Ultrafilter Space Methods in Infinite Ramsey Theory

Sławomir Solecki

University of Illinois at Urbana–Champaign

November 2014

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Λ-semigroups and colorings—a review

A **partial semigroup** is a set S with a binary operation from a *subset* of $S \times S$ to S such that, for $x, y, z \in S$, if one of the products $(xy)z$, $x(yz)$ is defined, then both are and are equal.

Λ a set, S a partial semigroup, and X a set
A **Λ-partial semigroup over S based on X** is an assignment to each $\lambda \in \Lambda$ of a function from a subset of X to S such that for $s_0, \dots, s_k \in S$ and $\lambda_0, \dots, \lambda_k \in \Lambda$ there exists $x \in X$ with $s_0\lambda_0(x), \dots, s_k\lambda_k(x)$ defined.

Assume we have a Λ-partial semigroup over S and based on X.

A sequence (x_n) of elements of X is **basic** if for all $n_0 < \cdots < n_l$ and $\lambda_0, \ldots, \lambda_l \in \Lambda$

$$\lambda_0(x_{n_0})\lambda_1(x_{n_1}) \cdots \lambda_l(x_{n_l}) \tag{1}$$

is defined in S.

Assume we additionally have a point based Λ-semigroup $\mathcal{A}$ over $(A, \wedge)$.

A coloring of S is **$\mathcal{A}$-tame on (x_n)** if the color of elements of the form (1) with the additional condition $\lambda_k(\bullet) \wedge \cdots \wedge \lambda_l(\bullet) \in \Lambda(\bullet)$, for all $k \leq l$, depends only on

$$\lambda_0(\bullet) \wedge \lambda_1(\bullet) \wedge \cdots \wedge \lambda_l(\bullet) \in A.$$

$\mathcal{A}$ and $\mathcal{B}$ are Λ-semigroups with $\mathcal{A}$ being over A and based on X and $\mathcal{B}$ being over B and based on Y.

A **homomorphism from $\mathcal{A}$ to $\mathcal{B}$** is a pair of functions f, g such that $f\colon X \to Y$, $g\colon A \to B$, g is a homomorphism of semigroups, and, for each $x \in X$ and $\lambda \in \Lambda$, we have

$$\lambda(f(x)) = g(\lambda(x)).$$

Theorem

Fix a finite set Λ. Let $\mathcal{S}$ be a Λ-partial semigroup over S, and let $\mathcal{A}$ be a point based Λ-semigroup. Let $(f, g)\colon \mathcal{A} \to \gamma\mathcal{S}$ be a homomorphism.

Then for each $D \in f(\bullet)$ and each finite coloring of S, there exists a basic sequence (x_n) of elements of D on which the coloring is $\mathcal{A}$-tame.

The goal:

produce homomorphisms
from point based Λ-semigroups $\mathcal{A}$ to $\gamma\mathcal{S}$ of interest.

New ones from old ones—tensor products

Fix a partial semigroup S.

Λ_0, Λ_1 finite sets

$\mathcal{S}_i$, for $i = 0, 1$, Λ_i-partial semigroups over S with $\mathcal{S}_i$ is based on X_i

Put

$$\Lambda_0 \star \Lambda_1 = \Lambda_0 \cup \Lambda_1 \cup (\Lambda_0 \times \Lambda_1).$$

Define

$$\mathcal{S}_0 \otimes \mathcal{S}_1$$

to be a $\Lambda_0 \star \Lambda_1$-partial semigroup over S based on $X_0 \times X_1$ as follows: with

$$\lambda_0,\ \lambda_1,\ (\lambda_0, \lambda_1) \in \Lambda_0 \star \Lambda_1$$

associate partial functions $X_0 \times X_1 \to S$ by letting

$$\begin{aligned} \lambda_0(x_0, x_1) &= \lambda_0(x_0), \\ \lambda_1(x_0, x_1) &= \lambda_1(x_1), \\ (\lambda_0, \lambda_1)(x_0, x_1) &= \lambda_0(x_0)\lambda_1(x_1). \end{aligned}$$

$\mathcal{S}_0 \otimes \mathcal{S}_1$ is a $\Lambda_0 \star \Lambda_1$-partial semigroup.

Proposition (S.)

Fix semigroups A and B. For $i = 0, 1$, let $\mathcal{A}_i$ and $\mathcal{B}_i$ be Λ_i-semigroups over A and B, respectively. Let

$$(f_0, g)\colon \mathcal{A}_0 \to \mathcal{B}_0 \;\; \text{and} \;\; (f_1, g)\colon \mathcal{A}_1 \to \mathcal{B}_1$$

be homomorphisms. Then

$$(f_0 \times f_1, g)\colon \mathcal{A}_0 \otimes \mathcal{A}_1 \to \mathcal{B}_0 \otimes \mathcal{B}_1$$

is a homomorphism.

Let $\mathcal{S}_i$, $i = 0, 1$, be Λ_i-partial semigroups over S based on X_i. Consider

$$\gamma\mathcal{S}_0 \otimes \gamma\mathcal{S}_1 \text{ and } \gamma(\mathcal{S}_0 \otimes \mathcal{S}_1).$$

Both are $\Lambda_0 \star \Lambda_1$-semigroups over γS.

The first one is based on $\gamma X_0 \times \gamma X_1$, the second one on $\gamma(X_0 \times X_1)$.

There is a natural map $\gamma X_0 \times \gamma X_1 \to \gamma(X_0 \times X_1)$ given by

$$(\mathcal{U}, \mathcal{V}) \to \mathcal{U} \times \mathcal{V},$$

where, for $C \subseteq X_0 \times X_1$,

$$C \in \mathcal{U} \times \mathcal{V} \iff \{x_0 \in X_0 \colon \{x_1 \in X_1 \colon (x_0, x_1) \in C\} \in \mathcal{V}\} \in \mathcal{U}.$$

Proposition (S.)

Let S be a partial semigroup. Let $\mathcal{S}_i$, $i = 0, 1$, be Λ_i-partial semigroups over S. Then

$$(f, \mathrm{id}_{\gamma S})\colon \gamma\mathcal{S}_0 \otimes \gamma\mathcal{S}_1 \to \gamma(\mathcal{S}_0 \otimes \mathcal{S}_1),$$

where $f(\mathcal{U}, \mathcal{V}) = \mathcal{U} \times \mathcal{V}$, is a homomorphism.

An application—Furstenberg–Katznelson Theorem for located words

A bit more of general theory

$\mathcal{A}$ a point based Λ-semigroup over a semigroup A

Fix a natural number r.
Associate with each $\vec{\lambda} \in \Lambda_{<r}$ an element $\vec{\lambda}(\bullet)$ of A by letting

$$\vec{\lambda}(\bullet) = \lambda_0(\bullet) \wedge \cdots \wedge \lambda_m(\bullet),$$

where $m < r$ is the length of $\vec{\lambda}$.

This way we get a finite set $\Lambda_{<r}(\bullet) \subseteq A$.

$\mathcal{S}$ a Λ-partial semigroup over a partial semigroup S
(x_n) a basic sequence in $\mathcal{S}$

A coloring of S is r-$\mathcal{A}$-**tame on** (x_n) if the color of elements of the form

$$\lambda_0(x_{n_0})\lambda_1(x_{n_1})\cdots\lambda_l(x_{n_l}),$$

for $n_0 < \cdots < n_l$ and $\lambda_0, \ldots, \lambda_l \in \Lambda$, with the additional condition

$$\lambda_k(\bullet) \wedge \cdots \wedge \lambda_l(\bullet) \in \Lambda_{<r}(\bullet) \ \text{ for all } k \le l$$

depends only on

$$\lambda_0(\bullet) \wedge \lambda_1(\bullet) \wedge \cdots \wedge \lambda_l(\bullet) \in A.$$

The following corollary is an apparent generalization of the theorem.

Corollary

Fix a finite set Λ and a natural number r. Let $\mathcal{S}$ be a Λ-partial semigroup, $\mathcal{A}$ a point based Λ-semigroup, and $(f, g)\colon \mathcal{A} \to \gamma\mathcal{S}$ a homomorphism. Then for each $D \in f(\bullet)$ and each finite coloring of S, there exists a basic sequences (x_n) of elements of D on which the coloring is r-$\mathcal{A}$-tame.

The corollary follows from the theorem and the two propositions.

Proof.

We have a homomorphism (f, g) from $\mathcal{A}$ to $\mathcal{S}$.
There is a homomorphism $\mathcal{A}^{\otimes r} \to (\gamma\mathcal{S})^{\otimes r}$ equal to (f^r, g) by the first proposition.
Note that $D \times X^{r-1} \in f^r(\bullet)$.
Since, by the second proposition, there is a homomorphism $(\gamma\mathcal{S})^{\otimes r} \to \gamma(\mathcal{S}^{\otimes r})$, we have a homomorphism

$$\mathcal{A}^{\otimes r} \to \gamma(\mathcal{S}^{\otimes r}),$$

and we are done by the theorem. $\square$

Katznelson–Furstenberg for located words

Recall the statement:

Fix a set F of finitely many types. Color, with finitely many colors, all words from $\mathbb{N}$ to $M+N$. There exists a sequence of variable words (x_n) from $\mathbb{N}$ to M with $x_n < x_{n+1}$ and such that the color of words of the form

$$x_{n_0}[i_0] + x_{n_1}[i_1] + \cdots + x_{n_l}[i_l],$$

with $n_0 < n_1 < \cdots < n_l$, depends only on the type of the sequence obtained from $(i_0, \ldots, i_l)$ by deleting all entries less than M, provided this type belongs to F.

The type of $(j_0, \ldots, j_k)$ is the sequence obtained from $(j_0, \ldots, j_k)$ by shortening each run of identical numbers to a single number.

Monoid Λ:

L, Γ finite disjoint sets, e an element not in $L \cup \Gamma$.

$$\Lambda = L \cup \Gamma \cup \{e\},$$

with

$$\lambda_0 \cdot \lambda_1 = \begin{cases} \lambda_0, & \text{if } \lambda_1 = e; \\ \lambda_1, & \text{if } \lambda_1 \in L \cup \Gamma, \end{cases}$$

is a monoid with the identity element e.

Semigroup A:

Γ disjoint from $\{0,1\}$

Let

$$A$$

be freely generated by $\Gamma \cup \{0,1\}$ subject to the relations

$$a \wedge a = a \ \text{ and } \ a \wedge 1 = 1 \wedge a = a.$$

Point based Λ-semigroup $\mathcal{A}$ over A:

Assignment to elements of Λ of functions $\{\bullet\} \to A$:
For $\lambda \in \Lambda$, let $\lambda(\bullet) \in A$ be

$$\lambda(\bullet) = \begin{cases} 0, & \text{if } \lambda = e; \\ 1, & \text{if } \lambda \in L; \\ \lambda, & \text{if } \lambda \in \Gamma. \end{cases}$$

This defines a point based Λ-semigroup over A called $\mathcal{A}$.

Proposition

U a compact semigroup, $V \subseteq U$ a compact subsemigroup, $H \subseteq V$ a compact two-sided ideal in V. Assume Λ acts on U by continuous endomorphisms so that V is L-invariant.
Then there exists a homomorphism $(f, g)\colon \mathcal{A} \to U_\Lambda$ with $f(\bullet) \in H$.

Partial semigroup:

$S = (L \cup \Gamma)$-words and variable $(L \cup \Gamma)$-words
$T = L$-words and variable L-words
$D =$ variable L-words

Note: $D \subseteq T \subseteq S$, D a two-sided ideal in T, T a subsemigroup of S

Action of Λ on S:

$$\lambda(x) = \begin{cases} x, & \text{if } x \text{ is a } (L \cup \Gamma)\text{-word or } \lambda = e; \\ x[\lambda], & \text{if } x \text{ is a variable } (L \cup \Gamma)\text{-word and } \lambda \in L \cup \Gamma. \end{cases}$$

Then $H = \gamma D$ is a compact two-sided ideal in $V = \gamma T$, which is a subsemigroup of $U = \gamma S$.

Note that V is L-invariant.

So by the last theorem and the corollary:

given $r > 0$, there is a basic sequence (x_n) in D such that the color of

$$\lambda_0(x_{n_0}) + \cdots + \lambda_l(x_{n_l}) = x_{n_0}[\lambda_0] + \cdots + x_{n_l}[\lambda_l]$$

depends only on

$$\lambda_0(\bullet) \wedge \cdots \wedge \lambda_l(\bullet) \in A$$

as long as

$$\lambda_0(\bullet) \wedge \cdots \wedge \lambda_l(\bullet) \in \Lambda_{<r}(\bullet).$$

Each finite set of types is included in $\Lambda_{<r}(\bullet)$ for some r.

A sketch of an application— the Hales–Jewett theorem for left-variable words

Monoid Λ:

$\Lambda = L \cup \{e\}$ with $e \notin L$, with multiplication

$$\lambda_0 \cdot \lambda_1 = \begin{cases} \lambda_0, & \text{if } \lambda_1 = e; \\ \lambda_1, & \text{if } \lambda_1 \in L, \end{cases}$$

is a monoid with the identity element e.

Semigroup A:

$A = \{0, 1\}$ with $i \wedge j = \min(i, j)$.

Assignment $\Lambda \to A$:

For $\lambda \in \Lambda$, let $\lambda(\bullet) \in A$ be

$$\lambda(\bullet) = \begin{cases} 0, & \text{if } \lambda = e; \\ 1, & \text{if } \lambda \neq e. \end{cases}$$

Proposition (S.)

U a compact semigroup, H a compact two-sided ideal in U, $G \subseteq H$ a right ideal. Assume Λ acts on U by continuous endomorphisms.
Then there exist

$$u \in H,\ \text{a homomorphism } g\colon A \to U,\ \text{and } v \in G$$

such that

$$\lambda(u) = g(\lambda(\bullet)) \ \text{ and } \ uv = u.$$

Some questions

Λ a monoid

$\widehat{\Lambda}$ the semigroup generated freely by Λ subject to the relations

$$e \wedge \lambda = \lambda \wedge e = e.$$

U a compact semigroup on which Λ acts by continuous endomorphisms

H a compact two sided ideal in U

Question. For what Λ, does there exist

$$u \in H \text{ and a homomorphism } g\colon \widehat{\Lambda} \to U$$

such that for each $\lambda \in \Lambda$

$$\lambda(u) = g(\lambda)?$$

The question amounts to asking for what Λ there exists a homomorphism

$$(f, g)\colon \mathcal{A} \to U_\Lambda \text{ with } f(\bullet) \in H,$$

where $\mathcal{A}$ is the point based Λ-semigroup over $\widehat{\Lambda}$ given by $\lambda(\bullet) = \lambda \in \widehat{\Lambda}$.

Fix $M > 0$. Let E be the monoid with composition of all non-decreasing functions $s\colon M \to M$ such that

$$s(0) = 0 \ \text{ and } \ s(i+1) \leq s(i) + 1, \text{ for all } i < M - 1.$$

Question. Does the question above have positive answer for $\Lambda = E$?

E as above acting on a compact semigroup U with a compact two-sided ideal H

$A = M$ with $i \wedge j = \min(i, j)$

For $s \in E$, let

$$s(\bullet) = M - (1 + \max s) \in A.$$

Question. Do there exist

$$u \in H \text{ and a homomorphism } g \colon A \to U$$

such that

$$s(u) = g(s(\bullet))?$$

A positive answer to this question implies the generalized Gowers' theorem.