COUNTABLE ABELIAN GROUP ACTIONS

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1. Introduction

In these lectures, we will be interested in actions of countable abelian groups on standard Borel spaces. Recall that if G is a countable group and X is a standard Borel space, an action $a : G \times X \to X$ is called *Borel* if for each $g \in G$, the map $x \to a(g, x)$ is Borel. When the action a is understood, we will call X a *Borel G-space* and write $g \cdot x$ or gx for $a(g, x)$.

If X is a Borel G-space, then E_G is defined to be the equivalence relation on X where xE_Gy iff there is $g \in G$ with $gx = gy$. By Lusin-Novikov, $E_G \subseteq X \times X$ is Borel. The Feldman-Moore theorem asserts that this is entirely general: if E is a Borel equivalence relation on X with countable classes, then there is a countable group G and a Borel G-action on X with $E_G = E$. We will call such an E a *countable* equivalence relation. If E is an equivalence relation with finite classes, we similarly call E *finite*.

A countable Borel equivalence relation E on X is said to be *hyperfinite* if there is a sequence $\langle E_n : n < \omega \rangle$ of finite Borel equivalence relations with $E_n \subseteq E_{n+1}$ and $E = \bigcup_n E_n$. The demand that the finite equivalence relations be increasing is essential, as the proof of the Feldman-Moore theorem shows that any countable Borel equivalence relation E is the union of Borel equivalence relations with classes of size at most two. A fundamental question is this: for which groups G do we have E_G hyperfinite for every Borel G-space? The first non-trivial result in this direction is the following.

Theorem 1.1 (Slaman-Steel [SS]). *Let X be a Borel $\mathbb{Z}$-space. Then $E_{\mathbb{Z}}$ is hyperfinite.*

The key lemma needed to prove this is the "marker lemma," a method of finding Borel subsets of X or "markers" that meet every orbit. Variations of the marker lemma technique have proven key in obtaining hyperfiniteness results, including the "clopen marker lemma" of Gao and Jackson that we'll prove in the next section. Repeated and careful use of the clopen marker lemma is used to construct "orthogonal marker regions," a key technical device used in the proof of hyperfiniteness for abelian group actions.

Theorem 1.2 (Gao-Jackson [GJ]). *Let X be a Borel G-space for G an abelian group. Then E_G is hyperfinite.*

Much of their proof focuses on the free part of the shift action on 2^G, where given $x \in 2^G$ and $g \in G$, we have $g \cdot x(h) = x(g^{-1}h)$. An element $x \in 2^G$ is *free* if for every $g \in G \setminus \{1\}$, we have $g \cdot x \neq x$, and the *free part* of the shift 2^G, denoted $F(G)$, is the collection of free elements. Note that $F(G)$ is an invariant Borel subset of 2^G, so is a Borel G-space in its own right. One feature of the proof that orbit equivalence on $F(\mathbb{Z}^{<\omega})$ is hyperfinite is a new proof of the fact that orbit equivalence on $F(\mathbb{Z}^2)$ is hyperfinite, a result originally proven by Weiss.

One can also view $F(\mathbb{Z}^2)$ as a Borel graph, where a *Borel graph* is a standard Borel space X with a Borel graph structure $\Gamma \subseteq X^2$. In the case of $F(\mathbb{Z}^2)$ we declare that $(x, y) \in \Gamma$ iff there is $g \in \{(0,1),(0,-1),(1,0),(-1,0)\}$ with $g \cdot x = y$. Given a graph (X,Γ), the *chromatic number* $\chi(X)$ of X is the smallest cardinal n so that there is a *graph coloring* $c : X \to n$, a coloring of X where $(x,y) \in \Gamma$ implies that $c(x) \neq c(y)$. If (X,Γ) is a Borel graph, the *Borel chromatic number* $\chi_B(X)$ is the smallest $n \leq \omega$ so that there is a Borel graph coloring $c : X \to n$. If there is no such coloring, we say X has uncountable Borel chromatic number. Using the axiom of choice, it is easy to see that $\chi(F(\mathbb{Z}^2)) = 2$. An argument using ergodic theory shows that $\chi_B(F(\mathbb{Z}^2)) \geq 3$, but the precise value was an open question for some time.

Theorem 1.3 (Gao-Jackson-Seward???)**.** $\chi_B(F(\mathbb{Z}^2)) = 3$.

If (X,Γ) is a Borel graph and τ is a compatible zero-dimensional Polish topology on X, the *continuous chromatic number* $\chi_c(X)$ is the least $n \leq \omega$ so that there is a continuous graph coloring $c : X \to n$. Notice that $F(\mathbb{Z}^2)$ inherits a compatible zero-dimensional Polish topology as a G_δ subset of $2^{\mathbb{Z}^2}$ with its product topology. Hence it makes sense to ask about the continuous chromatic number of $F(\mathbb{Z}^2)$.

Theorem 1.4 (Gao-Jackson-Seward???)**.** $\chi_c(F(\mathbb{Z}^2)) = 4$.

These lecture notes are organized as follows. In section 2 we prove the clopen marker lemma and use it to build equivalence relations whose classes are "almost squares." In section 3 we use the almost square equivalence relations of section 2 to produce a continuous 4-coloring of $(F(\mathbb{Z}^2),\Gamma)$, providing the upper bound for Theorem 1.4. In section 4, we mimic the methods of section 2 to construct "orthogonal marker regions." In section 5, we use the orthogonal marker regions from section 4 to produce a Borel 3-coloring of $(F(\mathbb{Z}^2,\Gamma)$, proving Theorem 1.3. And in section 6, we discuss "hyperaperiodic points" of $F(\mathbb{Z}^2)$ and use them to rule out the existence of a continuous 3-coloring, providing the lower bound for Theorem 1.4.

As the only group discussed in the remainder of these notes is $\mathbb{Z}^2$, we often write G in place of $\mathbb{Z}^2$.

2. Clopen markers and regions

While the markers produced in Slaman and Steel's proof of Theorem 1.1 are just Borel, we will need more precise control over the complexity of markers to produce a continuous 4-coloring of $F(\mathbb{Z}^2)$. We will produce various marker sets $C \subseteq F(\mathbb{Z}^2)$ which are clopen in the topology $F(\mathbb{Z}^2)$ inherits from $2^{\mathbb{Z}^2}$. If $s \in 2^{<\mathbb{Z}^2}$, then set $N_s = \{x \in 2^{\mathbb{Z}^2} : x|_{\mathrm{dom}(s)} = s\}$. Then a basis for the topology on $F(\mathbb{Z}^2)$ is given by $\{N_s \cap F(\mathbb{Z}^2) : s \in 2^{<\mathbb{Z}^2}\}$.

Endow $\mathbb{Z}^2$ with the maximum metric, i.e. $d((a,b),(c,d)) = \max(|a-b|,|c-d|)$. If $g \in \mathbb{Z}^2$, we write $|g|$ for $d(g,(0,0))$. We define a (possibly infinite valued) metric on $F(\mathbb{Z}^2)$ as follows: set $\rho(x,y) = d$ if there is $g \in \mathbb{Z}^2$ with $|g| = d$ and $g \cdot x = y$. If x and y are not in the same orbit, set $\rho(x,y) = \infty$.

The following lemma is from [GJ], where is was proven for $F(\mathbb{Z}^n)$. While we only state and prove the lemma for $n = 2$, the proof easily generalizes.

Lemma 2.1 (Clopen marker lemma, [GJ] Lemma 2.1)**.** *Let $d \geq 1$ be an integer. There is a clopen $C \subseteq F(\mathbb{Z}^2)$ so that*

(1) if $x,y \in C$, then $\rho(x,y) > d$, and

(2) if $x \in F(\mathbb{Z}^2)$, then there is $g \in \mathbb{Z}^2$ with $|g| \leq d$ and $g \cdot x \in C$.

Proof. Let $S \subseteq 2^{<\mathbb{Z}^2}$ be those s with $N_s \cap g \cdot N_s = \emptyset$ for every $g \in \mathbb{Z}^2 \setminus \{(0,0)\}$ with $|g| \leq d$. Enumerate $S = \{s_n : n < \omega\}$.

Define sets $\langle C_i : i < \omega \rangle$ as follows. We set $C_0 = N_{s_0}$. If $C_0, ..., C_{i-1}$ are determined, we set

$$C_i = C_{i-1} \cup \left(N_{s_i} \setminus \left(\bigcup_{|g| \leq d} g \cdot C_{i-1} \right) \right).$$

We then set $C = \left(\bigcup_{i<\omega} C_i \right) \cap F(\mathbb{Z}^2)$.

Say that $x \neq y \in C$. Then $x, y \in C_i$ for some least $i < \omega$. If both x and y are in N_{s_i}, then since $s_i \in S$ we must have $\rho(x,y) > d$. If $x \in C_{i-1}$ and $y \in C_i \setminus C_{i-1}$, then by the definition of C_i, we have $\rho(x,y) > d$. This proves item (1).

Now suppose $x \in F(\mathbb{Z}^2)$. Since $g \cdot x \neq x$ for any $g \in \mathbb{Z} \setminus \{(0,0)\}$, then in particular there is some $s_i \in S$ with $x \in N_{s_i}$. Let $i < \omega$ be the least such i. If $x \in C_i$, we are done. If not, then there is some $g \in \mathbb{Z}^2$ with $|g| \leq d$ so that $x \in g \cdot C_{i-1}$, so then $g^{-1} \cdot x \in C_{i-1} \subseteq C$. This proves item (2).

Lastly, we check that C is clopen. Certainly each C_i is clopen in $2^{\mathbb{Z}^2}$, so C is open. Note that by item (2), we have $F(\mathbb{Z}^2) = \bigcup_{|g| \leq d} g \cdot C$. But by item (1), this union is disjoint, so $F(\mathbb{Z}^2) \setminus S$ is also open. □

We will call the set C constructed in Lemma 2.1 a (d,d)-marker, where the first d refers to item (1) and the second to item (2) in the statement of the lemma. Once we have clopen (d,d)-markers, we can then use them to build other clopen markers with different properties.

Lemma 2.2 ([GJ] Lemma 2.2)**.** *Let C_0 be a clopen (d,d)-marker, and let $D > 2d$. Then there is $C_1 \subseteq C_0$ a clopen $(D-2d, D+d)$-marker.*

Proof. Use Lemma 2.1 to find $C \subseteq F(\mathbb{Z}^2)$ a clopen (D,D)-marker. We use C to produce C_1 as follows: since C_0 is a (d,d)-marker, each $x \in C$ is within distance d of some $y \in C_0$. Let $f : C \to C_0$ choose the lexicographically least such y (i.e. for some $g \in \mathbb{Z}^2$ with $|g| \leq d$, we have $gx \in C_0$, so choose the lexicographically least such g). Then set $C_1 = \mathrm{Im}(f)$. Now it is easy to verify that C_1 is a clopen $(D-2d, D+d)$-marker. □

We will use these two lemmas to break apart $F(\mathbb{Z}^2)$ into definable *d-almost squares*, squares whose edge lengths are in $\{d, d+1\}$ for some d. To make the notion of definability precise, suppose $R \subseteq E_G$ is an equivalence relation on $F(\mathbb{Z}^2)$. We say that R is *relatively clopen* if the set $\{(g,x) \in G \times F(\mathbb{Z}^2) : (x, gx) \in E\}$ is clopen as a subspace of $G \times F(\mathbb{Z}^2)$. This is the same as requiring that for every $g \in G$, the set $\{x \in F(\mathbb{Z}^2) : (x, gx) \in E\}$ is clopen as a subspace of $F(\mathbb{Z}^2)$. Our goal will be to prove the following theorem.

Theorem 2.3. *Let $d > 0$ be an integer. There is an equivalence relation $R_d \subseteq E_G$ which is relatively clopen and so that each equivalence class is a d-almost square.*

We will need to introduce some terminology to discuss subsets of $\mathbb{Z}^2$. Let $S \subseteq \mathbb{Z}^2$ be finite. Let $e_0 = (1,0)$ and $e_1 = (0,1)$. An *e_i-face* of S is a set of the form $\{(s_0, s_1) \in S : s_i = c \text{ and } e_i \cdot s \notin S\}$ for some $c \in \mathbb{Z}$. A *$-e_i$-face* is defined similarly, and an *i-face* is a face which is either an e_i-face or a $-e_i$-face. To each face we can associate a subset of G:

if F is an e_i-face, set $G_F = \{g = (g_0, g_1) \in G : g_i > 0\}$, and if F is a $-e_i$-face, we set $G_F = \{g = (g_0, g_1) \in G : g_i < 0\}$. If $F \subseteq S$ is a face, we can write $S = S_F^+ \sqcup S_F^-$, where $S_F^+ = \{s \in S : \exists x \in F \exists g \in G(g \cdot x = s \text{ and } g \in G_F)\}$. We call two faces *parallel* if they are both i-faces for some i. Given two i-faces, the *perpendicular distance* is the absolute value difference between the two constants c used in defining the two faces.

The key observation is that these definitions make sense for S a finite subset of some $F(\mathbb{Z}^2)$ equivalence class. Of course, we cannot specify the numerical value of one of the coordinates, but we can define an e_i-face of S to be a maximal subset F of S so that $x \in F$ implies $e_i \cdot x \notin F$ and that $x, y \in F$ implies that $gx = y$ for some $g = (g_0, g_1)$ with $g_i = 0$. Similarly for $-e_i$-faces. Given two i-faces F and F', we define the perpendicular distance to be the unique $d \in \mathbb{N}$ so that whenever $x \in F$, $y \in F'$, and $gx = y$ for $g = (g_0, g_1)$, we have $|g_i| = d$.

Typically, the subsets S for which we will be using the above terminology will arise as equivalence classes of some equivalence relation $R \subseteq E_G$. We will sometimes call the R-equivalence classes *R-marker regions*.

Lemma 2.4. *Let $D > 0$ be an integer and let $\Delta \gg D$. Then there is a relatively clopen equivalence relation $R \subseteq E_G$ so that for $S \subseteq F(\mathbb{Z}^2)$ an R-marker region, any pair of distinct parallel faces has perpendicular distance at least D, but less than Δ.*

Proof. Let $\Delta_2 \gg \Delta_1 \gg D$. Using Lemmas 2.1 and 2.2, let $M_1 \subseteq F(\mathbb{Z}^2)$ be a (Δ_1, Δ_1)-marker set, and let $M_2 \subseteq M_1$ be a $(\Delta_2 - 2\Delta_1, \Delta_2 + \Delta_1)$-marker set. Let $g_0, ..., g_{k-1}$, $g_0 = (0, 0)$, enumerate the elements of $\mathbb{Z}^2$ with $|g| \leq \Delta_2 + \Delta_1$. We can use M_2 and the g_i to partition M_1 into k pieces as follows. Set $A_0 = g_0 \cdot M_2 = M_2$. If $A_0, ..., A_{i-1}$ have been determined, set $A_i = ((g_i \cdot M_2) \cap M_1) \setminus \bigcup_{j<i} A_j$. By the second marker property of M_2, we have $M_1 = \bigcup_{i<k} A_k$, and by the first marker property, we have $\rho(x, y) > \Delta_2 - 2\Delta_1$ whenever $x \neq y \in A_i$ for $i < k$.

We now use M_1 to define an equivalence relation R_0, then use the partition $M_1 = \bigcup_{i<k} A_i$ to adjust R_0 to the desired R. Given $x \in F(\mathbb{Z}^2)$, write $B_x := \{y \in F(\mathbb{Z}^2) : \rho(x, y) \leq \Delta_1\}$. We set $R_0 = \{(x, y) : \forall z \in M_1 (x \in B_z \Leftrightarrow y \in B_z)\}$. The equivalence relation R will have a very similar description, but with the B_z replaced by slightly different B'_z. If $z \in A_0$, set $B'_z = B_z$. Suppose $z \in A_j$. We set B'_z to be some rectangle containing B_z so that

(1) If F and F' are the e_i-faces of B_z and B'_z, respectively, their perpendicular distance is at most $\Delta_1/10$. Similar for $-e_i$-faces.
(2) If $y \in A_\ell$ for some $\ell < j$ and $\rho(y, z) < 3\Delta_1$, then any parallel faces of B'_y and B'_z have perpendicular distance at least D.

Notice first that if $w \neq z \in A_j$, then since $\rho(w, z) > \Delta_2 - 2\Delta_1 \gg 3\Delta_1$, we can adjust B_z without worrying about how we adjust B_w. Also, since M_1 is a (Δ_1, Δ_1)-marker set, we know that for any $z \in A_j$, the number of y we have to worry about in item (2) is at most 9, meaning that for each face of B_z, the number of parallel faces we need to worry about is at most 18. So long as $\Delta_1/10 \gg 36D$, we can successfully adjust the face outward to a face of B'_z. To make the algorithm explicit, choose the least such outward adjustment that works for each face.

As promised, we now set $R = \{(x, y) : \forall z \in M_1 (x \in B'_z \Leftrightarrow y \in B'_z)\}$. First note that each R-equivalence class is contained in some B'_z, so any parallel faces have perpendicular distance at most $3\Delta_1 \approx \Delta$. Then note that if $y, z \in M_1$ and $B'_y \cap B'_z \neq \emptyset$, then $\rho(y, z) < 3\Delta_1$. In particular, distinct parallel faces F and F' of any R-equivalence class S either are subsets of the e_i and $-e_i$ faces of some B'_z or subsets of faces of some distinct B'_z and B'_y with

$\rho(y, z) < 3\Delta_1$. It follows that F and F' have perpendicular distance at least D. Lastly, since we used clopen marker sets and an explicit, bounded algorithm to cook up R, R is relatively clopen. □

Let us remark now that the precise numerical estimates aren't particularly important, but it is useful to have some numbers floating around when following these proofs. We will continue to plug in numbers that "work" for the proofs at hand; readers might find it helpful to plug in their own numbers.

Lemma 2.5. *Let $0 < D < \Delta$ be integers, and let $R_0 \subseteq E_G$ be a relatively clopen equivalence relation so that each R_0-marker region has distinct parallel faces at perpendicular distance between D and Δ. Then there is a relatively clopen equivalence relation $R \subseteq R_0$ so that each R-marker region is a rectangle with sides of length between D and Δ.*

Proof. Let S be an R_0-marker region, and let $F_0, \dots, F_{k_S - 1}$ enumerate the faces of S. Note that $k_S < \Delta^2$. We set $R = \{(x, y) : xR_0y \text{ and } \forall i < k_S(x \in S^+_{F_i} \Leftrightarrow y \in S^+_{F_i})\}$. By the assumption on R_0, each R-marker region is a rectangle with sides of length between D and Δ. And since R_0 was relatively clopen and R was produced using an explicit, bounded algorithm, R will also be relatively clopen. □

We will have proven Theorem 2.3 once we prove the following simple lemma.

Lemma 2.6. *Let $0 < d$ be an integer, let $0 < D < \Delta$ be integers with $D > d^2$, and let $R_0 \subseteq E_G$ be a relatively clopen equivalence relation so that each R_0-marker region is a rectangle with side lengths between D and Δ. Then there is a relatively clopen equivalence relation $R \subseteq R_0$ so that each R-marker region is a d-almost square.*

Proof. Let $S \subseteq F(\mathbb{Z}^2)$ be an R_0-marker region, and let $x \in S$ be the lower left corner. Let d_0 be least so that $(d_0, 0) \cdot x \notin S$, and let d_1 be least with $(0, d_1) \cdot x \notin S$. Then write $d_i = m_i d + n_i(d + 1)$, breaking apart the sides of S accordingly. □

Remark. We needed the upper bound Δ to ensure that our search for d_0 and d_1 was bounded. Otherwise we would obtain a relatively *open* equivalence relation.

3. A continuous four-coloring

In this short section, we use the almost-square marker regions constructed in the previous section to exhibit a continuous four-coloring of the graph $(F(\mathbb{Z}^2), \Gamma)$. Let $R \subseteq E_G$ be a relatively clopen equivalence relation whose classes are d-almost squares for some d yet to be specified. We start be defining a continuous, but incorrect 2-coloring $c : F(\mathbb{Z}^2) \to 2$ by starting at the lower left corner of each region, coloring it 0, then 2-coloring the region by proceeding upwards and rightwards from the corner. Of course, the issue is that adjacent elements beween different regions may be assigned the same color. Let $E = \{(x, y) \in \Gamma : c(x) = c(y)\}$.

Let us now show that any E-connected component has size at most 4 so long as d is at least 10 (our remark from earlier is in effect: the precise bound 10 isn't important, but it works). First note that it is impossible for any $x_0, x_1, x_2 \in F(\mathbb{Z}^2)$ with $x_{j+1} = e_i \cdot x_j$ for some e_i to have $(x_j, x_{j+1}) \in E$. Otherwise x_1 would be in both the e_i-face and the $-e_i$-face of its R-marker region, which is impossible since d is large enough. In particular, no $x \in F(\mathbb{Z}^2)$ is adjacent to 3 points. It follows that each connected E-component is an E-path. So towards a contradiction let $y_0, ..., y_4$ be an E-path of distinct points. Without loss of generality $y_1 = e_0 y_0$ and $y_2 = e_1 y_1$. Suppose $y_3 = -e_0 y_2$. Then we must have $(y_0, y_3) \in E$, so $y_0, ..., y_3$ forms a 4-cycle, and y_4 cannot be in the component. If $y_3 = e_0 y_2$, then $y_4 = \pm e_1 y_3$. If $y_4 = -e_1 y_3$, then we must have $(y_1, y_4) \in E$, making y_1 E-adjacent to 3 points, a contradiction. If $y_4 = e_1 y_3$, note first that $-e_1 y_3$ must be in a different R-marker region from either y_1 or y_3. But then either y_1 is in both the e_0 and $-e_0$-faces of its R-marker region, or y_3 is in both the e_1 and $-e_1$-faces of its R-marker regions. Either of these is a contradiction.

It is worth noting that [GJ] proves the analog of this result in n dimensions for $n \geq 2$, but for $n = 2$ this hands-on proof seems more illuminating.

Our goal is to correct the 2-coloring c to a 4-coloring $\gamma : F(\mathbb{Z}^2) \to 4$. Let

$$B_1 = \{x \in F(\mathbb{Z}^2) : \exists i \in 2((e_i \cdot x, x) \notin R \vee (-e_i \cdot x, x) \notin R)\}.$$

This set consists of the "boundaries" of the R-marker regions. Also set $C_i = c^{-1}(\{i\})$ for $i < 2$. Notice that E-connected components lie entirely within $C_0 \cap B_1$ or $C_1 \cap B_1$. Also notice that no element of $C_i \setminus B_1$ is Γ-adjacent to any element of $C_i \cap B_1$.

Let $\prec$ be the lexicographic order on $\mathbb{Z}^2$, i.e. $(a, b) \prec (c, d)$ if either $a < b$ or both $a = b$ and $c < d$. We can also view $\prec$ as giving a lexicographic ordering of each E_G-equivalence class. In particular, if $S \subseteq F(\mathbb{Z}^2)$ is a finite subset of an E_G-class, then let $c_S : S \to 2$ be defined as follows: let s' be the $\prec$-least element of S, and if $g = (g_0, g_1)$ is the unique group element with $g \cdot s' = s$, set $c_S(s) = g_0 + g_1 (\mathrm{mod} 2)$. If any pairs from S are Γ-adjacent, then c_S will give the elements different colors.

Now we can define γ in stages. For $x \in C_0 \setminus B_1$, set $\gamma(x) = c(x) = 0$. For $x \in C_0 \cap B_1$, let $[x]$ denote the E-connected component of x. On $[x]$, set $\gamma = c_{[x]}$. Now repeat these steps for $C_1 \setminus B_1$ and $C_1 \cap B_1$, using the colors 2 and 3 instead of 0 and 1. Then γ is a proper graph-coloring of $(F(\mathbb{Z}^2), \Gamma)$; since $\gamma^{-1}(j)$ is clopen for each $j \in 4$, we see that γ is continuous as desired.

4. Orthogonal marker regions

Suppose that $R, R' \subseteq E_G$ are both equivalence relations whose classes are rectangles, and let $s > 0$ be an integer. We call R and R' *s-orthogonal* if given F and F' parallel faces of R and R' respectively and points $x \in F$ and $x' \in F'$, we have $\rho(x, x') > s$. As a useful shorthand, if $F, F' \subseteq F(\mathbb{Z}^2)$, set $\rho(F, F') = \min\{\rho(x, x') : x \in F, x' \in F'\}$. Now suppose $R_d \subseteq E_G$ is a relatively clopen equivalence relation whose classes are d-almost squares. Our first goal in this section is to construct a relatively clopen $R \subseteq E_G$ with rectangular classes so that R_d and R are orthogonal to some specified degree. Our proof will mimic the proof of Theorem 2.3 via the proofs of Lemmas 2.4, 2.5, and 2.6. Here is the precise statement.

Theorem 4.1. *Let $R_d \subseteq E_G$ be a relatively clopen equivalence relation whose classes are d-almost squares. Then there is a relatively clopen equivalence relation $R \subseteq E_G$ whose classes are rectangles with side lengths between $9d$ and $12d$ which is Cd-orthogonal to R_d.*

Here C will be something like 10^{-10}, so we implicitly assume throughout that d is large enough for this to make sense. Once again, we abide by our running convention that we choose numbers that "work" for the purpose at hand.

Proof. We start by mimicking the proof of Lemma 2.4. Set $D = 200d$, $\Delta_1 = 400D$, and let $\Delta_2 \gg \Delta_1$. Let $M_1 \subseteq F(\mathbb{Z}^2)$ be a (Δ_1, Δ_1)-marker set, and let $M_2 \subseteq M_1$ be a $(\Delta_2 - 2\Delta_1, \Delta_2 + \Delta_1)$-marker set. Once again, we use M_2 to partition $M_1 = \bigsqcup_{i<k} A_i$. As before, set $B_x = \{y \in F(\mathbb{Z}^2) : \rho(x, y) \leq \Delta_1\}$, and set $R_0 = \{(x, y) : \forall z \in M_1(x \in B_z \Leftrightarrow y \in B_z)\}$. By replacing B_z with B'_z, we will obtain an equivalence relation R_1 so that parallel faces of the same R_1-class have distance between D and Δ, where $\Delta \approx 3\Delta_1$.

Here is where we encounter the first major difference between the proof of Lemma 2.4 and this proof: not only do we have to worry about the distance between parallel B'_y and B'_z faces, we also need to worry about the faces of R_d. For each $z \in M_1$, we set B'_z to be some rectangle containing B_z so that

(1) If F and F' are the e_i-faces of B_z and B'_z, respectively, then their perpendicular distance is at most $\Delta_1/10$. Similar for $-e_i$-faces.
(2) If $y, z \in M_1$ and $\rho(y, z) < 3\Delta_1$, then any parallel faces of B'_y and B'_z have perpendicular distance at least D.
(3) If F is a face of some R_d-class and F' is a parallel B'_z face with $\rho(F, F') < 3\Delta_1$, then F and F' have perpendicular distance at least Cd.

This is done by induction on $i < k$, where $z \in A_i$. For $i = 0$, let's consider the number of R_d-faces we need to worry about when adjusting the e_i-face of B_z to B'_z. It will be enough to consider R_d-classes contained in $C_z := \{x \in F(\mathbb{Z}^2) : \rho(x, z) \leq 5\Delta_1\}$. There are at most $100\Delta_1^2/d^2$ such classes, so at most $200\Delta_1^2/d^2 = 128 \cdot 10^9$. So long as $\Delta_1/10 = 8000d > 64 \cdot 10^9 \cdot Cd$, we can perform the adjustment. For $i > 0$, the reasoning is similar, also taking into account the previously adjusted B'_y for $y \in A_j$ for some $j < i$.

Upon completing the adjustments and obtaining the equivalence relation R_1, we now use Lemma 2.5 to obtain an equivalence relation $R_2 \subseteq R_1$ whose classes are rectangles with side lengths between D and Δ. Notice that by item (3), R_2 is still Cd-orthogonal to R_d. We now mimic the proof of Lemma 2.6 to obtain R. Say S is an R_2-class which is a $s \times t$-rectangle for some $D \leq s, t \leq \Delta$. First we write $s = s_0 + \cdots + s_\ell$ and $t = t_0 + \cdots + t_m$ where s_i and t_j are each between $10d$ and $11d$. We divide S into the corresponding $(\ell+1)(m+1)$ rectangles. Then we adjust the locations of these divisions by at most d to maintain orthogonality to R_d. As an example, consider s_0; we want to enlarge or shrink s_0 by at most d. Call the lower left corner of S $(0, 0)$. Any R_d-face we need to consider when adjusting s_0 will be contained in an R_d-class contained in the rectangle $[s_0 - 3d, s_0 + 3d] \times [-t/6, 7t/6]$, which has size at most $6d \times 4\Delta_1$. Hence there are at most $2 \cdot 10^7$ faces we need to consider, and the small size of C guarantees that we can perform the needed adjustment. The other adjustments are similar. ☐

It will be handy to construct relatively clopen equivalence relations $R \subseteq E_G$ which are orthogonal to several equivalence relations we might be considering. Let $R \subseteq E_G$ be a finite equivalence relation. Call R *bounded* if there is $n < \omega$ so that for every $(x, y) \in R$, there is $g \in G$ with $|g| < n$ and $gx = y$. We call R *decomposable* into rectangles with a certain property (e.g. having certain side lengths) if each R-class is a disjoint union of rectangles with the property. Note that if R is bounded and decomposable, then we can definably find $R' \subseteq R$ an equivalence relation whose classes are rectangles with the desired property. In particular, if R is relatively clopen, then R' will be as well.

Theorem 4.2. *Let $R_1, ..., R_k \subseteq E_G$ be bounded relatively clopen equivalence relations. Assume that each R_i is decomposable to rectangles whose side lengths are between d and $12d$.*

Furthermore, assume that for every ball B of radius $4 \cdot 10^5 \cdot d$, there are at most b indices i so that some face from an R_i-class meets B. Then we can find a relatively clopen $R \subseteq E_G$ whose classes are rectangles with side lengths between $9d$ and $12d$ and which is Cd/b-orthogonal to each R_i for $i < k$.

Proof. Let $R'_i \subseteq R_i$ be relatively clopen equivalence relations whose classes are rectangles with side lengths between d and $12d$. Now run the proof of Theorem 4.1; note that $4 \cdot 10^5 \cdot d = 5\Delta_1$ from that proof, so there are at most b times as many faces to consider. Similarly, there are at most b times as many faces to consider during the final adjustment in the last paragraph of that proof. □

Remark. The boundedness of the R_i is not essential, but it makes the proof conceptually simpler. Also, the various equivalence relations we will be constructing will all be bounded.

Our final goal for this section is to construct a triangular array of relatively clopen bounded equivalence relations $R^i_j \subseteq E_G$ for every $j \leq i < \omega$. To start, fix a rapidly increasing sequence of natural numbers $d_0 \ll d_1 \ll \cdots$. Define R^i_i to be a relatively clopen equivalence relation whose classes are d_i-almost squares as given by Theorem 2.3. Now fix $j < i < \omega$ and assume that R^k_ℓ has been defined whenever $\ell \leq k < i$ and also whenever $k = i$ and $j < \ell \leq i$. We assume that the following holds.

(1) Each $\tilde{R}^k_\ell$ for $\ell < k$ is decomposable to rectangles whose side lengths are between $9d_\ell$ and $12d_\ell$, where given R, we have $\tilde{R}$ obtained by using Lemma 2.5.
(2) If S is an R^k_ℓ-class and $\ell < k$, then there is an $R^k_{\ell+1}$-region S' so that for every face $F \subseteq S$, there is a face $F' \subseteq S$ with $\rho(F, F') < 12d_\ell$.
(3) In any ball B of radius $4 \cdot 10^5 \cdot d_\ell$, there are at most 2 indices k with $\ell < k \leq i-1$ so that some face of an R^k_ℓ-class meets B.
(4) For $\ell \leq k_1 < k_2$, we have that $R^{k_1}_\ell$ and $R^{k_2}_\ell$ are $Cd_\ell/2$-orthogonal, where C is the constant from Theorems 4.1 and 4.2.

We want to define R^i_j so that items (1) through (4) continue to hold. We first apply Theorem 4.2; we set $R_1, ..., R_k$ to be the equivalence relations $R^j_j, ..., R^{i-1}_j$, we set $d = d_j$, and we set $b = 2$. Doing this, we obtain a relatively clopen equivalence relation R whose classes are rectangles with side lengths between $9d_j$ and $12d_j$. Let $f : F(\mathbb{Z}^2) \to F(\mathbb{Z}^2$ be the Borel function so that $f(x)$ is the center of the R-class of x (when side lengths are even, favor the lower left). Now define R^i_j by setting $(x, y) \in R^i_j$ iff $(f(x), f(y)) \in R^i_{j+1}$.

Clearly R^i_j is clopen, bounded, and decomposable into rectangles with side lengths between $9d_j$ and $12d_j$, taking care of item (1). Let $S \subseteq F(\mathbb{Z}^2)$ be an R-class, and let $F \subseteq S$ be a face. We now check items (2) and (4). To check (4), let $S' \subseteq F(\mathbb{Z}^2)$ be an R^k_j-class for some $j \leq k < i$, and $F' \subseteq S'$ a face parallel to F. Notice that F is a union of faces of R-classes all parallel to F. Since R is $Cd_j/2$-orthogonal to R^k_j, we conclude that any member of F is ρ-distance at least $Cd_j/2$ from any member of F'. Therefore R^i_j is $Cd_j/2$-orthogonal. To check (2), let $x \in F$. Without loss of generality suppose $(e_0x, x) \notin R^i_j$. Then $(f(e_0x), f(x)) \notin R^i_{j+1}$. Consider a path consisting of geodesics from $f(e_0x)$ to e_0x, then x, then $f(x)$; at some point, two consecutive points of the path must be in different R^i_{j+1}-classes. Then either of these points is in a face F' which witnesses (2).

We now check item (3). Suppose there were three indices $j < k_1 < k_2 < k_3 = i$ and a ball B of radius at most $4 \cdot 10^5 \cdot d_j$ so that some faces F_1, F_2, F_3 of some respective $R_j^{k_a}$-classes meet B. Now use item (2) to find nearby faces F_1', F_2', F_3' of some respective $R_{j+1}^{k_a}$-classes. Now is where we use the assumption that $d_{j+1} \gg d_j$; we cannot use the fact that (3) holds inductively because we might have $k_1 = j+1$. However, some pair of faces, without loss of generality say F_1' and F_2', are parallel. So since (4) holds inductively, we have $\rho(F_1', F_2') > Cd_{j+1}/2 > 8 \cdot 10^5 \cdot d_j$. This contradiction shows that (3) holds.

5. A Borel three-coloring

We now use the array of equivalence relations $\langle R_j^i : j \leq i < \omega \rangle$ constructed in the last section to produce a Borel 3-coloring of $(F(\mathbb{Z}^2), \Gamma)$. For $n < \omega$, let $R_n = \bigcap_{N \geq n} R_0^N$. Each R_n is a bounded Borel equivalence relation ($2d_n$ works as a bound) with $R_n \subseteq R_{n+1} \subseteq E_G$. We now have an alternate proof to the following theorem of Weiss [W].

Theorem 5.1. *The equivalence relation E_G is hyperfinite.*

Proof. We will show that $\bigcup_n R_n = E_G$. Suppose $(x, y) \in E_G$, but there is no $n < \omega$ with $(x, y) \in R_n$. Then there are infinitely many $n < \omega$ with $(x, y) \notin R_0^n$. Consider a shortest path from x to y; in particular, this path is finite, so there is some point z on the path belonging to a face of some R_0^n-class for infinitely many $n < \omega$. But some pair of these faces must be parallel, contradicting the fact that for any $k_1, k_2 > 0$, we have that $R_0^{k_1}$ and $R_0^{k_2}$ are $Cd_0/2$-orthogonal. □

It will be useful to consider the *shortest path* metric ρ_Γ on E_G-classes: if $g = (g_0, g_1)$ and $gx = y$, set $\rho_\Gamma(x, y) = g_0 + g_1$. Given $S \subseteq F(\mathbb{Z}^2)$ and $n < \omega$, set $\partial^n(S) = \{x \in F(\mathbb{Z}^2) \setminus S :$ $\rho_\Gamma(x, S) \leq n$. For $n = 1$, we write $\partial(S)$ for $\partial^1(S)$. Fix $k < \omega$, and let S be an R_k-class. For $n \geq k$, denote by $[S]_n$ the R_n-saturation of S. Define the *index set* of S by

$$\mathrm{ind}(S) = \{n \geq k : \exists y \in \partial^{60}(S)(y \in [S]_{n+1} \setminus [S]_n)\}.$$

Of course, the precise value 60 is unimportant, but it works. We now have the following very useful fact.

Proposition 5.2. *For any $k < \omega$ and any R_k-class S, we have* $|\mathrm{ind}(S)| \leq 3$.

Proof. For sake of contradiction, suppose we had $y_0, ..., y_3 \in \partial^{60}(S)$ and distinct indices $n_0, ..., n_3$ so that $y_i \in [S]_{n_i+1} \setminus [S]_{n_i}$. It follows that each y_i is a member of some $R_0^{n_i}$-face. Since the n_i are distinct, we may assume that $k < n_1 < n_2 < n_3$. Now we repeatedly use property (2) of the triangular array $\langle R_j^i : j \leq i < \omega \rangle$. For each $i \in \{1, 2, 3\}$, find a face F_i of some $R_{k+1}^{n_i}$-class with $\rho(F_i, y_i) \leq 12(d_0 + \cdots + d_k)$. But some pair of the F_i are parallel, so by orthogonality must be at distance at least $Cd_{k+1}/2$. Since d_{k+1}. But since S is a k-class, we have $\rho(y_i, y_j) < 2d_k$ for any $i, j \leq 3$. Since $d_{k+1} \gg d_k$, we have a contradiction. □

We will adopt the notation that if S is an R_k-class, then the largest element of $\mathrm{ind}(S)$ is denoted n_2^S, then n_1^S the next largest, and so on. So $\mathrm{ind}(S) = \{n_2^S, n_1^S, n_0^S\}$, $\{n_2^S, n_1^S\}$, or $\{n_S^2\}$ depending on $|\mathrm{ind}(S)|$.

If S is an R_k-class, define $\varphi_S : \partial(S) \to j$ by setting $\varphi_S(y) = i + 1$ iff $y \in [S]_{n_i^S+1} \setminus [S]_{n_i^S}$. Now set

$$\mathrm{int}(S) = \{x \in S : \forall y \in \partial^{60}(S)(\rho_\Gamma(x, y) > 20 \cdot \varphi_S(y))\}.$$

If S is an R_k-class, let $b(S) = \max(\text{ind}(S))$. We call S *maximal* if for any $\ell < \omega$ and R_ℓ-class $S' \supsetneq S$, we have $b(S') > b(S)$. Every class is contained in some maximal class. To see this, note that for S an R_k-class, we have $b(S) \geq k$, so if S is non-maximal, then $[S]_\ell$ is maximal for some $k < \ell \leq b(S)$.

Lemma 5.3. *Say $S \subsetneq S'$ are R_k and R_ℓ-classes, respectively, with S maximal. Let $x \in \text{int}(S)$. If $y \in F(\mathbb{Z}^2)$ with $\rho_\Gamma(x, y) \leq 20$, then $y \in \text{int}(S')$.*

Proof. Note first that $y \in S$. Let $z \in \partial^{60}(S')$. First assume that $z \in \partial^{60}(S) \cap \partial^{60}(S')$. Since S is maximal, then we have $\varphi_S(z) \geq \varphi_{S'}(z)+1$, so $\rho_\Gamma(y, z) \geq \rho_\Gamma(x, z)-20 \geq \varphi_S(z)\cdot 20-20 \geq \varphi_{S'}(z)$. If $z \in \partial^{60}(S') \setminus \partial^{60}(S)$, then $\rho_\Gamma(y, z) \geq 60 \geq 20 \cdot \varphi_{S'}(z)$. □

We are now ready to produce a Borel 3-coloring of $(F(\mathbb{Z}^2), \Gamma)$. Set

$$Y = \bigcup \{\partial^5(\text{int}(S)) : S \text{ a maximal } R_k\text{-class }, k < \omega\}.$$

We claim that both $\Gamma|_Y$ and $\Gamma|_{F(\mathbb{Z}^2)\setminus Y}$ have finite connected components. We saw in Lemma 5.3 that $\Gamma|_Y$ has finite connected components. To see that $\Gamma|_{F(\mathbb{Z}^2)\setminus Y}$ has finite connected components, it is enough to note two things. First, for every $x \in F(\mathbb{Z}^2)$, the ball of radius 60 around x is contained in some R_k class for some $k < \omega$, and second, for every $k < \omega$, every R_k-class is contained in a maximal R_ℓ-class for some $\ell \geq k$. Furthermore, note that any two components of Y are at ρ_Γ-distance greater than 2, and same for components of $F(\mathbb{Z}^2) \setminus Y$.

Now we essentially reprove a lemma of Conley and Miller [CM]. Let $c : Y \to 2$ be a Borel 2-coloring, which exists since Y has finite connected components. Let $Y' = c^{-1}(\{0\})$, and consider $F(\mathbb{Z}^2) \setminus Y'$. We claim that $\Gamma|_{F(\mathbb{Z}^2)\setminus Y'}$ also has finite connected components. To see this, suppose $S \subseteq F(\mathbb{Z}^2) \setminus Y'$ is a connected component. First note that if $y \in S \cap Y$ and $x \in S$ with $(y, x) \in \Gamma$, then $x \notin Y$. If $C \subseteq F(\mathbb{Z}^2) \setminus Y$ is a connected component which meets S, then it follows that C is the only connected component of $F(\mathbb{Z}^2) \setminus Y$ which meets S and that $S \subseteq \{x \in F(\mathbb{Z}^2) : \rho_\Gamma(x, C) \leq 1\}$. Hence S is finite. Now let $\gamma : F(\mathbb{Z}^2) \setminus Y' \to \{1, 2\}$ be a Borel 2-coloring; then $f : F(\mathbb{Z}^2) \to 3$ given by

$$f(x) = \begin{cases} 0 & \text{if } x \in Y' \\ \gamma(x) & \text{if } x \in F(\mathbb{Z}^2) \setminus Y' \end{cases}$$

is the desired Borel 3-coloring.

6. There is no continuous four-coloring

The 3-coloring produced at the end of the last section is not continuous; the finite connected components we have to color are arbitrarily large. In this section, we will show that this feature of the coloring is not a fluke; no continuous 3-coloring of $(F(\mathbb{Z}^2), \Gamma)$ exists. The proof uses two main tools: hyperaperiodic points and a "contour integral" analysis of 3-colorings of Γ.

A point $x \in 2^{\mathbb{Z}^2}$ is *hyperaperiodic* if $\overline{\mathbb{Z}^2 \cdot x} \subseteq F(\mathbb{Z}^2)$. Equivalently, x is hyperaperiodic if for every $g \in \mathbb{Z}^2 \setminus \{(0, 0)\}$, there is some $R < \omega$ so that for every squarel $B \subseteq \mathbb{Z}^2$ of side length R, there is $h \in \mathbb{Z}^2$ with $h, gh \in B$ and $x(h) \neq x(gh)$.

Proposition 6.1. *There is a hyperaperiodic $x \in 2^{\mathbb{Z}^2}$.*

Proof. Enumerate $\mathbb{Z}^2 \setminus \{(0,0)\} = \{g_n : n < \omega\}$ in a way so that $|g_n| \leq n$. We will inductively define $x_n \in 2^{<\mathbb{Z}^2}$ as follows. Let $\mathrm{dom}(x_0)$ be a square of side length 5. For some $h_0 \in \mathrm{dom}(x_0)$ with $g_0 h_0 \in \mathrm{dom}(x_0)$, set $x_0(h_0) = 0$ and $x_0(g_0 h_0) = 1$. Let $B_0 = \{h_0, g_0 h_0\}$. Now assume x_{n-1} and $B_{n-1} \subseteq \mathrm{dom}(x_{n-1})$ have been defined, where $\mathrm{dom}(x_{n-1})$ is a square of side length 5^n, and $|B_{n-1}|/(25)^n < 1/12 = \sum_{k=1}^{\infty} 2/(25)^k$. Set

$$\mathrm{dom}(x_n) = \bigcup_{|a|,|b| \leq 2} (a,b) \cdot \mathrm{dom}(x_{n-1}),$$

and set

$$C = \mathrm{dom}(x_{n-1}) \cup \bigcup_{|a|,|b| \leq 2} (a,b) \cdot B_{n-1}.$$

On C, define $x_n(c,d) = x_{n-1}(a+c, b+d)$ for the unique (a,b) with $|a|, |b| \leq 2$ and $(a+c, b+d) \in \mathrm{dom}(x_{n-1})$. Now find $h_n \in \mathrm{dom}(x_n) \setminus C$ with $g_n h_n \in \mathrm{dom}(x_n) \setminus C$; this is possible since $|C|/25^{n+1} < 3/24$ and since $|g_n| \leq n$. Define $x_n(h_n) = 0$, $x_n(g_n h_n) = 1$, and $x_n(g) = 0$ for any $g \in \mathrm{dom}(x_n)$ with $x_n(g)$ still undefined. Set

$$B_n = \{h_n, g_n h_n\} \cup \bigcup_{|a|,|b| \leq 2} (a,b) \cdot B_{n-1}.$$

We now set $x = \bigcup_n x_n$. ☐

We will want to construct hyperaperiodic points with other desired properties. Luckily, we can easily do this by taking some hyperaperiodic $x \in 2^{\mathbb{Z}^2}$ and "cutting and pasting." To be precise, write $\mathbb{N} \setminus \{0\} = \bigcup_n I_n$, where $I_n = [2^n, 2^{n+1})$. Now let $\{p_n : n < \omega\}$ enumerate a subset of odd primes greater than 3 so that $p_n > 2^n$. Fix a hyperaperiodic $x \in 2^{\mathbb{Z}^2}$. We define $y \in 2^{\mathbb{Z}^2}$ as follows. Set $y(0,\ell) = 0$ for every $\ell \in \mathbb{Z}$. If $k \in I_n$, set $y(k,\ell) = x(k - 2^n, \ell \bmod p_n)$. If $k < 0$, set $y(k,\ell) = y(-k,\ell)$.

To see that y is also hyperaperiodic, fix $g = (g_0, g_1) \in \mathbb{Z}^2$, and set $g' = (g_0, -g_1)$. Let $R_x < \omega$ be large enough so that for every square B of side length R_x, there are $h, h' \in B$ so that $gh, g'h' \in B$ with $x(h) \neq x(gh)$ and $x(h') \neq x(g'h')$. We want to find a corresponding side length R_y to show that y is hyperaperiodic. Any R_y large enough so that every inteval $I \subseteq \mathbb{N}$ of length $R_y/2$ intersects some I_n in an interval of length R_x will work.

We will call a point $y \in 2^{\mathbb{Z}^2}$ constructed as above a *hyperaperiodic point with vertical periodicity.*

We now turn to the second component of the proof, which is an analysis of a certain "contour integral" for 3-colorings of $(F(\mathbb{Z}^2), \Gamma)$. Fix $c : F(\mathbb{Z}^2) \to 3$ be a 3-coloring of Γ. For $(x,y) \in \Gamma$, define

$$\delta_c(x,y) = \begin{cases} 0 & \text{if } c(x) = c(y), \\ 1 & \text{if } c(x) + 1 = c(y) \bmod 3, \\ -1 & \text{if } c(x) - 1 = c(y) \bmod 3. \end{cases}$$

Note that $\delta_c(x,y) = -\delta_c(y,x)$. The key observation is the following analog of the Cauchy integral theorem.

Lemma 6.2. *Let $x, y \in F(\mathbb{Z}^2)$ be the lower left and upper right corners, respectively, of some rectangle R. Let $x = x_0 = w_0$, and write $x_0, x_1, \ldots, x_n = y$ and $w_0, w_1, \ldots, w_n = y$ for* *the paths along the left and top edges of R and the bottom and right edges of R, respectively. Then* $\sum_{i=0}^{n-1} \delta_c(x_i, x_{i+1}) = \sum_{i=0}^{n-1} \delta_c(w_i, w_{i+1})$.

Proof. By breaking apart the rectangle into unit squares and summing, it is enough to prove the claim for R a unit square. This is a simple case-by-case analysis, keeping in mind that c is a proper graph coloring. □

We are now ready to prove that there is no continuous 3-coloring of $(F(\mathbb{Z}^2), \Gamma)$. For sake of contradiction, suppose $c : F(\mathbb{Z}^2) \to 3$ were a continuous graph coloring. Let $y \in F(\mathbb{Z}^2)$ be a hyperaperiodic point with veritcal periodicity. Set $Y = \overline{\mathbb{Z}^2 \cdot y} \subseteq F(\mathbb{Z}^2)$. Then each $c^{-1}(i) \cap Y$ is clopen; it follows that there is some $N < \omega$ so that for any $x, x' \in Y$ with $x|_{[-N,N]^2} = x'|_{[-N,N]^2}$, we have $c(x) = c(x')$. Find $k \in N$ so that $[-N, N] \subseteq I_n - k$ for some $n < \omega$.

Now consider the rectangle with lower left corner $(k, 0) \cdot y$ and upper right corner $(k + 2^n, d) \cdot y$ for $d > 0$ suitably large. Note that $[-N, N] \subseteq I_{n+1} - 2^n - k$, so in particular, we have for any $\ell \in \mathbb{Z}$ that

$$((k, \ell) \cdot y)|_{[-N,N]^2} = ((k, \ell + p_n) \cdot y)|_{[-N,N]^2},$$

and

$$((k + 2^n, \ell) \cdot y)|_{[-N,N]^2} = ((k + 2^n, \ell + p_{n+1}) \cdot y)|_{[-N,N]^2}.$$

As in Lemma 6.2, write $(k, 0) \cdot y = x_0 = w_0$, $(k + 2^n, d) \cdot y = x_m = w_m$, where $x_0, x_1, ..., x_m$ and $w_0, w_1, ..., w_m$ are the paths along the left and top edges of R and the bottom and right edges of R. Consider a sum along the left edge of the form $S = \sum_{i=0}^{p_n \cdot p_{n+1}-1} \delta(x_{\ell+i}, x_{\ell+i+1})$. We must have S a multiple of p_{n+1} since the entries in the sum repeat with period p_n. However, we cannot have $\sum_{i=0}^{p_n-1} \delta(x_{\ell+i}, x_{\ell+i+1}) = \pm p_n$ since $p_n > 3$ is prime. So S is not divisible by p_n. A similar analysis of a sum along the right edge of the form $T = \sum_{i=0}^{p_n \cdot p_{n+1}-1} \delta(w_{r+i}, w_{r+i+1})$ shows that T is divisible by p_n and not divisible by p_{n+1}. In particular $S \neq T$.

By making d suitably large, we can ensure that

$$\left| \left(\sum_{i=0}^{d-1} \delta(x_i, x_{i+1}) \right) - \left(\sum_{i=0}^{d-1} \delta(w_{m-d+i}, w_{m-d+i+1}) \right) \right| \gg 2^{n+1}$$

This makes it impossible for the contour integral around R to have value zero, contradicting Lemma 6.2.

References

[CM] C.T. Conley and B.D. Miller, A bound on measurable chromatic numbers of locally finite Borel graphs, preprint (2014).

[GJ] S. Gao and S. Jackson, Countable abelian group actions and hyperfinite equivalence relations, *Invent. Math.*, **201**, (2015), 309–383.

[SS] T. Slaman and J. Steel, Definable functions on degrees, *Cabal Seminar* 81–85, *Lecture Notes in Mathematics*, **1333**, (1988), 3755.

[W] B. Weiss, Measurable dynamics, Conference in Modern Analysis and Probability (R. Beals et al. eds.), Contemporary Mathematics, **26**, (1984), 395421.