha \mid \alpha < \kappa \rangle \in M$, where $\mathrm{dom}(f_\alpha) = \kappa$ for all α. Then
$$\langle [f_\alpha] \mid \alpha < \kappa \rangle \in \mathrm{Ult}(M,U).$$

Proof. $\langle [f_\alpha] \mid \alpha < \kappa \rangle = \langle i_U^M(f_\alpha)([id]) \mid \alpha < \kappa \rangle$. But
$$\langle i_U^M(f_\alpha) \mid \alpha < \kappa \rangle = i_U^M(\langle f_\alpha \mid \alpha < \kappa \rangle) \restriction \kappa \in \mathrm{Ult}(M,U).$$
So $\mathrm{Ult}(M,U)$ can compute $\langle [f_\alpha] \mid \alpha < \kappa \rangle$ from $[id]$ and $\langle i_U^M(f_\alpha) \mid \alpha < \kappa \rangle$. $\square$

Exercise 2. Prove (1)-(5) above.

Part (5) says that $U = U_{i_U^M}$, that is, U is derived from its own ultrapower embedding. In general, U_j captures only part of the information in j. The relationship is given by the following lemma.

Lemma 1.2. *Let $M \models \mathbf{ZFC}^-$ be transitive, and $j : M \to N$ be elementary, with $\kappa = \mathrm{crit}(j)$ and $\kappa \in \mathrm{wfp}(N)$. Then we have the commutative diagram*

$$\begin{tikzcd} M \arrow[rr, "j"] \arrow[drr, "i"'] & & N \\ & & \mathrm{Ult}(M,U_j) \arrow[u, "k"'] \end{tikzcd}$$

where $i = i_{U_j}^M$, and $k([f]) = k(i(f)(\kappa)) =_{df} j(f)(\kappa)$. Moreover,

(a) *$k \restriction \mathcal{P}(\kappa)^{\mathrm{Ult}} = id$ and $k \restriction (\kappa^+)^{\mathrm{Ult}} = id$,*

(b) *if $\mathcal{P}(\kappa)^M = \mathcal{P}(\kappa)^N$, then $k \restriction \mathcal{P}(\mathcal{P}(\kappa))^{\mathrm{Ult}} = id$ and $k \restriction (\kappa^{++})^{\mathrm{Ult}} = id$.*

Exercise 3. Prove Lemma 1.2.

So it follows that $\mathrm{Ult}(M,U_j)$ is isomorphic to $\mathrm{Hull}^N(\mathrm{ran}(j) \cup \{\kappa\})$, via κ.

It is often important to take ultrapowers of models of less than full $\mathbf{ZFC}^-$. In this case, we only get restricted forms of Los' Theorem. For example:

Exercise 4. Let M be transitive, rudimentarily closed, and $M \models \mathbf{AC}$. (A paradigm case is $M = L_\lambda$, for some limit λ.) Let U be an ultrafilter on $\mathcal{P}(\kappa)^M$. Then

(a) $\mathrm{Ult}(M,U)$ is well defined,

(b) Łoś' Theorem holds for Σ_0 formulae,

(c) $i_U^M : M \to \mathrm{Ult}(M,U)$ is Σ_1-elementary.

We turn now to the issue of the well-foundness of the "target model" for our embeddings. We are generally only able to use well-founded models, so this is a crucial issue. We want to stay in the realm of well-founded models!

Note first that if $j : M \to N$ where N is well-founded, then $\mathrm{Ult}(M,U_j)$ is well-founded, since it embeds into N.

A sufficient condition that $\mathrm{Ult}(M,U)$ be well-founded is given by

Definition 1.3. Let U be an ultrafilter on $\mathcal{P}(\kappa)^M$. Then U is **ω_1-complete** iff whenever $A_n \in U$ for all $n < \omega$, then $\bigcap_{n<\omega} A_n \neq \emptyset$.

(There is no requirement that $\langle A_n \mid n < \omega \rangle \in M$, which is why we do not demand $\bigcap_{n<\omega} A_n \in U$.)

Lemma 1.4. *Let M be transitive, rudimentarily closed, and U be an ω-complete ultrafilter on $\mathcal{P}(\kappa)^M$. Then $\mathrm{Ult}(M,U)$ is well-founded.*

Proof. Suppose $[f_{n+1}]\widetilde{\in}[f_n]$ for all n. Set

$$\alpha \in A_n \Longleftrightarrow f_{n+1}(\alpha) \in f_n(\alpha).$$

Then $\bigcap_{n<\omega} A_n = \emptyset$.(Since any element α in this intersection will produce a infinite descending $\in$-sequence in M:

$$f_0(\alpha) \ni f_1(\alpha) \ni f_2(\alpha) \ni \cdots \ni f_n(\alpha) \ni \cdots .$$

□

Corollary 1.5. *Let M be an transitive, rudimentarily closed, and closed under ω-sequences. Let U be an M-κ-complete ultrafilter on $\mathcal{P}(\kappa)^M$. Then $\mathrm{Ult}(M,U)$ is well-founded.*

Proof. If $\langle A_n \mid n < \omega \rangle$ is a counterexample to the well-foundedness, then $\langle A_n \mid n < \omega \rangle \in M$ by the closure of M under ω-sequences. This would then imply U is not M-κ-complete. □

Remark. Thus $\mathrm{Ult}(V,U)$ is well founded, and more generally, $\mathrm{Ult}(M,U)$ is well-founded when $U \in M$, and $M \vDash \mathbf{ZFC}$.

Exercise 5. Let U be an ultrafilter on $\mathcal{P}(\kappa)^V$. Then $\mathrm{Ult}(V,U)$ is well-founded iff U is ω-complete.

ω-completeness is only a sufficient condition for the well-foundedness of $\mathrm{Ult}(M,U)$. For if M is itself countable, and U is an ultrafilter on $\mathcal{P}(\kappa)^M$ which is non-principal, then U is not ω-complete. But now take any transitive $N \vDash \mathbf{ZFC}$ and ultrafilter W on $\mathcal{P}(\kappa)^N$ such that $\mathrm{Ult}(N,W)$ is well-founded. Let

$$\pi : H \to V_\theta.$$

where

$$\pi((M,U)) = (N,W)$$

and H is countable transitive. Then π restricts to $\pi : \mathrm{Ult}(M,U) \to \mathrm{Ult}(N,W)$, so $\mathrm{Ult}(M,U)$ is well-founded. But M is countable, so U is not ω-complete.

We conclude this section with a few more basic facts.

Proposition 1.6. *Let U be an M-ultrafilter on κ, then*

$$U \notin \mathrm{Ult}(M,U).$$

Proof. Let $i = i^M_U$, and note that $\mathrm{Ult}(M,U) \vDash i(\kappa)$ is stronly inaccessible. If $U \in \mathrm{Ult}(M,U)$, then the map

$$f \mapsto [f]_U \qquad (f \in \kappa^\kappa)$$

is in $\mathrm{Ult}(M,U)$, and mps κ^κ onto $i(\kappa)$, contrary to inaccessibility. □

Exercise 6. Let κ be strongly inaccessible. Show any stationary $S \subseteq \kappa$ can be partitioned into κ-many pairwise disjoint stationary sets.

[Hint: Otherwise, we get a stationary $T \subseteq S$ such that if U is the club filter on T, then U is a V-ultrafilter on κ. Now show $U \in \mathrm{Ult}(V,U)$.]

Proposition 1.7. *Let U be a V-nuf on κ, and $i = i^V_M$. Then*

(a) $2^\kappa < i(\kappa) < (2^\kappa)^+$;
(b) *if* $\mathrm{cof}(\alpha) \neq \kappa$, *then* $i(\alpha) = \sup_{\beta<\alpha} i(\beta)$;
(c) *if* $\mathrm{cof}(\alpha) \neq \kappa$ *and* $\forall \beta < \alpha (\beta^\kappa < \alpha)$, *then* $i(\alpha) = \alpha$.

Proof. For (a), note $2^\kappa \leq (2^\kappa)^{\mathrm{Ult}(V,U)} < i(\kappa)$, because $\mathcal{P}(\kappa) \subseteq \mathrm{Ult}(V,U)$, and $i(\kappa)$ is inaccessible there, But $f \to [f]_U$ $(f \in \kappa^\kappa)$ shows $|i(\kappa)| \leq 2^\kappa$.

For (b), if $[f] < i(\alpha)$, then $f(\nu) < \alpha$ for U a.e. ν. So $[f] < i(\beta)$.

(c) is also easy. □

Exercise 7. Prove Rowbottom's Theorem: Let U be a M-nuf on κ, where $M \models \mathbf{ZFC}$ is transitive. (M rudimentarily closed sufficient.) Let $f : [f]^n \to \gamma$, with $\gamma < \kappa$, and $f \in M$. Then there is an $A \in U$ such that f is constant on $[A]^n$.

[Hint: The proof is by induction on n. You need to prove a little more than what is stated, to cope with the possibility that $U \in M$.]

LECTURE 2

Iterated Ultrapowers

Let $M \models \mathbf{ZFC}^-$ be transitive, and suppose

$$M \models \mathcal{E} \text{ is a set of normal ultrafilters.}$$

For any $U \in \mathcal{E}$, we can form $M_1 = \mathrm{Ult}(M, U)$, with $i_U^M : M \to M_1$ elementary. We can then take any $W \in i_E^M(\mathcal{E})$ and form $\mathrm{Ult}(M_1, W) = M_2$, with $i_W^{M_1} : M_1 \to M_2$ elementary. (That is, we can do so if M_1 was well-founded. It would make perfect sense for ill-founded M_1, as well, but formally speaking, we have not consider that case.) In this way, we produce

$$M = M_0 \to M_1 \to M_2 \to \cdots \to M_\alpha \to M_{\alpha+1} \to \cdots$$

where we continue at limit steps by taking M_λ to be the direct limit of the M_α for $\alpha < \lambda$.

Definition 2.1. Let M be transitive, and $M \models \mathbf{ZFC}^- +$ "$\mathcal{E}$ is a set of nufs". A **linear iteration** of $(M, \mathcal{E})$ is a sequence $I = \langle U_\alpha \mid \alpha < \beta \rangle$ such that there are (unique) transitive M_α, $\alpha < \beta$, and $i_{\alpha\gamma} : M_\alpha \to M_\gamma$ for $\alpha \le \gamma < \beta$, with

(1) $M_0 = M$;
(2) $U_\alpha \in i_{0\alpha}(\mathcal{E})$ for $\alpha < \beta$;
(3) if $\alpha + 1 < \beta$, then

$$\begin{aligned} M_{\alpha+1} &= \mathrm{Ult}(M_\alpha, U_\alpha), \\ i_{\alpha,\alpha+1} &= i_{U_\alpha}^{M_\alpha}, \text{ and} \\ i_{\xi,\alpha+1} &= i_{\alpha,\alpha+1} \circ i_{\xi,\alpha} \text{ for } \xi < \alpha. \end{aligned}$$

(4) if $\lambda < \beta$ is a limit,

$$\begin{aligned} M_\lambda &= \text{direct limit of } M_\alpha,\ \alpha < \lambda, \text{ under } i_{\alpha\gamma}\text{'s} \\ i_{\alpha\lambda} &= \text{direct limit map, for } \alpha < \lambda. \end{aligned}$$

If I is a linear iteration of $(M, \mathcal{E})$, we write U_α^I, M_α^I, $i_{\alpha\gamma}^I$ for the associate ultrafilters , models, and embeddings. There is a unique "last model" associated to I:

$$\begin{aligned} M_\infty^I &= \text{direct limit of } M_\alpha^I, \text{ for } \alpha < \mathrm{lh}(I), \text{ if } \mathrm{lh}(I) \text{ is a limit.} \\ M_\infty^I &= \mathrm{Ult}(M_\alpha^I, U_\alpha^I)\text{,s if } \alpha + 1 = \mathrm{lh}(I). \end{aligned}$$

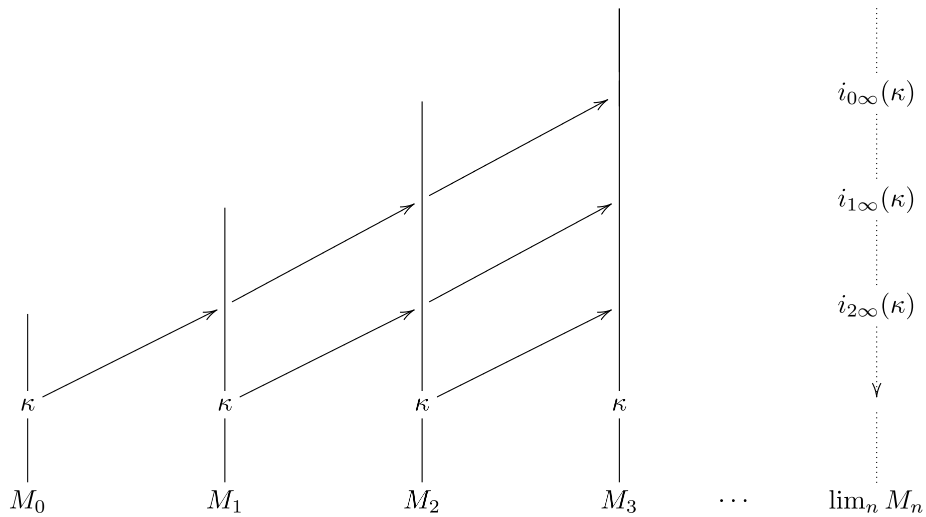
We let $I_{\alpha,\infty}^I : M_\alpha^I \to M_\infty^I$ be the canonical embedding, for $\alpha < \mathrm{lh}(I)$. Unlike the M_α^I, M_∞^I may be ill-founded.

Definition 2.2. Let $M \models \mathbf{ZFC}^- +$ "$\mathcal{E}$ is a set of nufs", M transitive. We say $(M, \mathcal{E})$ is **linearly iterable** iff for every linear iteration I of $(M, \mathcal{E})$, M_∞^I is well-founded.

Just as it sufficed for well-foundedness of single ultrapower, ω-completeness suffices for linear iterability.

Theorem 2.3. *Let $M \models \mathbf{ZFC}^- +$ "$\mathcal{E}$ is a set of nufs", with M transitive. Suppose every $U \in \mathcal{E}$ is ω-complete (in V). Then $(M, \mathcal{E})$ is linearly iterable.*

Remark. Suppose U is a nuf on κ, and set $M_0 = V$, and $M_{n+1} = \mathrm{Ult}(M_n, U)$ for all n. (By induction, $\mathcal{P}(\kappa)^{M_n} = \mathcal{P}(\kappa)$, so this make sense.) Then the direct limit of M_n is ill-founded, as the picture shows:



The moral is that in a legitimate iteration, you can't just pull the next ultrafilter out of your hat! It has to come form the last model.

The proof of Theorem 2.3 will rely on some lemmas of independent interest. Let us call $(M, \mathcal{E})$ s.t. $M \vDash \mathbf{ZFC}^- +$ "$\mathcal{E}$ is a set of nufs", and M transitive, a **good pair**. We say $(M, \mathcal{E})$ is **α-linearly iterable** iff whenever I is a linear iteration of $(M, \mathcal{E})$ and $\mathrm{lh}(I) < \alpha$, then M^I_∞ is well-founded.

Lemma 2.4. *Let $(M, \mathcal{E})$ be a good pair which is α-linearly iterable. Let $\pi : N \to M$ elementary, with $\pi(\mathcal{F}) = \mathcal{E}$. Then $(N, \mathcal{F})$ is α-linearly iterable.*

Proof. Let I be an iteration of $(N, \mathcal{F})$ with $\mathrm{lh}(I) < \alpha$. We can complete the diagram

$$\begin{tikzcd}
M_0 \arrow[r] & M_1 \arrow[r] & \cdots M_\xi \arrow[r] & \cdots M^J_\infty \\
N_0 \arrow[r] \arrow[u, "\pi_0"] & N_1 \arrow[r] \arrow[u, "\pi_1"] & \cdots N_\xi \arrow[r] \arrow[u, "\pi_\xi"] & \cdots N^I_\infty \arrow[u, "\pi_\infty"]
\end{tikzcd}$$

Here $M_0 = M$, $N = N_0$, $\pi_0 = \pi$, and $U_\xi = \pi_\xi(U^I_\xi)$ give us J.We get $\pi_{\xi+1}$ by setting

$$\pi_{\xi+1}([f]_{U^I_\xi}) = [\pi_\xi(f)]_{U^J_\xi},$$

and π_λ a limit because the diagram commutes. Since M_∞ is well founded, and N^I_∞ embeds into it, N^I_∞ is well-founded. $\square$

Failures of iterability reflect into the countable:

Lemma 2.5. *Let $(M, \mathcal{E})$ be a good pair. Equivalent are:*

(1) *$(M, \mathcal{E})$ is linearly iterabbble;*
(2) *whenever $\pi : N \to M$ with N countable and $\pi(\mathcal{F}) = \mathcal{E}$, then $(N, \mathcal{F})$ is ω_1-linearly iterable.*

Proof. Lemma 2.4 gives (1)→(2). Now assume (1) fails, and let I be an iteration of $(M, \mathcal{E})$ such that M^I_∞ is ill-founded. Let $\sigma : H \to V_\theta$ with H countable transitive, θ large, and $\sigma((N, \mathcal{F})) = (M, \mathcal{E})$, and $\sigma(J) = I$. Then

$$H \vDash J \text{ is an iteration of } (N, \mathcal{F}) \text{ with } M^J_\infty \text{ ill-founded.}$$

But the right hand side is absolute for well-founded models, so as $\mathrm{lh}(J) < \omega_1$, $(N, \mathcal{F})$ is not ω_1-linearly iterable. But setting $\pi = \sigma \restriction N$, this shows (2) fails. $\square$

The following lemma is the beg to our proof of Theorem 2.3. It is due to Jensen.

Lemma 2.6. *Let $M \vDash \mathbf{ZFC}^- +$ "U is a nuf on κ", with M transitive. Equivalent are:*

(1) *U is ω-complete;*
(2) *whenever $\pi : N \to M$ is elementary, with N countable and $\pi(W) = U$, then there is a σ such that*

$$\begin{tikzcd} M & \\ N \arrow[u, "\pi"] \arrow[r, "i_W^N"'] & \mathrm{Ult}(N, W) \arrow[ul, "\sigma"'] \end{tikzcd}$$

commutes.

Proof. (1)$\to$ (2). Let $\pi : N \to M$ with $\pi(W) = U$. Pick a "typical object" for $\mathrm{ran}(\pi)$, that is α such that

$$\alpha \in \bigcap_{A \in W} \pi(A).$$

This we can do because U is ω-complete. Now set

$$\sigma([f]^N_M) = \pi(f)(\alpha).$$

$\square$

Exercise 8. Show σ is well defined, elementary, and $\pi = \sigma \circ i^N_W$.

Exercise 9. Prove (2) $\to$ (1).

The map σ in (2) of Lemma 2.6 is called "π-realization" of $\mathrm{Ult}(N, W)$. The Lemma 2.6 says that if U is ω-complete, then countable fragment of $\mathrm{Ult}(M, U)$ can be "realized" back in M.

Proof of Theorem 2.3. Let $(M, \mathcal{E})$ be a good pair, and every $U \in \mathcal{E}$ be ω-complete. Suppose $(M, \mathcal{E})$ is not linearly iterable. Let, by Lemma 2.5, $\pi : N \to M$ with N countable, and $\pi(\mathcal{F}) = \mathcal{E}$, and I a countable iteration of $(N, \mathcal{F})$ such that M^I_∞ is ill-founded. Repeatedly using Lemma 2.6,we get

$$\begin{tikzcd} M & & & & \\ N \arrow[u, "\pi_0"] \arrow[r] & N_1 \arrow[ul, "\pi_1"] \arrow[r] & N_2 \arrow[ull, "\pi_2"] \arrow[r] & \cdots \arrow[r] & M^I_\infty \arrow[ullll, "\pi_\infty"'] \end{tikzcd}$$

(π_λ for λ limit comes from the commutativity of the diagram.) Since M is well-founded, so is M^I_∞, a contradiction. $\square$

Corollary 2.7 (ZFC)**.** *If $\mathcal{E}$ is a set of nufs, then $(V, \mathcal{E})$ is linearly iterable.*

Corollary 2.8. *Let $M \vDash$ **ZFC** + "$\mathcal{E}$ is a family of nufs", with M transitive, and $\omega_1 \in M$. Then $(M, \mathcal{E})$ is linearly iterable.*

Proof Sketch. Working inside M, where Corollary 2.7 holds, construct a "universal" linear iteration $I = \langle U_\alpha \mid \alpha < \lambda \rangle$ with (a) $\mathrm{cof}(\lambda) = \omega$, and (b) whenever $W \in i_{0\alpha}(\mathcal{E})$, then $i_{\alpha\beta}(W) = U_\beta$ for cofinally many β. This implies that every $W \in i_{0\infty}(\mathcal{E})$ is ω-complete. Thus $(M^I_\infty, i_{0\infty}(\mathcal{E}))$ is linearly iterable in V. Since $i_{0\infty} : M \to M^I_\infty$, $(M, \mathcal{E})$ is linearly iterable in V. $\square$

Exercise 10. Prove that every $W \in i_\infty(\mathcal{E})$ is ω-complete, granted (a) and (b).

Exercise 11. Where is $M \vDash$ **ZFC** (rather than just $M \vDash$ **ZFC**$^-$) used in the proof of Corollary 2.8?

Some Applications

A. Regularity properties of definable sets of reals.

Theorem 2.9 (Gaifman, Rowbottom)**.** *If there is a measurable cardinal, then for all reals x, $\omega_1^{L[x]}$ is countable.*

Proof. Let U be a nuf on κ. Clearly, there is an M such that $(M, \{U\})$ is a good pair. By Theorem 2.3, $(M, \{U\})$ is iterable. Now let $x \in \mathbb{R}$, and let $\pi : N \to M$ be elementary, with N countable transitive, $x \in N$, and $\pi(W) = U$. So $(N, \{W\})$ is a good pair, and iterable. Let

$$\alpha = (\omega_1^{L[x]})^N.$$

It is enough to see $\alpha = \omega_1^{L[x]}$. But let I be the unique linear iteration of N of length ω_1, and $i : N \to N_\infty^I$ the canonical embedding. Then

$$\alpha = i(\alpha) = (\omega_1^L[x])^{N_\infty^I}.$$

Since $\omega_1 \subseteq N_\infty^I$, this implies $\alpha = \omega_1^{L[x]}$. $\square$

Theorem 2.10 (Solovay)**.** *If there is a measurable cardinal, then all $\underset{\sim}{\Sigma}^1_2$ sets of reals are Lebesgue measurable, have the Baire Property, and have the Perfect Set Property.*

Proof. See [[**2**],]. The proof uses Theorem 2.9. $\square$

Determinacy is the fundamental regularity property, and we have

Theorem 2.11 (Martin)**.** *If there is a measurable cardinal, then all $\underset{\sim}{\Pi}^1_1$ games are determined.*

A proof using iterated ultrapowers can be given, but we shall not go that far a field now.

B. Correctness and Generic Absolutness.

Theorem 2.12. *Let $(M, \{U\})$ be a good pair which is linearly iterable; Then*

$$(HC^M, \in) \prec_{\Sigma_1} (HC, \in).$$

Proof. Here $HC = \{x \mid |TC(x)| < \omega_1\}$ is the class of hereditarily countable sets. Σ_1-over-HC is equivalent to Σ^1_3.

Let I be the unique iteration of M of length ω_1. Then

$$HC^M = HC^{M_\infty^I} \prec_{\Sigma_1} HC,$$

using $\omega_1 \subset M_\infty^I$ and Shoenfield absoluteness. $\square$

Theorem 2.13 (Martin, Solovay)**.** *Let κ be measurable, and G be $\mathbb{P}$-generic for some $\mathbb{P}$ with $|\mathbb{P}| < \kappa$. Then*

$$(HC, \in)^V \prec_{\Sigma_2} (HC, \in)^{V[G]}.$$

Proof. By Tarski-Vaught, it is enough to see that if $x \in HC^V$, and

$$p \Vdash^{\mathbb{P}} (HC, \in) \models \varphi[\check{x}, \tau].$$

where φ is Π_1, then for some $y \in V$, $(HC, \in)^V \models \varphi[x, y]$. But by Löwenheim-Skolem, we can get an iterable good pair $(N, \{U\})$ such that N is countable, and (for q, $\mathbb{Q}$, σ the collapses of p, $\mathbb{P}$, τ)

$$N \models (q \Vdash^{\mathbb{Q}} (HC, \in) \models \varphi[\check{x}, \sigma]).$$

Let $i : N \to N_\infty^I$ be an iteration map, with $\omega_1 \subseteq N_\infty^I$. We may assume $N \models$ "V_x exists", and $\mathbb{Q} \in V_\alpha^N$, where $\alpha < \kappa$. It follows that $V_\alpha^N = V_\alpha^{N_\infty^I}$ is countable, and hence there is in V a $\mathbb{Q}$-generic g over N_∞^I such that $q \in g$. Note here that $i(\langle q, \mathbb{Q}, \sigma\rangle) = \langle q, \mathbb{Q}, \sigma\rangle$. Thus $(HC, \in)^{N_\infty^I[g]} \models \varphi[\check{x}, \sigma_g]$. Since φ is Π_1 and $\omega_1 \subseteq N_\infty^I$, Shoenfield absoluteness implies $(HC, \in)^V \models \varphi[\check{x}, \sigma_g]$. So σ_g is the desired y. $\square$

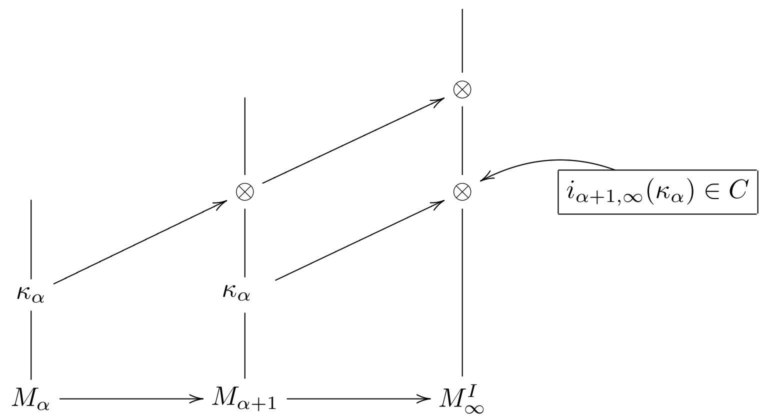
We conclude this section with exercises on two basic features of linear iteration.

What is the analog of $\mathrm{Ult}(M, U) = \left\{i_U^M(f)(\kappa) \mid f \in M\right\}$, for $\kappa = \mathrm{crit}(U)$? We can generate M_∞^I from $\mathrm{ran}(i_{0,\infty}^I)$ together with all the critical points, as follows:

Exercise 12. Let $I = \langle U_\alpha \mid \alpha < \beta \rangle$ be a linear iteration of M, with $\kappa_\alpha = \mathrm{crit}(U_\alpha)$, and set $C = \{i_{\alpha+1,\infty}(\kappa_\alpha) \mid \alpha < \beta\}$. (Note the "$\alpha + 1$" here!) Then

$$M^I_\infty = \left\{ i_{0,\infty}(f)(a) \mid a \in C^{<\omega} \right\}.$$

Picture:



Notice that every $\gamma \in C$ comes from a unique stage in I, in fact, $\gamma = i_{\alpha+1,\infty}(\kappa_\alpha)$ for α least such that $\gamma \in \mathrm{ran}(i_{\alpha+1,\infty})$. (Simply because $\kappa_\alpha \notin \mathrm{ran}(I_{\alpha,\alpha+1})$.) Ordinals in C "belonging to the same measure" are indiscernible, in the following sense:

Exercise 13. Let $I = \langle U_\alpha \mid \alpha < \beta \rangle$ be a linear iteration of M, and $\kappa_\alpha = \mathrm{crit}(U_\alpha)$. For $U \in M$, put

$$C_u = \{i_{\alpha+1,\infty}(\kappa_\alpha) \mid U_\alpha = i_{0\alpha}(U)\}.$$

Let $\gamma_0 < \cdots < \gamma_{n-1}$ and $\delta_0 < \cdots < \delta_{n-1}$ with each $\gamma_i, \delta_i \in C_U$. Let $t \in \mathrm{ran}(i^I_{0,\infty})$. Then

$$M^I_\infty \vDash \varphi[\gamma_0, \cdots, \gamma_{n-1}, t] \Longleftrightarrow M^I_\infty \vDash \varphi[\delta_0, \cdots, \delta_{n-1}, t]$$

for all wff $\varphi(v_0, \cdots, v_{n-1})$.

[Hint: Let $\gamma_i = i_{\alpha_i+1,\infty}(\kappa_{\alpha_i})$. Let J be the iteration of M of length n where you just hit U and its images. Consider the diagram

$$\begin{tikzcd}
M^I_0 \arrow[r] & M^I_{\alpha_0} \arrow[r] & M_{\alpha_0+1} \arrow[r] & M_{\alpha_1} \arrow[r] & M_{\alpha_1+1} \arrow[r] & \cdots \arrow[r] & M^I_\infty \\
M^J_0 \arrow[r] \arrow[u] \arrow[ur] & M^J_1 \arrow[r] \arrow[ur] \arrow[urr] & M^J_2 \arrow[r] \arrow[urr] & \cdots \arrow[r] & M^J_n \arrow[urr] & &
\end{tikzcd}$$

Make sense of the diagram, and use it to show that if $v_i = \mathrm{crit}(U^J_i)$ for $i < n$, then $M^I_\infty \vDash \varphi[\gamma_0, \cdots, \gamma_{n-1}, i_{0\infty}(U)]$ iff $M^J_n \vDash \varphi[\nu_0, \cdots, \nu_{n-1}, i_{0n}(U)]$. Since a similar equivalence for the δ_i's, we're done.]

LECTURE 3

Canonical Inner Models and Comparison

?? Some of the main applications of iterated ultrapowers lie in inner model theory.

We begin with models constructed from coherent sequences of normal ultrafilters. ([**3**]) Roughly speaking, a "coherent sequence" is one in which the nufs occur in order of strength, without leaving gaps. The strength order is the "Mitchell order" $\triangleleft$, where for U and W nufs,

$$U \triangleleft W \quad \text{iff} \quad U \in \mathrm{Ult}(V, W).$$

Clearly, $U \triangleleft W \to \mathrm{crit}(U) \leq \mathrm{crit}(W)$, and $\mathrm{crit}(U) < \mathrm{crit}(W) \to U \triangleleft W$. Thus the interesting case is when $\mathrm{crit}(U) = \mathrm{crit}(W)$. By Proposition 1.6, $U \not\triangleleft U$. In fact,

Lemma 3.1. $\triangleleft$ *is well-founded.*

Proof. Let $U \triangleleft W$ and $\mathrm{crit}(U) = \mathrm{crit}(W) = \kappa$. Then

$$i_U^V(\kappa) = i_U^{\mathrm{Ult}(V,W)}(\kappa) < i_W^V(\kappa).$$

The first equality holds because V and $\mathrm{Ult}(V, W)$ have the same $f : \kappa \to \kappa$. The inequality holds because $i_W^V(\kappa)$ is inaccessible in $\mathrm{Ult}(V, W)$.

So the map $U \mapsto i_U^V(\kappa)$ maps $\{U \mid \mathrm{crit}(U) = \kappa\}$ into the ordinals, witness $\triangleleft$ is well-founded. □

The argument actually shows $\triangleleft\restriction \{U \mid \mathrm{crit}(U) = \kappa\}$ has rank $< (2^\kappa)^\kappa$

It is consistent that $\triangleleft$ is not linear, in fact, there can be $2^{(2^\kappa)}$ nufs on κ which are $\triangleleft$-minimal. ([Kunen-Pairs,??]) But in the canonical inner models, $\triangleleft$ is linear.

Definition 3.2. A **coherent sequence of nufs** is a function $\mathcal{U}$ such that $\mathrm{dom}(\mathcal{U}) \subseteq \mathrm{Ord} \times \mathrm{Ord}$, and

(1) $(\kappa, \beta) \in \mathrm{dom}(\mathcal{U}) \Longrightarrow \mathcal{U}(\kappa, \beta)$ is a nuf on κ;
(2) $(\kappa, \beta) \in \mathrm{dom}(\mathcal{U}) \wedge \gamma < \beta \Longrightarrow (\kappa, \gamma) \in \mathrm{dom}(\mathcal{U})$,
(3) Letting $o^{\mathcal{U}}(\kappa) = \sup \{\beta \mid (\kappa, \beta) \in \mathrm{dom}(\mathcal{U})\}$, and $i : V \to \mathrm{Ult}(V, \mathcal{U}(\kappa, \beta))$, we have

$$o^{i(\mathcal{U})}(\kappa) = \beta,$$

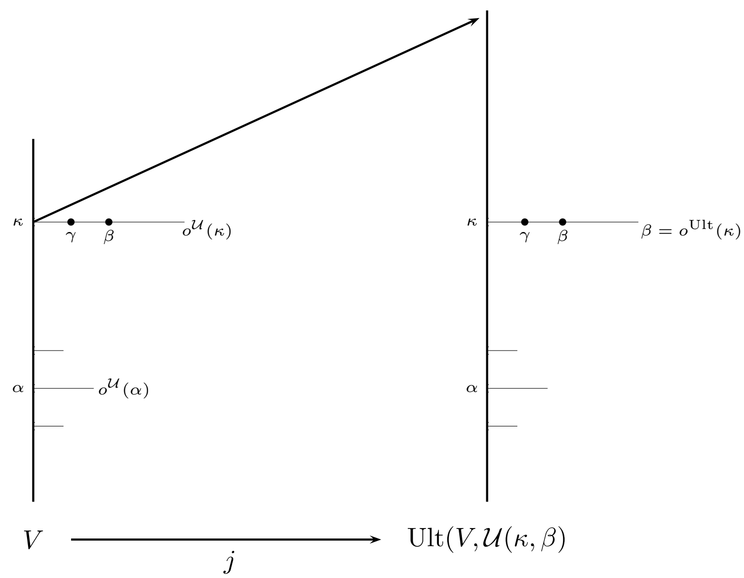
and

$$i(\mathcal{U})(\kappa, \gamma) = U(\kappa, \gamma), \qquad \text{for all } \gamma < \beta.$$

Remark. Let's write $\mathcal{U} \restriction (\kappa, \gamma)$ for $\mathcal{U} \restriction \{(\alpha, \tau) \mid \alpha < \kappa \text{ or } (\alpha = \kappa \wedge \tau < \gamma)\}$. Condition (3) can then be expressed

$$i^V_{\mathcal{U}(\kappa,\beta)}(\mathcal{U}) \restriction (\kappa, \beta + 1) = \mathcal{U} \restriction (\kappa, \beta).$$

Picture:



$$i(\mathcal{U})(\alpha,\gamma) = \begin{cases} \mathcal{U}(\alpha,\gamma), & \text{if } \alpha < \kappa, \text{ or } \alpha = \kappa \wedge \gamma < \beta; \\ \text{undefined}, & \text{if } \alpha = \kappa \wedge \gamma \geq \beta \end{cases}$$

For $\mathcal{U}$ a coherent segment of nufs, we set

$$L[\mathcal{U}] = L[A],$$

where

$$A = \{(\beta,\gamma,X) \mid X \in \mathcal{U}(\beta,\gamma)\}\,.$$

This gives us that $L[\mathcal{U}] = \mathbf{ZFC} +$ "$V = L[\mathcal{U}]$", and $\mathcal{U}(\alpha,\beta) \cap L[\mathcal{U}] \in L[\mathcal{U}]$ for all (β,γ) in $\mathrm{dom}(\mathcal{U})$. It follows that

$$L[\mathcal{U}] \vDash \mathcal{U}(\beta,\gamma) \text{ is a nuf on } \beta$$

for all $(\beta,\gamma) \in \mathrm{dom}(\mathcal{U})$, where on the right we write "$\mathcal{U}(\beta,\gamma)$" for "$\mathcal{U}(\beta,\gamma) \cap L[\mathcal{U}]$", as we shall do when context makes the meaning clear.

It is still open (almost 40 years after [**3**]) whether

$$L[\mathcal{U}] \vDash \mathcal{U} \text{ is a coherent sequence of nufs.}$$

(Later developments reduced the importance of this question.) The problem is that ultrapowers computed in $L[\mathcal{U}]$ may diverge too much from those computed in V. However, [**3**] did show

Theorem 3.3. *Suppose there is an elementary $j : V \to M$ with $\mathrm{crit}(j) = \kappa$ and $V_{\kappa+2} \subseteq M$. Then there is a coherent segment U of nufs such that*

$$L[\mathcal{U}] \vDash U \textit{ is coherent } \wedge \exists\alpha(o^U(\alpha) = \alpha^{++}).$$

Proof Sketch. Let κ, j and M be as in the hypothesis. We define a "maximal coherent sequence below $o(\alpha) = \alpha^{++}$ as follows.

Suppose we have defined $\mathcal{U} \restriction (\alpha,\beta)$, where $\alpha < \kappa$, so that $L[\mathcal{U} \restriction (\alpha,\beta)] \vDash \mathcal{U} \restriction (\alpha,\beta)$ is coherent. If $o(\alpha) = \alpha^{++}$ holds in $L[\mathcal{U} \restriction (\alpha,\beta)]$, we have the desired $\mathcal{U}$, and we stop the construction. So suppose $o(\alpha) < \alpha^{++}$ in $L[\mathcal{U} \restriction (\alpha,\beta)]$. Now pick a nuf W on α such that

$$L[\mathcal{U} \restriction (\alpha,\beta)^\frown\langle W\rangle] \vDash \mathcal{U} \restriction (\alpha,\beta)^\frown\langle W\rangle \text{ is coherent.}$$

and set

$$\mathcal{U}(\alpha,\beta) = W,$$

if there is such a W. (**AC** is used here.) If there is no such W, we set $o^{\mathcal{U}}(\alpha) = \beta$, and go on to defining $\mathcal{U}(\alpha',0)$ for some $\alpha' > \alpha$.

This defines our $\mathcal{U}$, with $\mathrm{dom}(\mathcal{U}) \subseteq \kappa \times \kappa$. It is enough to see
Claim. The construction reaches some $(\alpha, \beta) \in \kappa \times \kappa$ such that $L[\mathcal{U} \restriction (\alpha, \beta)] \models o^{\mathcal{U}}(\alpha) = \beta = \alpha^{++}$.

Proof. Suppose not. Now work in M, where $j(\mathcal{U})$ is a maximal sequence below $j(\kappa)$. Let $\beta = o^{j(\mathcal{U})}(\kappa)$, and let

$$W = U_j = \{A \subseteq \kappa \mid \kappa \in j(A)\}.$$

It is enough to show

$$(*) \qquad L[j(\mathcal{U}) \restriction (\kappa, \beta)^\frown \langle W \rangle] \models j(\mathcal{U}) \restriction (\kappa, \beta)^\frown \langle W \rangle \text{ is coherent,}$$

for then $j(\mathcal{U})$ is not maximal in M. Note here the crucial fact: $W \in V_{\kappa+2}$, so $W \in M$! This is where we use the strength of j.

To prove $(*)$, we consider the diagram

$$\begin{tikzcd}
L[j(\mathcal{U}) \restriction (\kappa, \beta)] \arrow[r, "j"'] \arrow[dr, "i"'] & L[j(j(\mathcal{U}) \restriction (\kappa, \beta))] \\
& \mathrm{Ult}(L[j(\mathcal{U}) \restriction (\kappa, \beta)], U_j) \arrow[u, "k"]
\end{tikzcd}$$

□

Exercise 14.

(a) Give precise definitions of i and k;
(b) Show $j(\mathcal{U}) \restriction (j(\kappa), 0) = j(j(\mathcal{U})) \restriction (j(\kappa), 0)$;
(c) Show $k \restriction (\beta + 1) = \text{identity}$;
(d) Prove $(*)$.

This complete our sketch of the proof of Theorem 3.3. □

Thus, granted large cardinals in V, there are inner models $L[\mathcal{U}]$ such that $L[\mathcal{U}] \models \mathcal{U}$ is coherent., and $L[\mathcal{U}] \models$ "There are many measurable cardinals". We now show such models are canonical, for example, every real number in such a model is ordinal definable in simple way.

Definition 3.4. A **measures-premouse** is a pair $(M, \mathcal{U})$ such that

$$M \models \mathbf{ZFC}^- + \text{“}\mathcal{U} \text{ is a coherent sequence of nufs”} + \text{“}V = L[\mathcal{U}].$$

A **measures-mouse** is a linearly iterable measures premouse.

Notice that a measures-premouse is a good pair, after we forget the order on $\mathcal{U}$. So linear iterability for it makes sense, and our earlier results apply. If I is an iteration of $\mathfrak{M} = (M, \mathcal{U})$, then we write $\mathfrak{M}^I_\alpha = (M^I_\alpha, i^I_{0\alpha}(\mathcal{U}))$ and $\mathfrak{M}^I_\infty = (M^I_\infty, i^I_{0\infty}(\mathcal{U}))$.

Definition 3.5. Let $\mathfrak{M} = (M, \mathcal{U})$ and $\mathfrak{N} = (N, \mathcal{W})$ be measures-premouse. We say that $\mathfrak{M}$ is an **initial segment of** $\mathfrak{N}$, and write $\mathfrak{M} \trianglelefteq \mathfrak{N}$, iff for all $\kappa \in M$,

(a) $o^{\mathcal{U}}(\kappa) = o^{\mathcal{W}}(\kappa)$, and
(b) $\mathcal{U}(\kappa, \beta) = \mathcal{W}(\kappa, \beta) \cap M$ for all $\beta < o^{\mathcal{U}}(\kappa)$,
(c) $\mathrm{Ord}^{\mathfrak{M}} \leq \mathrm{Ord}^{\mathfrak{N}}$

Remark. Let $A_{\mathcal{U}} = \{(\beta, \gamma, X) \mid X \in \mathcal{U}(\beta, \gamma)\}$, and similiarly for $A_{\mathcal{W}}$. So $M = L_\alpha[A_{\mathcal{U}}]$ for some α. Clauses (a)-(c) say tht $\alpha \leq \mathrm{Ord}^{\mathfrak{N}}$, and

$$(L_\alpha[A_{\mathcal{U}}], \in, A_{\mathcal{U}}) = (L_\alpha[A_{\mathcal{W}}], \in, A_{\mathcal{W}} \cap L_\alpha[A_{\mathcal{W}}]).$$

The key to inner model theory at any level is a **comparison process**, a method by which two mice can be simultaneously iterated so that an iteration of one is an initial segment of an iteration of the other. At the level of measures-mice, linear iteration suffices for comparison, and we get

Lemma 3.6 (Comparison Lemma for Measures-Mice)**.** *Let $\mathfrak{M}$ and $\mathfrak{N}$ be measures-mice which are sets; Then there are linear iteration I and J such that $\mathfrak{M}^I_\infty \trianglelefteq \mathfrak{N}^J_\infty$ or $\mathfrak{N}^J_\infty \trianglelefteq \mathfrak{M}^I_\infty$.*

Remark. There is a version of Lemma 3.6 which holds for proper class $\mathfrak{M}$, $\mathfrak{N}$. Then I and J might be a proper class.

Proof. We define initial segment I_ν and J_ν of I and J, by induction on ν. Set $I_0 = \emptyset = J_0$. Let

$$\mathfrak{M}_\infty^{I_\nu} = (P, \mathcal{H})$$

and

$$\mathfrak{N}_\infty^{J_\nu} = (Q, \mathcal{L})$$

be the two last models, where at stage $\nu = 0$ we set $\mathfrak{M}^{I_0} = \mathfrak{M}$ and $\mathfrak{N}^{I_0} = \mathfrak{N}$. We may assume $(P, \mathcal{H}) \ntrianglelefteq (Q, \mathcal{L})$ and $(Q, \mathcal{L}) \ntrianglelefteq (P, \mathcal{H})$, and otherwise we can set $I = I_\nu$ and $J = J_\nu$, and our comparison has succeeded.

We now obtain $I_{\nu+1}$ and $J_{\nu+1}$ by **iterating away the least disagreement** between $(P, \mathcal{H})$ and $(Q, \mathcal{L})$. Namely, let (κ, β) be lexicographically least such that either

(a) $(\kappa, \beta) \in \mathrm{dom}(\mathcal{H}) \triangle \mathrm{dom}(\mathcal{L})$,

or

(b) $(\kappa, \beta) \in \mathrm{dom}(\mathcal{H}) \cap \mathrm{dom}(\mathcal{L})$, and $\mathcal{H}(\kappa, \beta) \cap P \cap Q \neq \mathcal{L}(\kappa, \beta) \cap P \cap Q$.

If (b) holds, then we set

$$\begin{aligned} I_{\nu+1} &= I_\nu \frown \langle \mathcal{H}(\kappa, \beta) \rangle, \\ J_{\nu+1} &= J_\nu \frown \langle \mathcal{L}(\kappa, \beta) \rangle. \end{aligned}$$

If (a) holds, and $(\kappa, \beta) \in \mathrm{dom}(\mathcal{H})$, we set

$$\begin{aligned} I_{\nu+1} &= I_\nu \frown \langle \mathcal{H}(\kappa, \beta) \rangle, \\ J_{\nu+1} &= J_\nu. \end{aligned}$$

If (a) holds, and $(\kappa, \beta) \in \mathrm{dom}(\mathcal{L})$, we set

$$\begin{aligned} I_{\nu+1} &= I_\nu, \\ J_{\nu+1} &= J_\nu \frown \langle \mathcal{L}(\kappa, \beta) \rangle. \end{aligned}$$

This defines $I_{\nu+1}$ and $J_{\nu+1}$. For λ a limit, we let $I_\lambda = \bigcup_{\nu<\lambda}$ and $J_\lambda = \bigcup_{\nu<\lambda} J_\nu$.

It is enough to show our process terminate. In fact

Claim. For some $\nu < \max(|\mathrm{TC}(M)|, |\mathrm{TC}(N)|)^+$, $\mathfrak{M}_\infty^{I_\nu} \trianglelefteq \mathfrak{N}_\infty^{J_\nu}$ or $\mathfrak{N}_\infty^{J_\nu} \trianglelefteq \mathfrak{M}_\infty^{I_\nu}$.

proof of the claim. Let $\theta = \max(|\mathrm{TC}(M)|, |\mathrm{TC}(N)|)^+$. It is easy to see $\theta = \mathrm{dom}(I_\theta) = \mathrm{dom}(J_\theta)$. Let us write M_α, $i_{\alpha\beta}$ for the models and embeddings of I_θ, and N_α and $j_{\alpha\beta}$ for those of J_θ. Now let

$$\pi : S \to V_\gamma,$$

where $\gamma >> \theta$, S is transitive, $|S| < \theta$, everything relevant is in $\mathrm{ran}(\pi)$. We can arrange that for some $\alpha < \theta$,

$$\pi(\alpha) = \theta,$$

and

$$\pi \restriction \alpha = id.$$

Moreover $\pi \restriction \mathrm{TC}(\mathfrak{M}) \cup \mathrm{TC}(\mathfrak{N}) \cup \{\mathfrak{M}, \mathfrak{N}\} = id$. It is easy to see then from the absoluteness of our process that

$$\pi^{-1}(I_\theta) = I_\alpha,$$

and

$$\pi^{-1}(J_\theta) = J_\alpha.$$

Subclaim. $\pi \restriction M_\alpha = i_{\alpha,\infty}$ and $\pi \restriction N_\alpha = j_{\alpha,\infty}$

proof of the subclaim. Note $M_\alpha = M_\infty^{I_\alpha} = \pi^{-}(M_\infty^{I_\theta})$, and for all $\beta < \alpha$, $i_{\alpha\beta} = \pi^{-1}(i_{\beta,\infty})$. So if $x \in M_\alpha$, let $x = i_{\beta\alpha}(\bar{x})$ where $\beta < \alpha$, and then

$$\begin{aligned} \pi(x) &= \pi(i_{\beta\alpha}(\bar{x})) \\ &= \pi(i_{\beta\alpha})(\bar{x}) \\ &= i_{\beta\infty}(\bar{x}) \\ &= i_{\alpha,\infty}(x) \end{aligned}$$

The same proof shows that $\pi \upharpoonright N_\alpha = j_{\alpha,\infty}$. □

So we have $\alpha = \mathrm{crit}(\pi) = \mathrm{crit}(i_{\alpha,\infty}) = \mathrm{crit}(j_{\alpha,\infty})$. Now notice the critical points used in I_θ are strictly increasing, and therefore

$$\alpha = \mathrm{crit}(i_{\alpha,\alpha+1}) < \mathrm{crit}(i_{\alpha+1,\infty}).$$

Similarly,

$$\alpha = \mathrm{crit}(j_{\alpha,\alpha+1}) < \mathrm{crit}(j_{\alpha+1,\infty}).$$

This is the crucial use of coherence! By coherence, $\mathfrak{M}_{\alpha+1}$ and $\mathfrak{N}_{\alpha+1}$ agree on all measures with crit $\leq \alpha$, and this agreement will never be disturbed later.

Exercise 15. Provide the details here.

So we had a diagreement of type (b) at α. Let U and W be the disagreeing ultrafilters, i.e. $I_{\alpha+1} = I_\alpha{}^\frown\langle U\rangle$ and $J_{\alpha+1} = J_\alpha{}^\frown\langle W\rangle$. We must have a set $A \subseteq \alpha$ such that $A \in M_\alpha \cap \mathfrak{N}_\alpha$ and $A \in U \triangle W$. Say $A \in U$ and $A \notin W$, then

$$\alpha \in i_{\alpha,\alpha+1}(A) = i_U^{M_\alpha}(A),$$

and

$$\alpha \notin j_{\alpha,\alpha+1}(A) = i_W^{\mathfrak{N}_\alpha}(A).$$

So

$$\alpha \in i_{\alpha,\infty}(A),$$

and

$$\alpha \notin j_{\alpha,\infty}(A),$$

because neither $i_{\alpha+1,\infty}$ nor $j_{\alpha+1,\infty}$ moves α. Since $i_{\alpha,\infty}(A) = \pi(A) = j_{\alpha,\infty}(A)$, we have a contradiction.

This finish the proof of the **Claim**. □

This finish the proof of Lemma 3.6. □

It is easy to see that linear iterability is π_1-definable in the language of set theory. So we get:

Theorem 3.7. *Auusme $V = L[\mathcal{U}]$, where $\mathcal{U}$ is a coherent sequence of nufs. Then* **CH** *holds and $\mathbb{R}$ admits a Δ_2^{HC} well-order.*

Proof. Let $<^{\mathcal{U}}$ be "the" order of construction in $L[\mathcal{U}]$. (It's unique up to how we order the formulae. Fix one recursive ordering of formulae.) Then for $x, y \in \mathbb{R}$

$(*)$ $\quad x <^{\mathcal{U}} y$ iff $\exists(M,\mathcal{W}) \in \mathrm{HC}((M,\mathcal{W})$ is a measures-premouse $\wedge\, (M,\mathcal{W})$ is ω_1-iterable $\wedge\, M \vDash x <^{\mathcal{W}} y.)$

The ($\Longrightarrow$) direction comes from Löwenheim-Skolem: if $x <^{\mathcal{U}} y$, we have some α such that

$$(L_\alpha[\mathcal{U}], \in, \mathcal{U}) \vDash \mathbf{ZFC}^- + x <^{\mathcal{U}} y,$$

and then we can take $(M,\mathcal{U})$ to be the transitive collapse of a countable elementary submodel of $(L_\alpha[\mathcal{U}], \in, \mathcal{U})$. For the ($\Longleftarrow$) direction of $(*)$, suppose $(M,\mathcal{W})$ is as on the right hand side, but $y \leq^{\mathcal{U}} x$. Let $(L_\alpha[\mathcal{U}], \in, \mathcal{U}) \vDash y \leq^{\mathcal{U}} x$. Now bt Lemma 3.6, we can compare $\mathfrak{M} = (M, W)$ with $\mathfrak{N} = (N, \mathcal{U})$. The comparison maps do not move x and y, since they are reals. This is a contradiction.

The same proof shows that the order-type of $<^{\mathcal{U}} \upharpoonright \mathbb{R}$ is ω_1, so **CH** holds. □

Exercise 16. (a) Provide the details of the proof that $\mathfrak{M}$ and $\mathfrak{N}$ cannot be compared.
(b) Provide the details of the proof that **CH** holds in $L[\mathcal{U}]$.

Exercise 17. Assume $V = L[\mathcal{U}]$, where $\mathcal{U}$ is a coherent sequence of nufs.
(a) Show that $2^\alpha = \alpha^+$, except possibly when $\exists\kappa(\kappa < \alpha \wedge \alpha^+ \leq o^{\mathcal{U}}(\kappa))$.
(b) Show that if $o\mathcal{U}(\kappa) \leq \kappa^{++}$ for all κ, then **GCH** holds.
(c) Show that $o^{\mathcal{U}}(\kappa) \leq \kappa^{++}$ for all κ.
(d) (Harder) Show that **GCH** holds. (The hard case is $2^\alpha = \alpha^+$ when $\alpha = \kappa^+$ and $o^{\mathcal{U}}(\kappa) = \kappa^{++}$.)

The proof of Theorem 2.13 also shows that every real in a measure-mouse is ordinal definable in a simply way:

Exercise 18. Let $x \in \mathbb{R} \cap M$, where $(M,\mathcal{U})$ is a measure-mouse. Show that x is $\Delta_2^{\mathrm{HC}}(\{\alpha\})$, for some $\alpha < \omega_1$.

As corollaries to Theorem 3.7 and its proof, we get limitations on the consequences of (many) measurable cardinals we draw in Lecture 2. For example

Corollary 3.8. *Suppose $\exists j : V \to M(\mathrm{crit}(j) = \kappa \wedge V_{\kappa+2} \subseteq M)$. Then there is a model of* $\mathbf{ZFC} + \exists\alpha(o(\alpha) = \alpha^{++})$ *in which not all Δ_2^{HC} sets are Lebesgue measurable.*

Similarly, Martin-Solovay's generic absoluteness result (Theorem 2.13) does not extend to Σ_3^{HC}, even assuming many measurables. For let

$$\varphi = \forall x \in \mathbb{R} \exists \mathfrak{M}(\mathfrak{M} \text{ is a measures-mouse } \wedge x \in \mathfrak{M}).$$

One can calculate that φ is Π_3.

Exercise 19. Let $L[\mathcal{U}] \vDash \mathcal{U}$ is coherent, then

(1) $\mathrm{HC}^{L[\mathcal{U}]} \vDash \varphi$,
(2) whenever x is Cohen-generic $/L[\mathcal{U}]$, $\mathrm{HC}^{L[\mathcal{U}][x]} \vDash \neq \varphi$,
(3) φ is Π_3.

Finally, the correctness of models of the form $L[\mathcal{U}]$, $\mathcal{U}$ is coherent, is limited. For suppose there is $j : V \to M$ with $V_{\mathrm{crit}(j)+2} \subseteq M$. Then there is a Σ_2 sentence ψ such that

$$\mathrm{HC} \vDash \psi,$$

but whenever $\mathfrak{M}$ is a measures-mouse

$$\mathrm{HC}^{\mathfrak{M}} \nvDash \psi.$$

Exercise 20. (For the student who knows more inner model theory.) Prove this.

LECTURE 4

Extenders

Given $j : M \to N$ elementary, it may not be the case that $N = \mathrm{Hull}^N(\mathrm{ran}(j) \cup \{\kappa\})$, for $\kappa = \mathrm{crit}(j)$. That is we may not have $N \cong \mathrm{Ult}(M,\mathcal{U})$. In order to capture j in general, we need a certain system of ultrafilters, called an extender.

Definition 4.1. $[X]^n = \{a \subseteq X \mid |a| = n\}$, and $[x]^{<\omega} = \bigcup_{n<\omega} [X]^n$. For $a \subseteq \mathrm{Ord}$ with $|a| = n$, we write $a_i = i^{th}$ element of a in its increasing enumeration.

Now suppose $j : M \to N$ with $M \models \mathbf{ZFC}^-$ transitive. Suppose $\lambda \subseteq \mathrm{wfp}(N)$, and $\lambda \leq^N j(\kappa)$. For $a \subseteq [\lambda]^{<\omega}$ and $X \subseteq [\kappa]^{|a|}$ with $X \in M$, we put

$$X \in E_a \quad \text{iff} \quad a \in j(X).$$

Definition 4.2. $E_j^\lambda = \{(a, X) \mid X \in E_a\}$ is the (κ, λ)**-extender** derived from j.

Remark. The restriction $\lambda \leq j(\kappa)$ is not really needed. Extenders satisfying it are called “short extender”, but since we have no use here for “long” extenders, we have dropped the qualifier “short” in these lectures.

Clearly, $E_{\{\kappa\}} = U_j$. By allowing typical objects beyond κ to generate measures, however, we may be able to capture more of j.

As before, we have

(1) E_a is an ultrafilter on $\mathcal{P}([\kappa]^{|a|})^M$, and non-principal iff $a \not\subseteq \kappa$,
(2) E_a is M-κ-complete. We can form $\mathrm{Ult}(M, E_a)$, and we have

$$\begin{tikzcd} M \arrow[rr, "j"] \arrow[dr, "i"'] & & N \\ & & \mathrm{Ult}(M, E_a) \arrow[u, "k_a"'] \end{tikzcd}$$

commutes, where $k_a([f]) = j(f)(a)$. (So $k_a([id]) = a$.) the range of k_a is $\mathrm{Hull}^N(\mathrm{ran}(j) \cap a)$.

If $a \subseteq b$, there is a natural map

$$i_{ab}(x) = k_b^{-1}(k_a(x)).$$

Since $[\lambda]^{<\omega}$ is directed under inclusion, and the maps commute (i.e. $i_{ac} = i_{bc} \circ i_{ab}$ if $a \subseteq b \subseteq c$), we can set

$$\mathrm{Ult}(M, E) = \text{direct limit of } \mathrm{Ult}(M, E_a)\text{'s under } i_{ab}\text{'s}.$$

We can piece the k_a's together into

$$k : \mathrm{Ult}(M, E) \to N$$

given by $k(i_{a,\infty}(x)) = k_a(x)$, for $i_{a,\infty} : \mathrm{Ult}(M, E_a) \to \mathrm{Ult}(M, E)$ the direct limit map. The following commutative diagram summarizes things:

$$\begin{tikzcd} M \arrow[rr, "j"] \arrow[drr, "i_E^M"'] \arrow[ddr, "i_{E_b}^M"'] & & N \\ & & \mathrm{Ult}(M, E) \arrow[u, "k"'] \\ & \mathrm{Ult}(M, E_b) \arrow[ur, "i_{b,\infty}"'] & \end{tikzcd}$$

Note $\mathrm{ran}(k) = \mathrm{Hull}^N(ran(j) \cup \lambda)$, so that $k \restriction \lambda$ =identiy.

Properties of E_j^λ

For $a \subseteq b$ and $b \in [\lambda]^{<\omega}$, and $X \subseteq [\kappa]^{|a|}$, we think of X as a predicate of $|a|$-tuples, and let X^{ab} be the result of "adding dummy variables" corresponding to ordinals in $b - a$. That is, for

$$a = \{b_{i_1}, \cdots, b_{i_n}\} \text{ with } i_1 < \cdots < i_n,$$

we put

$$X^{ab} = \Big\{u \in [\kappa]^{|b|} \mid \{u_{i_1}, \cdots, u_{i_n}\} \in X\Big\}.$$

Similarly, if $f : [\kappa]^{|a|} \to M$,

$$f^{ab}(u) = f(\{u_{i_1}, \cdots, u_{i_n}\}), \text{ for } u \in [\kappa]^{|b|}.$$

We then have the following properties of $E = E_j^\lambda$, where $j : M \to N$ with $crit(j) = \kappa$ and $\lambda \leq^N j(\kappa)$:

(1) Each E_a is an M-κ-complete ultrafilter on $\mathcal{P}([\kappa]^{|a|})^M$,

(2) (Compatibility) If $a \subseteq b$, then $\forall X \in M$

$$X \in E_a \quad \text{iff} \quad x^{ab} \in E_b,$$

(3) (M-normaity) If $f \in M$ with $\mathrm{dom}(f) = [\kappa]^{|a|}$, and

$$f(u) < u_i, \text{ for } E_a \text{ a.e. } u,$$

then there is a $\xi < a_i$ such that letting $\xi = (a \cup \{\xi\})_k$,

$$f^{a,a\cup\{\xi\}}(u) = u_k, \text{ for } E_{a\cup\{\xi\}} \text{ a.e. } u,$$

If in addition $\mathcal{P}(\kappa)^M = \mathcal{P}(\kappa)^N$

(4) (M-amenability) If $a \in [\lambda]^{<\omega}$ and $a \subseteq \mathcal{P}([\kappa]^{|a|})$ with $a \in M$ and $M \vDash |a| \leq k$, then $E_a \cap a \in M$.

Definition 4.3. Given $M \vDash \mathbf{ZFC}^-$, transitive, and we call a system $E = \langle E_a \mid a \in [\lambda]^{<\omega}$ satisfying (1)-(4) above a **(κ, λ)-pre-extender over** M (or just an **M-pre-extender**). We write $\kappa = \mathrm{crit}(E)$ and $\lambda = \mathrm{lh}(E)$.

Notice that in definition 4.3, we have thrown away j and N. In particular, if $Q \vDash \mathbf{ZFC}^-$ is transitive, and $\mathcal{P}(\kappa)^Q = \mathcal{P}(\kappa)^M$, then E is an M-pre-extender iff E is a Q-pre-extender.

If E is a (κ, λ)-pre-extender over M, then we define $\mathrm{Ult}(M, E)$ as follows:

The elements are equivalence classes $[a, f]_E^M$, where for $f, g \in M$ with domains $[\kappa]^{|a|}$ and $[\kappa]^{|b|}$,

$$\langle a, f\rangle \sim \langle b, g\rangle \text{ iff for } E_{a\cup b} \text{ a.e. } u,\ f^{a,a\cup b}(u) = g^{b,a\cup b}(u),$$

and

$$[a, f]_E^M \widetilde{\in} [b, g]_E^M \text{ iff for } E_{a\cup b} \text{ a.e. } u,\ f^{a,a\cup b}(u) \in g^{b,a\cup b}(u),$$

Then we set

$$\mathrm{Ult}(M, E) = (\big\{[a, f]_E^M \mid a \in [\lambda]^{<\omega} \wedge f \in M\big\}, \widetilde{\in}).$$

Let also $i_E^M : M \to \mathrm{Ult}(M, E)$ be given by

$$i_E^M(x) = [\{0\}, \lambda u.x],$$

we have

(1) Łoś Theorem: given $\langle a_0, f_0\rangle, \cdots, \langle a_n, f_n\rangle$ and $\varphi(v_0, \cdots, v_n)$ and letting $b = \bigcup_{i\leq n} a_i$,

$$\mathrm{Ult}(M, E) \vDash \varphi[[a_0, f_0]_E^M, \cdots, [a_n, f_n]_E^M] \text{ iff for } E_b \text{ a.e. } u\ M \vDash \varphi[f_0^{a_0,b}(u), \cdots, f_n^{a_n,b}(u)].$$

(2) i_E^M is elementary,

(3) $\mathrm{crit}(i_E^M) = \kappa$,

(4) letting $id(u) = u$ for all $u \in [\kappa]^{|a|}$,

$$[a, id]_E^M = a,$$

and

$$[a, f]_E^M = i_E^M(f)(a).$$

(5) $X \in E_a$ iff $a \in i_E^M(X)$.

Exercise 21. Prove (1)-(5).

The $\mathrm{Ult}(M,E)$ is the Skolem-closure of $\mathrm{ran}(i_E^M)\cup\lambda$ inside $\mathrm{Ult}(M,E)$, and E is the (κ,λ)-pre-extender derived from $\mathrm{Ult}(M,E)$. By amenability, $\mathcal{P}(\kappa)^M = \mathcal{P}(\kappa)^{\mathrm{Ult}(M,E)}$, the closure of $\mathrm{Ult}(M,E)$ under sequences is given by:

Lemma 4.4. *Let E be a (κ,λ)-pre-extender over M, and $\alpha \leq \kappa$. Suppose M is closed under α-sequences, and ${}^{\alpha}\lambda \subseteq \mathrm{Ult}(M,E)$. Then $\mathrm{Ult}(M,E)$ is closed under α-sequences.*

Proof. Let $[a_\beta, f_\beta] \in \mathrm{Ult}(M,E)$ for all $\beta < \alpha$. Let $i = i_E^M$. Then

$$\langle i(f_\beta) \mid \beta < \alpha\rangle = i(\langle f_\beta \mid \beta < \alpha\rangle) \restriction \alpha \in M$$

and $\langle a_\beta \mid \beta < \alpha\rangle \in M$ as $\lambda_\alpha \subseteq M$. Thus $\langle i(f_\beta)(a_\beta) \mid \beta < \alpha\rangle \in M$. □

Exercise 22. Let κ be measurable. Show there is a (κ,λ)-extender over V such that $\mathrm{Ult}(V,E)$ is not closed under ω-sequences.

We have $\lambda \subseteq \mathrm{wfp}(\mathrm{Ult}(M,E))$, essentially by normality. How to guarantee $\mathrm{Ult}(M,E)$ is fully well-founded?

Definition 4.5. Let E be a (κ,λ)-pre-extender over M. We say E is ω**-complete** iff whenever $X_i \in E_a$ for all $i < \omega$, then there is an $f : \bigcup_{i<\omega} a_i \to \kappa$ such that

$$f\text{''}a_i \in X_i$$

for all $i < \omega$.

One sometimes calls f a "fiber" for $\langle (a_i, X_i) \mid i \in \omega\rangle$.

Lemma 4.6. *Let E be an ω-complete extender over M; Then $\mathrm{Ult}(M,E)$ is well-founded.*

Proof. Suppose $[a_{i+1}, g_{i+1}] \in [a_i, g_i]$ for all i. By meeting the right measure one sets, we can find a fiber f such that $g_{i+1}(f\text{''}a_{i+1}) \in g_i(f\text{''}a_i)$ for all i. This is a contradiction. □

Exercise 23. Let E be an ω-complete pre-extender over M, and $E \in M$. Let $\pi : N \to M$ be elementary, with N countable transitive, and $\pi(F) = E$. Show thatthere is a σ such that

$$\begin{tikzcd}
M & \\
N \arrow[u, "\pi"] \arrow[r, "i_F^N"'] & \mathrm{Ult}(N,F) \arrow[ul, "\sigma"']
\end{tikzcd}$$

commutes. (That is, countable fragments of $\mathrm{Ult}(M,E)$ can be realized back in M.)

Exercise 24. (a) Let E be an pre-extender over V; Then $\mathrm{Ult}(V,E)$ is well-founded iff E is ω-complete.

(b) There is a pre-extender over V such that $\mathrm{Ult}(V,E)$ is ill-founded.

Definition 4.7. E is a (κ,λ)**-extender over** M iff E is a (κ,λ)-pre-extender over M, and $\mathrm{Ult}(M,E)$ is well-founded.

In contrast pre-extender-hood, there are M,Q, and E such that E is a (κ,λ)-extender over M, and $\mathcal{P}(\kappa)^M = \mathcal{P}(\kappa)^Q$, but E is not a (κ,λ)-extender over Q, because $\mathrm{Ult}(Q,E)$ is ill-founded. If $j : M \to N$ where N is well-founded, then E_j is indeed an extender over M:

Lemma 4.8. *Let $j : M \to N$ where M is transitive, and E be the (κ,λ)-extender derived from j, Then the diagram*

$$\begin{tikzcd}
M \arrow[r, "j"] \arrow[dr, "i_E^M"'] & N \\
& \mathrm{Ult}(M,E) \arrow[u, "k"']
\end{tikzcd}$$

commutes, where $k([a,f]_E^M) = j(f)(a)$. Moreover, $k \restriction \lambda =$identity.

Of course, we began this section by essentially proving Lemma 4.8. But there we had defined $\mathrm{Ult}(M,E)$ in a slightly different way. You can think of Lemma 4.8 as saying that the two constructions give the same $\mathrm{Ult}(M,E)$. We leave this proof as an informal exercise.

It follows that if N is well-founded, so is $\mathrm{Ult}(M,E)$.

Computing Large Cardinal Strength

If U is a nuf on κ, then $V_{\kappa+2} \not\subseteq \mathrm{Ult}(V,U)$. With extenders, more is possible.

Lemma 4.9. *Let $j : V \to N$, $\mathrm{crit}(j) = \kappa$, and $\kappa < \lambda \leq j(\kappa)$. Suppose $V_\alpha \subset N$, and $|V_\alpha|^+ < \lambda$ Let E be the (κ, λ)-extender derived from j, then $V_\alpha \subseteq \mathrm{Ult}(V,E)$.*

Proof. Let $k : \mathrm{Ult}(V,E) \to N$ be the factor map, as in Lemma 4.8. Let $\beta = |V_\alpha|$, so $\beta < \lambda$. Let $(\beta, R) \cong (V_\alpha, \in)$. Since $\beta, \alpha \in \mathrm{ran}(k)$, we can pick $R \in \mathrm{ran}(k)$. But then $k^{-1}(R) = R$, as $k \restriction \beta = id$. Since $R \in \mathrm{Ult}(V,E)$, $V_\alpha \subseteq \mathrm{Ult}(V,E)$. □

Definition 4.10. Let E be a V-extender; then strength(E) =largest α such that $V_\alpha \subseteq \mathrm{Ult}(V,E)$.

Exercise 25. Let E be a V-extender; then $E \notin \mathrm{Ult}(V,E)$, and therefore strength$(E) \leq \mathrm{lh}(E)$.

Corollary 4.11. *Let $: V \to N$ where N is transitive, $\kappa = \mathrm{crit}(j)$, $\kappa < \lambda \leq j(\kappa)$. Suppose λ is inaccessible, and $V_\lambda \subseteq N$. Let E be the (κ, λ)-extender derived from j; then* strength$(E) = \lambda$.

Definition 4.12. A V-extender E is **nice** iff strength$(E) = \mathrm{lh}(E)$, and strength(E) is strongly inaccessible.

In the sequel, we shall use extenders (and iteration trees built from them) to extend the results of Martin-Solovay on correctness and generic absoluteness from Lecture 2. For these applications, nice extenders suffice. On the other hand, one can certainly not skip over the non-nice extenders in the bottom-up analysis of inner model theory.

We conclude this lecture with some simple lemmas on capturing large cardinal properties via extenders.

Definition 4.13.

(a) κ is β-strong iff $\exists j : V \to M$ (M transitive $\wedge \mathrm{crit}(j) = \kappa \wedge V_\beta \subseteq M$).
(b) κ is superstrong iff $\exists j : V \to N$ (M transitive $\wedge \mathrm{crit}(j) = \kappa \wedge V_{j(\kappa)} \subseteq M$).
(c) κ is λ-supercompact iff $\exists j : V \to M$ (M transitive $\wedge \mathrm{crit}(j) = \kappa \wedge^{\lambda} M \subseteq M$).

Proposition 4.14. *If κ is 2^κ-supercompact, then κ is a limit of superstrong cardinals.*

Proof. Let $j : V \to M$, $\mathrm{crit}(j) = \kappa$, and ${}^{2^\kappa} M \subseteq M$. So $j \restriction V_{\kappa+1} \in M$. Let E be the $(\kappa, j(\kappa))$ extender derived from j. We then have $E \in M$.

Exercise 26. $M \vDash \kappa$ is superstrong, as witnessed by E.

The exercise easily yields the proposition. □

Exercise 27. Let κ be superstrong. Show $\exists \alpha < \kappa \forall \beta < \kappa (\alpha$ is β-strong).

Definition 4.15. Let $\kappa < \delta$ and $A \subseteq V_\delta$; then κ **is A-reflecting in** δ iff

$$\forall \beta < \delta \exists j : V \to M(\mathrm{crit}(j) = \kappa \wedge V_\beta \subseteq M \wedge j(A) \cap V_\beta = A \cap V_\beta).$$

Definition 4.16. δ is **Woodin** iff $\forall A \subseteq V_\delta \exists \kappa < \delta(\kappa$ is A-reflecting in $\delta)$.

Proposition 4.17. *Suppose δ is Woodin; Then δ is strongly inaccessible, and there are arbitrarily large $\kappa < \delta$ such that $\forall \beta < \delta(\kappa$ is β-strong).*
The least Woodin cardinal is not Mahlo.

We leave the easy proof to the reader.
The following notations is quite useful.

Definition 4.18. Let E be a (κ, λ)-pre-extender, and $X \subseteq \lambda$; then $E \restriction X = \{(a, Y) \mid a \in [X]^{<\omega} \wedge Y \in E_a\}$

Mostly we use this when $X = \eta \leq \lambda$. If $\kappa < \eta \leq \lambda$, then $E \restriction \eta$ is itself a (κ, λ)-pre-extender.

Remark. We don't have to say "pre-extender over M", because pre-extender hood only depends on $\mathcal{P}(\kappa)^M$, which is determined by E itself. ($\mathcal{P}(\kappa)^M = E_{\{\kappa\}} \cup \{\kappa - A \mid A \in E_{\{\kappa\}}\}$.)

Lemma 4.19. *Let κ be superstrong; then κ is a Woodin limit of Woodin cardinals.*

Proof. Let $j : V \to M$ witness κ is superstrong. We show first κ is Woodin. So let $A \subseteq V_\kappa$.
Claim. $M \vDash \exists \alpha < j(\kappa)(\alpha$ is $j(A)$-reflecting in $j(\kappa))$.

PROOF OF THE CLAIM. Take $\alpha = \kappa$. Let $\kappa < \beta < j(\kappa)$. We may assume β is inaccessible in M. Let $E = E_j \restriction \beta$, so that $E \in M$. We have

$$A \cap V_\kappa = j(A) \cap V_\kappa,$$

so

$$j(A) \cap V^M_{j(\kappa)} = j(j(A)) \cap V^M_{j(\kappa)},$$

so

$$j(A) \cap V^M_\beta = i^M_E(j(A)) \cap V^M_\beta.$$

(To see the last line, note that

$$i^M_E(j(A)) \cap V^M_\beta = i^M_E(A) \cap V^M_\beta,$$

because $\beta < i^M_E(\kappa)$, and $i^M_E(A) \cap V^M_\beta = j(A) \cap V^M_\beta$, because $E = E_j \restriction \beta$.)

This gives the claim. $\square$

Pulling the claim back to V, we get $V \models \exists \alpha < \kappa(\alpha$ is A-reflecting in $\kappa)$. Since A was arbitrary, κ is Woodin. But now it easily follows that

$$M \models \kappa \text{ is Woodin.}$$

so

$$V \models \kappa \text{ is a limit of Woodins.}$$

$\square$

We have then the following consistency strength hierarchy on those properties. (Note that κ measurable iff κ is $\kappa + 1$-strong.)

- κ is 2^κ-supercompact.
- κ is superstrong.
- κ is Woodin.
- κ is β strong for all β.
- $\vdots$
- κ is $\kappa + 2$ strong.
- κ is measurable.

LECTURE 5

Linear Iteration via Extenders

Everything works much as it did wiht linear iteration via nufs, so we shall go quickly here.

Definition 5.1. An extender pair is an $(M, \mathcal{E})$ such that M is transitive, and $M \models \mathbf{ZFC}^- + \mathcal{E}$ is a set of V-extenders.

A linear iteration of $(M, \mathcal{E})$ is a sequence $\langle E_\alpha \mid \alpha < \beta \rangle$ determining M_α's and $j_{\alpha\gamma} : M_\alpha \to M_\gamma$, as before. $E_\alpha \in i_{0\alpha}(\mathcal{E})$, and $M_{\alpha+1} = \mathrm{Ult}(M_\alpha, E_\alpha)$. direct limits are taken at limit stages. M^I_∞ is the "last model" associated to the iteration I. $(M, \mathcal{E})$ is α-linearly iterable iff M^I_∞ is well-founded, for all I with $\mathrm{lh}(I) < \alpha$.

The following facts are proved just as they were for linear iterations of nuf-pairs.

(1) Let $(M, \mathcal{E})$ be an extender pair, $\pi : N \to M$ elementary, $\pi(\mathcal{F}) = \mathcal{E}$. Then if $(M, \mathcal{E})$ is α-linearly iterable, so is $(N, \mathcal{F})$. (cf Lemma 2.4)
(2) Let $(M, \mathcal{E})$ be an extender pair. Then $(M, \mathcal{F})$ is ω_1-linearly iterable. (cf. Lemma 2.5)
(3) Let $(M, \mathcal{E})$ be an extender pair such that every $E \in \mathcal{E}$ is ω-complete. Then $(M, \mathcal{E})$ is linearly iterable. (cf. Theorem 2.3, and Exercise 23?.)
(4) Let $(M, \mathcal{E})$ be an extender pair such that $M \models \mathbf{ZFC}$ and $\omega_1 \in M$. Then $(M, \mathcal{E})$ is linearly iterable. (cf. Corollary 2.8)

Applications

In Lecture 2, we showed that if there is a linearly iterable nuf pair $(M, \{U\})$ with "one measurable cardinal", then

(i) $\underset{\sim}{\Pi}^1_1$ determinacy holds.
(ii) $\underset{\sim}{\Sigma}^1_2$ sets are Lebesgue Measurable, etc.
(iii) M is $\underset{\sim}{\Sigma}^1_2$ correct.
(iv) M is Σ^1_3 correct in $M[G]$, when G is M-generic for $\mathbb{P} \in V^M_\kappa$, with $\kappa = \mathrm{crit}(U)$, and $M \models \mathbf{ZFC}$.

We showed that you can't add I to any of the subscripts in (i)-(iv), in Lecture **??**. This is true nomatter how many measures your nuf pair $(M, \mathcal{E})$ is assumed to have. It remains true if we replace $(M, \mathcal{E})$ with an extender pair.

The basic reason is that linear iterability is $\Pi^{\mathrm{HC}}_1 = \Pi^1_2$. Thus for any sentence φ, the statement

$$\psi = \text{"There is a countable, linearly iterable extender-pair } (M, \mathcal{E}) \text{ such that } (M, \mathcal{E}) \models \varphi.$$

(Think of φ as saying "There are extenders witnessing superstrongness", if you like.) If there are linearly iterable $(M, \mathcal{E}) \models \varphi$, then they cannot al be Σ^1_3 correct. For let $(M, \mathcal{E}) \models \varphi$ with $\mathrm{Ord} \cap M$ minimal; then $M \not\models \psi$.

This is not to say that the existence of linearly iterable extender pairs $(M, \mathcal{E})$ with "many extenders in $\mathcal{E}$" does not lead to strengthening of (i)-(iv) above. You just can't go all the way to $\underset{\sim}{\Pi}^1_2$ in (i), or to $\underset{\sim}{\Sigma}^1_3$ in (ii) or (iii), or to Σ^1_4 in (iv). Instead, one needs to replace $\underset{\sim}{\Pi}^1_1$ in (i) by some level Γ of $\underset{\sim}{\Delta}^1_2$, and $\underset{\sim}{\Sigma}^1_2$ in (ii) and (iii) by a corresponding level $\circlearrowleft \Gamma$ of $\underset{\sim}{\Delta}^1_3$. (At the moment, we don't see how to extend (iv).) Such strengthenings of (i)-(iv) were developed by D.A.Martin, his students, and others in the 1970's and early 1980's. (See for example ??)

Canonical Extender Models

If E and F are V-extenders, then again

$$E \lhd F \quad \text{iff} \quad E \in \mathrm{Ult}(V, F).$$

A coherent sequence of extenders will again be a sequence linearly ordered by $\lhd$, without leaving gaps. Here we face the complication that there certaninly E with $\eta < \mathrm{lh}(E)$ such that E is equivalent to $E \upharpoonright \eta$. in that $\mathrm{Ult}(V, E) = \mathrm{Ult}(V, E \upharpoonright \eta)$

Exercise 28. If $\mathrm{crit}(E) < \gamma$ and $\gamma + 2 = \mathrm{lh}(E)$, then $\mathrm{Ult}(V, E \upharpoonright (\gamma + 1) = \mathrm{Ult}(V, E)$.

So we can put the "same" extender on our sequence with different lengths. Some indexing convention is needed. Here is one from [**4**].

Definition 5.2. A **coherent sequence of non-overlapping extenders** is a function $\mathcal{E}$ with domain of the form $\{(\kappa, \beta) \mid \beta < o^{\mathcal{E}}(\kappa)\}$ such that

(1) If $o^{\mathcal{E}}(\kappa) > 0$, then $\forall \lambda < \kappa (o^{\mathcal{E}}(\lambda) < \kappa)$, and if $\beta < o^{\mathcal{E}}(\kappa)$, then
(2) $\mathcal{E}(\kappa, \beta)$ is a $(\kappa, \kappa + 1 + \beta)$ extender over V,
(3) $i_{\mathcal{E}(\kappa,\beta)}(\mathcal{E}) \upharpoonright (\kappa + 1, 0) = \mathcal{E} \upharpoonright (\kappa, \beta)$.

The "non-overlapping" part is clause (1). It guarantees that $\mathcal{E}$ is simple enough that iterability suffices for canonicity (e.g., for comparison). On the other hand, it prevents $L[\mathcal{E}]$ from satisfying more than "There is a strong cardinal". If we want a theory of canonical inner models with for example, Woodin cardinals, then drop clause (1) above, but at the same time we must generalize the notion of linear iterability.

Moving to extenders yields one simplification: the functions witnessing coherence are now trivial, as if $\gamma < \beta$, then $\gamma = [\{\gamma\}, id]_{\mathcal{E}(\kappa,\beta)}$. So we get

Proposition 5.3. *If $\mathcal{E}$ is a coherent sequence of non-overlapping extenders, then*

$$L[\mathcal{E}] \vDash \mathcal{E} \textit{ is a coherent sequence of non-overlapping extenders.}$$

Remark. $L[\mathcal{E}] = L[A]$, where $A = \{(\kappa, \beta, a, x) \mid (a, x) \in \mathcal{E}(\kappa, \beta)\}$.

Also, it becomes a little easier to show that large cardinal properties go down to $L[\mathcal{E}]$. For example,

Theorem 5.4. *Suppose there is a strong cardinal; then there is a proper class $\mathcal{E}$ such that*

$$L[\mathcal{E}] \vDash \mathcal{E} \textit{ is a coherent sequence of non-overlapping extenders,}$$

and

$$L[\mathcal{E}] \vDash \mathcal{E} \textit{ there is a strong cardinal.}$$

PROOF SKETCH. Construct a maximal non-overlapping coherent sequence, defining $\mathcal{E}(\kappa, \beta)$ by induction on the lexicographic order on the (κ, β)'s. □

Exercise 29. Give a real proof of Theorem 5.4.

Definition 5.5. An **extenders-premouse** is a pair $(M, \mathcal{E})$ such that M is transitive and

$$M \vDash \mathbf{ZFC}^{-} \mathcal{E} \text{ is a coherent sequence of non-overlapping extenders.}$$

An **extenders-mouse** is a linearly iterable extenders-premouse.

The initial segment relation $\mathfrak{M} \trianglelefteq \mathfrak{N}$ on extenders-premice is defined just as before. We get as before:

Theorem 5.6 (Comparison Lemma)**.** *Let $\mathfrak{M}$ and $\mathfrak{N}$ be set size extender-mice. Then there are linear iterations I and J of $\mathfrak{M}$ and $\mathfrak{N}$ such that*

$$\mathfrak{M}^I_\infty \trianglelefteq \mathfrak{N}^J_\infty \textit{ or } \mathfrak{N}^J_\infty \trianglelefteq \mathfrak{M}^I_\infty.$$

Corollary 5.7. *Suppose $V = L[\mathcal{E}]$, where $\mathcal{E}$ is a coherent sequence of non-overlapping extenders. Then* **CH** *holds, and $\mathbb{R}$ admits a Δ^{HC}_2 well order.*

We get other corollary parallel to those in Lecture **??** as well. For example,

Corollary 5.8. Con(**ZFC**+ *"There is a strong cardinal.")*$\Longrightarrow$ Con(**ZFC**+ *"There is a strong cardinal"+"$\mathbb{R}$ admits a Δ^{HC}_2 well order.")*

The Comparison Lemma, Theorem 5.6, is proved just as Lemma 3.6, the Comparison Lemma for measures-mice, was proved. I and J are constructed by iteration away the least disagreement.

Iteration Trees of Length ω

Suppose
$$M_0 \vDash E_0 \text{ is an extender and } \lambda = \text{strength}(E_0).$$
Set
$$M_1 = \text{Ult}(M_0, E),$$
and suppose
$$M_1 \vDash E_1 \text{ is an extender, and } \text{crit}(E_1) < \lambda.$$
If we were iterating in linear fashion as before, our next model would be $M_2 = \text{Ult}(M_1, E_1)$. But there is another possibility. Since $V_\lambda^{M_0} = V_\lambda^{M_1}$ and $\text{crit}(E_1) < \lambda$, E_1 is a pre-extender over M_0, and we could set
$$M_2 = \text{Ult}(M_0, E_1),$$
and continue iterating from there. (Assuming that M_2 is well-founded!) For example, one can show that for $\lambda_1 = \text{strength}(E_1)^{M_1}$, we have that $V_{\lambda_1}^{M_1} = V_{\lambda_1}^{M_2}$. (This is done below.) So if
$$M_2 \vDash E_2 \text{ is an extender}$$
and $\text{crit}(E_2) < lambda_1$, then we could set
$$M_3 = \text{Ult}(M_1, E_2),$$
and if M_3 were well-founded, continue from there. Our picture so far is

$$\begin{tikzcd}
 & M_1 \arrow[r] & M_3 \\
M_0 \arrow[ur] \arrow[dr] & & \\
 & M_2 &
\end{tikzcd}$$

If we could find extenders with the right pattern of strengths and critical points, with all the relevant ultrapowers well-founded, we might be able to generate an "alternating chain":

$$\begin{tikzcd}
 & M_1 \arrow[r] & M_3 \arrow[r] & M_5 & \cdots\cdots \\
M_0 \arrow[ur] \arrow[dr] & & & & \\
 & M_2 \arrow[r] & M_4 \arrow[r] & M_6 & \cdots\cdots
\end{tikzcd}$$

where
$$M_{n+1} = \text{Ult}(M_{n-1}, E_n)$$
with
$$M_n \vDash E_n \text{ is an extender.}$$
It turns out that if $M_0 = V$, and $\exists\delta(\delta \text{ is woodin})$, then there are indeed such alternating chains, and in fact, many other interesting and useful "iteration tree".

In this section, we shall restrict ourselves to models of full **ZFC**. Whenever we speak of $\text{Ult}(M, E)$, we assume that M is transitive and $M \vDash$ **ZFC**, enen if this not explicitly stated.

Definition 6.1. For M, N transitive models of **ZFC**,

$$M \underset{\alpha}{\frown} N \quad \text{iff} \quad V_\alpha^M = V_\alpha^N.$$

The following helps prepare agreement-of-models in an iteration tree.

Lemma 6.2. *Let M and N be transitive models of* **ZFC**, *and $M \underset{\kappa+1}{\frown} N$. Suppose*

$$M \vDash E \textit{ is an extender}$$

and $\mathrm{crit}(E) = \kappa$. *Then*

(1) *E is a pre-extender over N,*
(2) $\mathrm{Ult}(, E) \underset{i_E(\kappa)+1}{\frown} \mathrm{Ult}(N, E)$, *and*
(3) $i_E^M \restriction V_{\kappa+1}^M = i_E^N \restriction V_{\kappa+1}^N$.

Proof. Sketch of proof Let $f : [\kappa]^{|a|} \to V_{\kappa+1}^M = V_{\kappa+1}^N$. Then $f \in M$ iff $f \in N$. This implies that the two ultrapowers agree on their common image of $V_{\kappa+1}^M = V_{\kappa+1}^N$. □

Exercise 30. Think through the details.

In order to simplify some points we shall for now only consider iteration trees formed using extenders which are nice in the model they are taken from. (Recall that E is nice iff $\mathrm{lh}(E) = \mathrm{strength}(E)$ is strongly inaccessible.) From Lemma 4.4, we get at once

Lemma 6.3. *Suppose $M \vDash E$ is a nice (κ, λ)-extender, and $M \underset{\kappa+1}{\frown} N$. LEt $\alpha \leq \kappa$, and suppose both M and N are closed under α-sequences, then* $\mathrm{Ult}(N, E)$ *is closed under α-sequences.*

In the situation of Lemma 6.3, if $\alpha \geq \omega$, then we can conclude that $\mathrm{Ult}(N, E)$ is well-founded.

Definition 6.4. Let $\alpha \leq \omega$. T is a **tree order on** α iff

(1) (α, T) is a partial order,
(2) $(nTm) \Longrightarrow n < m$, for all $m, n < \alpha$,
(3) $\{n \mid nTm\}$ is linearly ordered by T, for all $m < \alpha$, and
(4) $0\ Tn$ for all n such that $0 < n < \alpha$.

We write $\mathrm{pd}_T(n+1)$=largest m such that $mT(n+1)$.

Definition 6.5. Let $\alpha \leq \omega$. A **nice iteration tree of length** α **on** M is a pair $\mathcal{T} = \langle T, \langle (M_n, E_n) \mid n < \alpha \rangle \rangle$ such that for all $n, m < \alpha$

(1) T is a tree order on α,
(2) $M_0 = M$,
(3) $M_n \vDash E_n$ is a nice extender,
(4) $n < m \Longrightarrow \mathrm{lh}(E_n) < \mathrm{lh}(E_m)$,
(5) if $n+1 < \alpha$, then $M_{n+1} = \mathrm{Ult}(M_k, E_n)$, where k = least i such that $M_i \underset{\mathrm{crit}(E_n)+1}{\frown} M_n$, moreover $\mathrm{pd}_T(n+1) = k$ in this case.

Notation. For $\mathcal{T}$ as above, we set $\mathrm{lh}(\mathcal{T}) = \alpha$. If $\mathcal{T}$ is an iteration tree, we write $M_n^{\mathcal{T}}$ and $E_n^{\mathcal{T}}$ for the models an extenders of $\mathcal{T}$. Note that if T is the associated tree order, then there are canonical embeddings

$$i_{nm}^{\mathcal{T}} : M_n^{\mathcal{T}} \to M_m^{\mathcal{T}} \quad (\text{for } nTm)$$

between the models earlier on a given branch and thise later. (Here $i_{k,m} : M_k \to \mathrm{Ult}(M_k, E_{m-1})$ is the canonical ultrapower embedding if $k = \mathrm{pd}_T(m)$, and $i_{k,m} = i_{\mathrm{pd}_T(m),m} \circ i_{k,\mathrm{pd}_T(m)}$ otherwise.)

The following lemma records the agreement between models in a nice iteration tree.

Lemma 6.6. *Let $\langle T, \langle (M_n, E_n) \mid n < \alpha \rangle \rangle$ be a nice iteration tree. Let $k \leq n$; then*

(a) $M_k \underset{\mathrm{lh}(E_k)}{\frown} M_n$, *but*
(b) *it is not the case that $M_k \underset{\mathrm{lh}(E_k)+1}{\frown} M_n$, if $k < n$.*

Moreover, for any n, $\mathrm{pd}_T(n+1)$ *is the least i such that* $\mathrm{crit}(E_n) < \mathrm{lh}(E_i)$.

Proof. We prove (a). Fix k. The proof is by induction on n. The case $n = k$ is clear. Not let $\lambda = \mathrm{lh}(E_k)$. We have

$$M_n \underset{\lambda}{\frown} M_k$$

by induction, and

$$M_n \underset{i_{e_n}(\kappa)+1}{\frown} M_{n+1}$$

by Lemma 6.2, where $\kappa = \mathrm{crit}(E_n)$. But

$$\mathrm{lh}(E_k) \leq \mathrm{lh}(E_n) \leq i_{E_n}(\kappa),$$

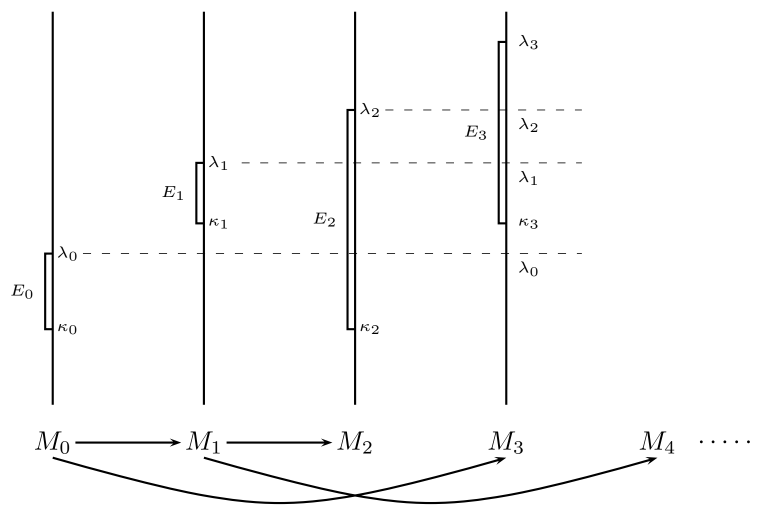
by clause (4) of Definition 6.5, and the fact that we use only short extenders. Thus $M_{n+1} \underset{\lambda}{\frown} M_k$, completing the induction step.

The rest is an exercise. □

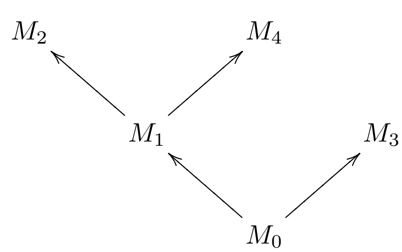
Exercise 31. Complete the proof of Lemma 6.6.

Remark. Our nice iteration trees are fairly special in several ways. It isn't necessary to use only nice extenders. It isn't necessary that the strengths of the extenders be increasing, as in clause (4) of Definition 6.5. It isn't necessary that $\mathrm{pd}_T(n+1)$ be the least i such that $M_i \underset{\mathrm{crit}(E_n)+1}{\frown} M_n$, it need only be some such i. A more general notion of iteration tree is needed in many contexts. The present restrictions make several points cleaner. In particular, 6.6 has the simple statement above.

Here is a diagram illustrating Lemma 6.6. In the diagram, $\mathrm{crit}(E_i) = \kappa_i$ and $\mathrm{strength}^{M_i}(E_i) = \lambda_i$.



Another picture of the same iteration tree:



As the first picture shows, the process of determing and iteration tree is linear: M_{n+1} is determined by $\langle (M_i, E_i) \mid ileqn \rangle$, and then we are free to choose E_{n+1} in order to continue. As the second picture shows, the embeddings between models may fall into a non-linear structure.

If $\mathcal{T}$ has length $n+1$, then we set

$$M^{\mathcal{T}}_{\infty} = \mathrm{Ult}(M_k, E^{\mathcal{T}}_n), \text{ where } k = \text{the least } i \text{ such that } \mathrm{crit}(E^{\mathcal{T}}_n) < \mathrm{lh}(E^{\mathcal{T}}_i),$$

and we call $M^{\mathcal{T}}_{\infty}$ be the **last model** of $\mathcal{T}$. Nothing forces it to be well-dounded, but if we are to continue from $\mathcal{T}$, it had better be! In this connection, we have immediately from Lemma 6.3:

Lemma 6.7. *Let $\mathcal{T}$ be a nice iteration tree of lenght$\leq \omega$, and $\eta < \inf\{\mathrm{crit}(E^{\mathcal{T}}_i) \mid i+1 < \mathrm{lh}(\mathcal{T})\}$. Suppose $M^{\mathcal{T}}_0$ is closed under η-sequences. Then*

(a) $\forall n < \mathrm{lh}(\mathcal{T})(M^{\mathcal{T}}_n$ *is closed under η-sequences), and*
(b) *if $\mathrm{lh}(\mathcal{T}) < \omega$, then $M^{\mathcal{T}}_{\infty}$ is closed under η-sequences,*
(c) *if $\mathrm{lh}(\mathcal{T}) < \omega$, then $M^{\mathcal{T}}_{\infty}$ is well-founded.*

Exercise 32. Prove Lemma **??**. (It is easy.)

How do we continue an iteration tree of length ω?

Definition 6.8. Let $\mathcal{T} = \langle T, \langle M_n, E_n \mid n < \omega \rangle\rangle$ be an iteration tree, and let b be a branch of T. Then

$$M^{\mathcal{T}}_b = \mathrm{dir}\lim_{k \in b} M^{\mathcal{T}}_k$$

under the $i^{\mathcal{T}}_{k,l}$ for $k, l \in b$. We say b **is well-founded** iff $M^{\mathcal{T}}_b$ is well-founded.

What we would like, in order to continue from $\mathcal{T}$, is a confinal-in-ω well-founded barnch of $\mathcal{T}$. If $M^{\mathcal{T}}_0 = V$, then we can find such a branch:

Theorem 6.9. *Let $\mathcal{T}$ be a nice iteration tree on V, with $\mathrm{lh}(\mathcal{T}) = \omega$. Then there is a confinal-in-ω well-founded barnch of $\mathcal{T}$.*

Remark. (1) So for example, the tree

$$\begin{tikzcd}
M_1 & M_2 & M_3 & M_4 & \cdots\cdots \\
& V = M_0 \arrow[ul] \arrow[u] \arrow[ur] \arrow[urr] \arrow[urrr] & & &
\end{tikzcd}$$

is impossible. In other words, the Mitchell order $\lhd$ on nice extenders is well-founded. (In fact, it is well-founded on arbitrary short extenders.)

(2) More generally, if $\mathcal{T} = \langle T, \cdots \rangle$ is a nice iteration tree on V of length ω, then T has an infinite branch. Andretta ([**1**]) has shown that this is the only restriction on T, provide there is a Woodin cardinal. That is, if there is a Woodin cardinal, and T is a tree order on ω having an infinite branch, then there is a nice iteration treeon V whose tree order is T.

By Theorem 6.9, if

$$\begin{tikzcd}
& M_1 \arrow[r] & M_3 \arrow[r] & \cdots \arrow[r] & M^{\mathcal{T}}_{odd} \\
V = M_0 \arrow[ur] \arrow[dr] & & & & \\
& M_2 \arrow[r] & M_4 \arrow[r] & \cdots \arrow[r] & M^{\mathcal{T}}_{even}
\end{tikzcd}$$

is a nice alternating chain on V, then either $M^{\mathcal{T}}_{odd}$ or $M^{\mathcal{T}}_{even}$ is well-founded. It is open whether one of the two must be ill-founded! More generally, we have

Big Open Problem. Is there a nice iteration tree on V of length ω having distinct cofinal well-founded branches?

A negative answer would mean that every such iteration tree can be continued in a **unique** way. This would have many useful consequences, as we shall see below.

PROO OF THEOREM 6.9. Let $\mathcal{T} = \langle T, \langle M_n, E_n \mid n < \omega \rangle \rangle$ be a counterexample. The first step is to localize the bad-ness of $\mathcal{T}$ in countably many ordinals. (Compare Lemma 2.5.) This is done in the

Claim. There are ordinals α_n, for $n < \omega$, such that for all n, m

$$nTm \Longrightarrow i^{\mathcal{T}}_{nm}(\alpha_n) > \alpha_m.$$

PROOF OF THE CLAIM. First pick η such that whenever b is a cofinal barnch of T, then $i^{\mathcal{T}}_b(\eta)$ is in the ill-founded part of $M^{\mathcal{T}}_b$. (Here $i^{\mathcal{T}}_b : M^{\mathcal{T}}_0 \to M^{\mathcal{T}}_b$ is the direct limit map.)

For $n < \omega$, let

$$\mathscr{B}_n = \{b \mid b \text{ is a confinal branch of } T \text{ and } n \in b\}.$$

Let

$$X = \{(n, f) \mid f : \mathscr{B}_n \to \eta\}.$$

For $(n, f), (m, g) \in X$, we let

$$(n, f) < (m, g) \quad \text{iff} \quad mTn \text{ and } \forall b \in \mathscr{B}_n (f(b) < g(b)).$$

It is easy to see that $<$ is a well-founded realtion on X.

Now if $\mathscr{B}_n = \emptyset$, so that T is well-founded below n, we put

$$\alpha_n = |n|_T = \text{ rank of } T \text{ below } n.$$

Then if $\mathscr{B}_n = \emptyset$ and nTm, we have $\mathscr{B}_m = \emptyset$, and since $\alpha_n < \omega_1$,

$$i_{nm}(\alpha_n) = \alpha_n > \alpha_m,$$

as desired.

So we may assume $\mathscr{B}_0 \neq \emptyset$, otherwise we're done. For each $b \in \mathscr{B}_0$, pick α^b_n for $n < \omega$ such that

$$i_{nm}(\alpha^b_n) > \alpha^b_m$$

whenever nTm and $n, m \in b$. We may assume $\alpha^b_0 = \eta$ for all b. Now for n such that $\mathscr{B}_n \neq \emptyset$, set

$$\alpha_n = \omega_1 + |(n, h)|_{i_{0,n}(<)}$$

where $h(b) = \alpha^b_n$ for all $b \in \mathscr{B}_n$. Note at this point that $h \in M_n$, since M_n is $2^{\aleph_0}$-closed. Then we have , if nTm and $\mathscr{B}_m \neq \emptyset$:

$$\begin{aligned} i_{n,m}(\alpha_n) &= \omega_1 + |(n, i_{nm}(\lambda b \in \mathscr{B}_n . \alpha^b_n))|_{i_{0m}(<)} \\ &= \omega_1 + |(n, \lambda b \in \mathscr{B}_n . i_{nm}(\alpha^b_n))|_{i_{0m}(<)} \\ &> \omega_1 + |(m, \lambda b \in \mathscr{B}_m . \alpha^b_m)|_{i_{0m}(<)} \\ &= \omega_1 + \alpha_m. \end{aligned}$$

This yields the claim. □

Ordinal s $\langle \alpha_n \mid n \in \omega \rangle$ as in the claim are said to witness that $\mathcal{T}$ is **continuously ill-founde**.

To simplify the rest of the proof a bit, we shll assume there are arbitrarily large ξ such that $V_\xi \models$ **ZFC**. We leave is as an exercise to dispense with this assumption.

Let $\langle \alpha_n \mid n < \omega \rangle$ witness that $\mathcal{T}$ is continuously ill-founded. Let $\eta_0 > \alpha_0$ be such that $V_{\eta_0} \models$ **ZFC**, and $\mathcal{T} \in V_\eta$. Now let N_0 be countable and transitive, and

$$\pi_0 : N_0 \to V_{\eta_0}$$

elementry, with $\mathcal{T} \in \operatorname{ran}(\pi_0)$, and $\langle \alpha_n \mid n < \omega \rangle \in \operatorname{ran}(\pi_0)$. Let

$$\pi(F_n) = E_n,$$

and

$$\pi(\beta_n) = \alpha_n$$

for all n. Note $\pi(T) = T$, and $\mathcal{U} = \langle T, \langle N_k, F_k \mid k < \omega \rangle\rangle$ is an iteration tree on N_0, where

$$N_{k+1} = \mathrm{Ult}(N_i, E_k),\ \text{for } i = \mathrm{pd}_T(k+1).$$

Moreover, $\langle \beta_n \mid n < \omega \rangle$ witness that $\mathcal{U}$ is continuously ill-founded.

We now define $(P_m, \in, \eta_m)$ and π_m by induction on m so that

(1) $\pi_m : N_m \to V_{\eta_m}^{P_m}$,

(2) $P_m \vDash \mathbf{ZFC}$, P_m is closed under ω-sequences and $\left\{\alpha \mid \eta_m < \alpha < \mathrm{Ord}^{P_m} \wedge V_\alpha^{P_m} \vDash \mathbf{ZFC}\right\}$ has order-type $\geq \pi_m(\theta_m)$,

(3) for all $k \leq m$

  (a) $P_k \sim_{\mathcal{T}} P_m$, where $\mathcal{T} = \sup \pi_k"\mathrm{lh}(F_k)$,

  (b) $\pi_k \upharpoonright V_{\mathrm{lh}(F_k)}^{N_k} = \pi_m \upharpoonright V_{\mathrm{lh}(F_k)}^{N_k}$ and

(4) $P_m \in P_{m-1}$, if $m > 0$.

If we do this, clause (4) yields the desired contradiction.

$m = 0$: We have π_0 and η_0 already. Let $P_0 = V_\tau$, where $V_\tau \vDash \mathbf{ZFC}$ and $\{\alpha \mid \eta_0 < \alpha < \tau \wedge V_\alpha \vDash \mathbf{ZFC}\}$ has order type at least $\pi_0(\theta_0)$.

$m = n+1$: Let $E = \pi_n(F_n)$. Let $E = \pi_n(F_n)$. Let $\kappa = \mathrm{pd}_T(n+1)$, and $\tau = \sup \pi_k"\mathrm{lh}(F_k)$. $\mathrm{crit}(F_n) < \mathrm{lh}(F_k)$, so $\mathrm{crit}(E) < \tau$. Also, $P_m \sim_\tau P_k$. Hence we may set

$$Q = \mathrm{Ult}(P_k, E).$$

We have that Q is closed under ω-sequences, and hence well-founded. Let $j : P_k \to Q$ be the canonical embedding. Set

$$\gamma = j(\eta_k),$$

we can now find σ such that the diagram

$$\begin{tikzcd}
V_{\eta_k}^{P_k} \arrow[r, "j"] & V_\gamma^Q & \\
N_k \arrow[u, "\pi_k"] \arrow[r, "i_{k,n+1}^{\mathcal{U}}"'] & \mathrm{Ult}(N_k, F_n) \arrow[u, "\sigma"'] & = N_{n+1}
\end{tikzcd}$$

commutes. Namely, we set

$$\sigma([a, f]_{F_n}^{N_k}) = [\pi_n(a), \pi_k(f)]_E^{P_k}.$$

Exercise 33. Prove that σ is well-defined, elementary, and that the diagram commutes.

Exercise 34. Show that $\sigma \upharpoonright V_{\mathrm{lh}(F_n)^{N_{n+1}}} = \pi_n \upharpoonright V_{\mathrm{lh}(F_n)}^{N_n}$.

Together, these exercise are called the Shift Lemma. The key to proving them is that π_n and π_k agree on $V_{\mathrm{lh}(F_k)}^{N_k}$, by induction.

We could now set $P_{n+1} = Q$ and $\eta_{n+1} = \gamma$ and $\pi_{n+1} = \sigma$, and satisfy (1)-(3). In order to satisfy (4) as well, we replace Q by a Skolem hull of itself. Notice that Q has

$$j(\pi_k(\theta_k)) = \sigma(i_{k,n+1}(\theta_k)) > \sigma(\theta_{n+1})$$

many $\alpha > \gamma$ such that $V_\alpha^Q \vDash \mathbf{ZFC}$, in order type. Note also $\sigma \in Q$, as Q is ω-closed. Let

$$\mu = \sigma(\theta_{m+1})^{th} \text{ ordinal } \alpha > \gamma \text{ such that } V_\alpha^Q \vDash \mathbf{ZFC}.$$

Put

$$P_{n+1} = \text{ transitive collapse of } \mathrm{Hull}^{V_\mu^Q}(\{\sigma\} \cup V_{\sup \sigma"\mathrm{lh}(F_n)}^Q),$$

and

$$\pi_{n+1} = \text{ image of } \sigma \text{ under the collapse,}$$
$$\eta_{n+1} = \text{ image of } \gamma \text{ under the collapse.}$$

Notice that $P_{n+1} \in V^{Q}_{\mathrm{lh}(E)} = V^{P_n}_{\mathrm{lh}(E)}$, as $sup\sigma$"$\mathrm{lh}(F_n) < \mathrm{lh}(E)$, because $\mathrm{lh}(E)$ is inaccessible in Q. This gives (4). We leave (1)-(3) to the reader.

Theorem 6.9 $\square$

LECTURE 7

Iteration Trees of Transfinite Length

It is often important to continue iterating into the transfinite. The way we continue a tree $\mathcal{T}$ of limit length λ, i.e.: pick a branch b which has been visited cofinally often below λ, and such that $\mathfrak{M}_b^{\mathcal{T}}$ is wellfounded. Set $\mathfrak{M}_\lambda^{\mathcal{T}} = \mathfrak{M}_b^{\mathcal{T}}$, and continue. To be more precise:

Definition 7.1. Let $\gamma \in \mathrm{Ord}$. We call $\mathcal{T}$ a **tree order on** γ iff

(1) T is a strict partial order of γ,
(2) $\forall \beta < \gamma$ (T wellorders $\{\alpha \mid \alpha T \beta\}$),
(3) $\forall \alpha, \beta$ ($\alpha T \beta \implies \alpha < \beta$),
(4) $\forall \alpha$ ($0 < \alpha \implies 0 T \alpha$),
(5) $\forall \alpha$ (α is a successor ordinal iff α is a T-successor), and
(6) $\forall \lambda < \gamma$ (λ is a limit ordinal $\implies \{\alpha \mid \alpha T \lambda\}$ is $\in$-cofinal in λ).

Definition 7.2. Let $\gamma \in \mathrm{Ord}$. A **nice iteration tree of length** γ **on** M is a system

$$\mathcal{T} = \langle T, \langle (M_\alpha, E_\alpha) \mid \alpha < \gamma \rangle, \langle i_{\alpha\beta} \mid \alpha T \beta \rangle \rangle$$

such that

(1) T is a tree order on γ,
(2) $M_0 = M$,
(3) $M_\alpha \models E_\alpha$ is a nice extender,
(4) $\alpha < \beta \implies \mathrm{lh}(E_\alpha) < \mathrm{lh}(E_\beta)$,
(5) if $\alpha + 1 < \gamma$, then $M_{\alpha+1} = \mathrm{Ult}(M_\xi, E_\alpha)$, where ξ is least such that

$$M_\xi \underset{\mathrm{crit}(E_\alpha)+1}{\frown} M_\alpha.$$

(6) if $\lambda < \gamma$ is a limit ordinal, then

$$\begin{aligned} M_\lambda &= \operatorname{dir}\lim_{\alpha T \lambda} M_\alpha, \\ i_{\alpha\lambda} &= \text{canonical embedding, for } \alpha T \lambda. \end{aligned}$$

Is it always possible to continue a nice iteration tree on V? At successor steps, yes.

Theorem 7.3. *Let $M \models$ **ZFC** be transitive and closed under ω-sequences. Let $\mathcal{T}$ be a nice iteration tree on M of length $\alpha + 1$, and let $\xi \leq \alpha$ be such that $\mathfrak{M}_\xi^{\mathcal{T}} \underset{\mathrm{crit}(E_\alpha^{\mathcal{T}})+1}{\frown} \mathfrak{M}_\alpha^{\mathcal{T}}$. Then $\mathrm{Ult}(\mathfrak{M}_\xi^{\mathcal{T}}, E_\alpha^{\mathcal{T}})$ is well-founded.*

Proof. We need the following exercise:

Exercise 35. Let $\mathcal{T}$ be a nice iteration tree and $\alpha < \mathrm{lh}(\mathcal{T})$. Show

$$\mathfrak{M}_\alpha^{\mathcal{T}} = \{ i_{0\alpha}^{\mathcal{T}}(f)(a) \mid f \in \mathfrak{M}_0^{\mathcal{T}} \wedge a \in [\nu]^{<\omega} \}$$

where $\nu = \sup\{ \mathrm{lh}(E_\xi^{\mathcal{T}}) \mid (\xi+1) T \alpha \text{ or } \xi + 1 = \alpha \}$.

The exercise generalizes the fact that

$$\mathrm{Ult}(M, E) = \{ i_E^M(f)(a) \mid f \in M \wedge a \in [\mathrm{lh}(E)]^{<\omega} \}.$$

It says that $\mathfrak{M}_\alpha^{\mathcal{T}}$ is Skolem-generated by $\mathrm{ran}(i_{0\alpha}^{\mathcal{T}})$ together with ordinals below the sup of the lengths of extenders used on the branch 0-to-α.

Now let ξ, α be as in the theorem, and set $N = \mathrm{Ult}(M_\xi^{\mathcal{T}}, E_\alpha^{\mathcal{T}})$. Let $i : M \to N$ be the canonical embedding. ($i = \pi \circ i_{0\xi}^{\mathcal{T}}$, where $\pi : M_\xi^{\mathcal{T}} \to N$.) Let $\lambda = \mathrm{lh}(E_\alpha^{\mathcal{T}})$. The proof of Exercise 35 easily yields

$$N = \{ i(f)(a) \mid a \in [\lambda]^{<\omega} \wedge f \in M \}.$$

Note here that although N may be illfounded, $i(\kappa) \in \mathrm{wfp}(N)$, where $\kappa = \mathrm{crit}(E_\alpha^{\mathcal{T}})$. This is because $V_{i(\kappa)+1}^N = V_{i(\kappa)+1}^{\mathrm{Ult}(M_\alpha^{\mathcal{T}}, E_\alpha^{\mathcal{T}})}$, and $V_{i(\kappa)+1}^{\mathrm{Ult}(M_\alpha^{\mathcal{T}}, E_\alpha^{\mathcal{T}})}$ can be computed in $M_\alpha^{\mathcal{T}}$, which thinks $E_\alpha^{\mathcal{T}}$ is an extender.

Now pick $\langle f_k \mid k < \omega \rangle$ such that there are $a_k \in [\lambda]^{<\omega}$ with $i(f_{k+1})(a_{k+1}) \in^N i(f_k)(a_k)$ for all k. Note $\langle f_k \mid k \in \omega \rangle \in M$! Thus

$$i(\langle f_k \mid k \in \omega \rangle) = \langle i(f_k) \mid k \in \omega \rangle \in N.$$

Now pick γ such that

$$N \models \gamma \in \mathrm{Ord} \wedge \lambda < \gamma \wedge \langle i(f_k) \mid k \in \omega \rangle \in V_\gamma.$$

Working in N, we have H, π such that

$$N \models N \text{ is transitive, } |H| = \lambda, \text{ and } \pi : H \to V_\gamma$$
$$\text{with } \pi \restriction \lambda = id \text{ and } \langle i(f_k) \mid k \in \omega \rangle \in \mathrm{ran}(\pi).$$

Now $(\mathrm{ran}(\pi), \in^N)$ is illfounded in V, hence $(H, \in^N)$ is illfounded in V. But $H \in V_{i(\kappa)}^N \subseteq \mathrm{wfp}(N)$, a contradiction. □

Insofar as continuing nice trees on V at limit steps goes, the main result is the following

Theorem 7.4. *Let $\mathcal{T}$ be a nice iteration tree on V of countable limit length λ. Suppose that for all limit $\eta < \lambda$, $\{\alpha \mid \alpha T \eta\}$ is the* unique *cofinal wellfounded branch of $\mathcal{T} \restriction \eta$. Then $\mathcal{T}$ has a cofinal, wellfounded branch.*

Remark. In other words, if $\mathcal{T}$ has made the only choice it could make at limit $\eta < \lambda$, then there is a choice for it to make at λ.

We shall sketch the proof of Theorem 7.4 in an appendix to this lecture. It is much like the proof of Theorem **??**.

This leads us to one of the biggest open problems in the subject.

Definition 7.5. Nice-UBH is the statement: Every nice iteratioin tree on V of limit length has at most one cofinal, wellfounded branch.

Definition 7.6. Generic-nice-UBH is the statement: $V[G] \models$ nice-UBH,whenever G is set generic over V.

Whether nice-UBH, or better generic-nice-UBH, are true are very important questions. Of course, the more useful answer would be "yes". The reason is that we would then get, via Theorem 7.4, an *iteration strategy* for V. We now explain that concept more precisely.

Let $M \models$ **ZFC** be transitive, and $\theta \in \mathrm{Ord}$. The (nice) **iteration game of length θ on** M is played as follows: there are two players, I and II. They cooperate to produce an iteration tree $\mathcal{T}$ on $\mathfrak{M}$. At successor rounds $\alpha + 1$, player I extends $\mathcal{T}$ by picking a nice $E_\alpha^{\mathcal{T}}$ from $\mathfrak{M}_\alpha^{\mathcal{T}}$, and setting $\mathfrak{M}_{\alpha+1}^{\mathcal{T}} = \mathrm{Ult}(M_\xi^{\mathcal{T}}, E_\alpha^{\mathcal{T}})$ for ξ least such that $\mathrm{crit}(E_\alpha^{\mathcal{T}}) < \mathrm{lh}(E_\xi^{\mathcal{T}})$. (If the ultrapower is illfounded, the game ends, and I has won.) At limit rounds $\lambda < \theta$, II extends $\mathcal{T}$ by picking b cofinal in λ such that $\mathfrak{M}_b^{\mathcal{T}}$ is wellfounded. If II fails to do this, I wins.

If after θ rounds, I has not yet won, then II wins.

We call this game $G_{\mathrm{nice}}(M, \theta)$. A winning strategy for II in $G_{\mathrm{nice}}(M, \theta)$ is called a **θ-iteration strategy** for M. We say M **is θ-iterable (for nice trees)** iff there is a θ-iteration strategy for M. We have

Theorem 7.7. *If nice-UBH holds, then V is ω_1-iterable for nice trees.*

Proof. Player II's strategy in $G_{\mathrm{nice}}(V, \omega_1)$ is: at round λ, pick the unique cofinal wellfounded branch of $\mathcal{T} \restriction \lambda$. □

$\omega_1 + 1$-iterability is much more useful than ω_1-iterability. We have

Theorem 7.8. *If generic-nice-UBH holds, then V is κ-iterable for nice trees, where κ is the least measurable cardinal.*

Exercise 36. Prove Theorem 7.8. [You need to know something about preservation of large cardinals under small forcing to do this one, so it's really only for the more advanced students.]

The main result known in the direction of proving UBH is the following. For $\mathcal{T}$ a nice tree, set

$$\delta(\mathcal{T}) = \sup\{\mathrm{lh}(E^{\mathcal{T}}_\alpha) \mid \alpha < \mathrm{lh}(\mathcal{T})\},$$

$$\mathfrak{M}(\mathcal{T}) = \bigcup_{\alpha<\mathrm{lh}(\mathcal{T})} V^{M^{\mathcal{T}}_\alpha}_{\mathrm{lh}(E^{\mathcal{T}}_\alpha)}.$$

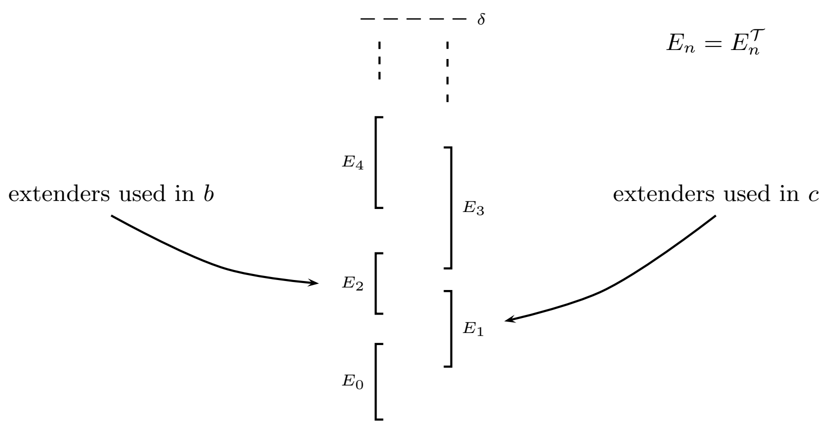
So $\mathrm{Ord} \cap \mathfrak{M}(\mathcal{T}) = \delta(\mathcal{T})$, and $\mathfrak{M}(\mathcal{T}) = V^{M^{\mathcal{T}}_b}_{\delta(\mathcal{T})}$ for any cofinal branch b of $\mathcal{T}$ such that $\delta(\mathcal{T}) \in M^{\mathcal{T}}_b$.

Theorem 7.9. *Let $\mathcal{T}$ be a nice iteration tree of limit length, $\delta = \delta(\mathcal{T})$, and suppose b and c are distinct cofinal branches of $\mathcal{T}$ such that $\delta \in M^{\mathcal{T}}_b \cap M^{\mathcal{T}}_c$. Let $A \subseteq \delta$ and $A \in M^{\mathcal{T}}_b \cap M^{\mathcal{T}}_c$. Then*

$$(\mathfrak{M}(\mathcal{T}), \in, A) \models \text{“} \exists\kappa\ (\kappa \textit{ is } A\textit{-reflecting in } \mathrm{Ord})\text{”}.$$

Remark. Another way this is often stated is: $\delta(\mathcal{T})$ is Woodin with respect to all $A \in M^{\mathcal{T}}_b \cap M^{\mathcal{T}}_c$, with respect to extenders in $\mathfrak{M}(\mathcal{T})$.

Proof of Theorem 7.9. (Sketch). Let us consider the special case $\mathcal{T}$ is an alternating chain, and b is its even branch and c is its odd branch. Note that the extenders of $\mathcal{T}$ overlap in the following “zipper” pattern



That is, letting $\kappa_n = \mathrm{crit}(E_n)$ and $\lambda_n = \mathrm{lh}(E_n)$: $\kappa_n < \kappa_{n+1} < \lambda_n$ for all n. Now let $A \subseteq \delta$ and $A \in M^{\mathcal{T}}_b \cap M^{\mathcal{T}}_c$. Pick m large enough that $A \in \mathrm{ran}(i_{m,b}) \cap \mathrm{ran}(i_{m+1,c})$.

Claim. For any $n \geq m$, $i_{E_n}(A \cap \kappa_n) \cap \lambda_n = A \cap \lambda_n$.

(That is, i_{E_n} shifts A to itself below the next critical points. It doesn't matter whether we write $i^{\mathfrak{M}(\mathcal{T})}_{E_n}$ or $i^{M^{\mathcal{T}}_{n-1}}_{E_n}$ here, since the ultrapowers are the same below the image of κ_n.)

Proof of Claim. Suppose e.g. $n+1 \in b$. Let $A = i_{m,b}(\bar{A})$. Then $i_{m,n-1}(\bar{A}) \cap \kappa_n = A \cap \kappa_n$, because $\mathrm{crit}(i_{n+1,b}) = \kappa_n$. So $i_{n-1,b}(A \cap \kappa_n)$ agrees with A below $i_{n-1,b}(A \cap \kappa_n)$. But $i_{n-1,b}(A \cap \kappa_n)$ agrees with $i_{E_n}(A \cap \kappa_n)$ below λ_n, because $\mathrm{crit}(i_{n+1,b}) \geq \lambda_n$. Consider

$$i_{E_n \restriction \kappa_{n+1}} \circ \cdots \circ i_{E_{m+1} \restriction \kappa_{m+2}} \circ \cdots i_{E_m \restriction \kappa_{m+1}} = j.$$

Note $E_i \restriction \kappa_{i+1} \in M(\mathcal{T})$, because $\kappa_{i+1} < \mathrm{lh}(E_i)$! It is routine to show that j witnesses κ_m is A-reflecting to β in $M(\mathcal{T})$. □

Exercise 37. (1) Complete the proof of this.

(2) Complete the proof of Theorem 7.9 as follows: let b and c be distinct cofinal branches of $\mathcal{T}$. Find $\langle \alpha_n \mid n \in \omega \rangle$ cofinal in λ such that

$$\alpha_{2n} + 1 \in b, \text{''} uad \alpha_{2n+1} + 1 \in c$$

and

$$\mathrm{crit}(E_{\alpha_k}) < \mathrm{crit}(E_{\alpha_{k+1}}) < \mathrm{lh}(E_{\alpha_k})$$

for all k. I.e. we have the zipper pattern embedded in the two branches. Now argue as above. □

Remark. For more detail, see [**?**].

Remark. So if $\mathcal{T}$ has distinct cofinal branches, then $\mathrm{lh}(\mathcal{T})$ has cofinality ω. This is also easy to see from the fact that every branch of an iteration tree is *closed* below its sup (as a set of ordinals).

Corollary 7.10. *Suppose nice-UBH fails. Then there is a proper class model with a Woodin cardinal.*

Proof. Let $\mathcal{T}$ be nice on V, and have distinct cofinal wellfounded branches b and c. Then $L(M(\mathcal{T})) \subseteq M_b^{\mathcal{T}} \cap M_c^{\mathcal{T}}$. So by Theorem 7.9,

$$L(M(\mathcal{T})) \models \delta(\mathcal{T}) \text{ is Woodin.}$$

□

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