Fluid-Solid Interaction — CNA Working Group — Ian Tice

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§0 - Introduction

We consider a domain $\Omega \subseteq \mathbb{R}^3$, filled with a viscous incompressible fluid and a finite number of rigid solid objects, given via the closed sets $S^i(t)$, $i=1,\dots,n$.

Our aim is to:
1) formulate the eqns of motion
2) compute the energy law and devise a weak form of the equations
3) construct global weak solutions and deal w/ problem of collisions.

Along the way we will make a particular effort to deal with S^i not necessarily smooth, which is useful in dealing w/ collisions.

Refs:
- Feireisl (J. Evolution Eqns, 2003) – global, collisions.
- Gunzburger, Lee, Seregin (J. Math. Fluid Mech., 2000) – local, global w/o collisions
- Sueur (CMP, 2012) – Kato-type analysis

§1 - Classical formulation of the problem

§1.1 - Rigid body kinematics and the continuity eqn.

We assume that at $t=0$ we have n disjoint bodies $S^i(0) \subset \Omega$, $i=1,\dots,n$. To simplify things, for now let's just consider one $S(0) \subset \Omega$. The generalization to n will be obvious.

We will assume the rigid body is given by the set $S(t) \subset \Omega$, and that $S(t)$ is the image of $S(0) \subset \Omega$ under the linear isometry
$$S(0) \ni y \mapsto \eta(y,t) = z(t) + Q(t)(y-y_0) \quad \text{for some } z(t), y_0 \in \Omega$$
and $Q(t) \in SO(3)$ with $Q(0)=I$, $y_0 = z(0)$. At $t=0$ the body has density $\rho_0 : S(0) \to (\rho_-, \infty)$ for some $\rho_- > 0$. At $t>0$ the density $\rho(\cdot,t) : S(t) \to (\rho_-,\infty)$ is given by
$$\boxed{\rho(\eta(y,t),t) = \rho_0(y).}$$

Then $\quad M := \int_{S(t)} \rho(x,t)\,dx \overset{x=\eta(y,t)}{=} \int_{S(0)} \rho_0(y)\,dy$ gives the constant mass of the body. Let's define the center of mass by

$$\bar{x}(t) = \frac{1}{M}\int_{S(t)} \rho(x,t)\,x\,dx.$$

It's convenient then to set $y_0 = \bar{x}(0)$ and $z(t) = \bar{x}(t)$ so that $\eta(y,t) = \bar{x}(t) + Q(t)(y-y_0)$ maps the center of mass at $t=0$ to the c.o.m. @ t:

$$\eta(y_0,t) = \bar{x}(t).$$

Now, the velocity at a point $x = \eta(y,t) \in S(t)$ is then

$$v(x,t) = \frac{d}{dt}\eta(y,t) = \dot{\bar{x}}(t) + \dot{Q}(t)(y-y_0),$$ which we can compute in terms of x,t via $x = \eta = \bar{x} + Q(y-y_0) \Rightarrow$

$$v(x,t) = \dot{\bar{x}}(t) + \dot{Q}(t)\left[Q^{-1}(t)(x-\bar{x}(t))\right]$$
$$= \dot{\bar{x}}(t) + \dot{Q}(t)Q^{T}(t)(x-\bar{x}(t)).$$

Note that $Q(t) \in SO(3) \Rightarrow \dot{Q}Q^T$ is skew-symmetric, so for each $t\geq 0$ $\exists!\ w(t)\in\mathbb{R}^3$ s.t. $\dot{Q}(t)Q^T(t)z = w(t)\wedge z \quad \forall z\in\mathbb{R}^3$. This $w(t)$ is the rotation vector: $w(t)$ points in the direction of the axis of rotation (right hand rule dictates the choice) and $|w(t)|$ gives the angular speed.

Let $\bar{v}(t) = \dot{\bar{x}}(t)$ = velocity of center of mass. Then

$$\boxed{v(x,t) = \bar{v}(t) + w(t)\wedge(x-\bar{x}(t)) \quad \forall x\in S(t).}$$

We will derive the dynamics of $\bar{v}$ and w. Note that one may recover $Q(t)$ from $w(t)$ by solving $\begin{cases}\dot{Q} = A(w)Q, \\ Q(0) = I\end{cases}$ where $A(w)$ is the skew-sym. matrix corresponding to the operator $z \mapsto w\wedge z$. Thus, having solved for $\bar{v}, w$, we can recover $\bar{x}$, Q, and η.

It will be useful later to know that the continuity equation also holds within $S(t)$. So, let's show that with ρ, v defined as above, they satisfy $\partial_t\rho + \operatorname{div}(\rho v) = 0$.

We compute

$$\eta = \bar{x} + Q(y - \bar{x}(0)) \Rightarrow \quad y = \bar{x}(0) + Q^{-1}(t)\,(x - \bar{x}(t))$$

$$\Rightarrow \quad \rho(x,t) = \rho_0\left(\bar{x}(0) + Q^T(t)\,(x - \bar{x}(t))\right)$$

$$\partial_t \rho(x,t) = \nabla\rho_0(\text{---}) \cdot \left[\dot{Q}^T(x-\bar{x}) - Q^T\bar{v}\right]$$

$$\partial_i \rho(x,t) = \partial_j \rho_0(\text{---})\, \partial_i (Q^T x)_j = \partial_j \rho_0\, \partial_i (Q_{\ell j} x_\ell) = \partial_j \rho_0\, Q_{ij}$$

$$\Rightarrow \quad \nabla\rho(x,t) = Q(t)\,\nabla\rho_0(\sim), \quad \text{so}$$

$$\nabla\rho \cdot v = Q\nabla\rho_0 \cdot (\bar{v} + \dot{Q}Q^T(x-\bar{x})). \qquad \text{OTOH,}$$

$$\operatorname{div} v = \operatorname{div}(\bar{x} + w \wedge (x - \bar{x})) = 0.$$

$$\therefore \quad \partial_t\rho + \operatorname{div}(\rho v) = \nabla\rho_0(\text{---}) \cdot \left[Q^T\bar{v} + Q^T\dot{Q}Q^T(x-\bar{x}) + \dot{Q}^T(x-\bar{x}) - Q^T\bar{v}\right]$$

$$= \nabla\rho_0(\sim)\,[0] \quad \text{since} \quad QQ^T = I \Rightarrow \dot{Q}Q^T + Q\dot{Q}^T = 0$$

$$= 0. \qquad \Rightarrow \quad \dot{Q}^T = -Q^T\dot{Q}Q^T$$

Hence ρ satisfies continuity eqn in $S(t)$:

$$\boxed{\partial_t\rho + \operatorname{div}(\rho v) = 0 \quad \text{in } S(t).}$$

§1.2 – The inertia tensor

We define the inertia tensor associated to $S(t)$ as

$$J(t) = \int_{S(t)} \rho(x,t)\left(|x - \bar{x}(t)|^2 I - (x - \bar{x}(t)) \otimes (x - \bar{x}(t))\right) dx.$$

Why? From above, we know the velocity at $x \in S(t)$ is

$v(x,t) = \bar{v}(t) + w(t) \wedge (x - \bar{x}(t))$. Then the total kinetic energy associated with $S(t)$ is

$$K(t) = \int_{S(t)} \frac{1}{2}\rho(x,t)\,|\bar{v}(t) + w(t) \wedge (x - \bar{x}(t))|^2\, dx$$

$$= \int_{S(t)} \frac{1}{2}\rho|\bar{v}|^2 + \rho\bar{v} \cdot w \wedge (x - \bar{x}) + \frac{1}{2}\rho|w \wedge (x - \bar{x})|^2$$

$$= \frac{1}{2}|\bar{V}|^2 M + \bar{V}\cdot w\wedge\int_{S(t)} \rho(x-\bar{x}) + \frac{1}{2}\int_{S(t)} \rho\, |w\wedge(x-\bar{x})|^2$$

$$= \frac{1}{2}|\bar{V}|^2 M + \bar{V}\cdot w\wedge\left(\bar{x}M - M\bar{x}\right) + \frac{1}{2}\int_{S(t)} \rho\, |w\wedge(x-\bar{x})|^2$$

$$= \frac{1}{2}M|\bar{V}(t)|^2 + \frac{1}{2}\int_{S(t)} \rho(x,t)\, |w(t)\wedge(x-\bar{x}(t))|^2\, dx$$

$:= K_{trans}(t) + K_{rot}(t)$, which means the kinetic energy splits into translational and rotational sub-parts.

We can write $|A\wedge B|^2 = |A|^2|B|^2 - (A\cdot B)^2$, so

$$K_{rot}(t) = \frac{1}{2}\int_{S(t)} \rho(x,t)\left[|x-\bar{x}(t)|^2|w(t)|^2 - \left((w(t))\cdot(x-\bar{x}(t))\right)^2\right] dx$$

$$= \frac{1}{2}\int_{S(t)} \rho(x,t)\left[|x-\bar{x}(t)|^2 I - (x-\bar{x}(t))\otimes(x-\bar{x}(t))\right] dx\; w(t)\cdot w(t)$$

$$\boxed{K_{rot}(t) = \frac{1}{2}J(t)w(t)\cdot w(t)}$$

where $J(t)$ is the inertia tensor defined above.
Note: Cauchy-Schwarz $\Rightarrow$ $J(t)\geq 0$, so $K_{rot}(t)\geq 0$.

Q: Does J actually vary?
A: Yes, but nicely.

$$x = \eta(y,t) = \bar{x}(t) + Q(t)(y-y_0) \qquad y\in S(0)$$

So, $x - \bar{x}(t) = Q(t)(y-y_0)$

Also, $\tilde{\rho}(y) = \rho(x(y,t),t)$.

Change of coords: $x = \eta(y,t)$, $y\in S(0)$.

$$J(t) = \int_{S(t)} \rho(x,t)\left[|x-\bar{x}(t)|^2 I - (x-\bar{x}(t))\otimes(x-\bar{x}(t))\right] dx$$

$$= \int_{S(0)} \tilde{\rho}(y)\left[|Q(t)(y-y_0)|^2 I - Q(t)(y-y_0)\otimes Q(t)(y-y_0)\right] dy$$

But

$$(Qz\otimes Qz)_{ij} = Q_{ik}z_k\, Q_{j\ell}z_\ell = (Qz\otimes z)_{i\ell}\, Q^T_{\ell j} = \left(Qz\otimes z\,Q^T\right)_{ij}$$

So

$$\boxed{J(t) = Q(t)\, J(0)\, Q^T(t) = Q(t) J(0) Q^{-1}(t).}$$

$J(t)$ is conjugate to $J(0)$ via $Q(t)$.

§1.3 Momenta

Now let's compute the momenta, starting with linear.

$$v(x,t) = \bar{v}(t) + w(t) \wedge (x - \bar{x}(t)) \Rightarrow p(x,t) = \rho(x,t)\left[\bar{v}(t) + w(t) \wedge (x - \bar{x}(t))\right]$$

$$\Rightarrow \quad P(t) = \int_{S(t)} \rho(x,t)\left[\bar{v}(t) + w(t) \wedge (x - \bar{x}(t))\right] dx$$

$$= M\bar{v}(t) + w(t) \wedge \left(M\bar{x}(t) - M\bar{x}(t)\right)$$

$$= M\bar{v}(t)$$

So, $\boxed{P(t) = M\bar{v}(t)}$, i.e. linear momentum comes just from the center of mass moving.

Now for angular momentum, computed w.r.t. $x = 0$.

$$L(t) = \int_{S(t)} x \wedge p(x,t)\, dx = \int_{S(t)} \left[\rho(x,t)\, x \wedge \bar{v}(t) + x \wedge \left(w(t) \wedge (x - \bar{x}(t))\right)\right] dx$$

$$= M\bar{x}(t) \wedge \bar{v}(t) + \int_{S(t)} \rho(x,t)\bar{x}(t) \wedge \left(w(t) \wedge (x - \bar{x}(t))\right) dx + \int_{S(t)} \rho(x,t)\,(x - \bar{x}(t)) \wedge \left(w(t) \wedge (x - \bar{x}(t))\right) dx$$

$$= M\bar{x}(t) \wedge \bar{v}(t) + \bar{x}(t) \wedge \left(w(t) \wedge \left(\int_{S(t)} \rho(x,t)\,(x - \bar{x}(t))\,dx\right)\right) + \int_{S(t)} \rho(x,t)\,(x - \bar{x}(t)) \wedge \left(w(t) \wedge (x - \bar{x}(t))\right) dx$$

$$= M\bar{x}(t) \wedge \bar{v}(t) + \bar{x}(t) \wedge \left(w(t) \wedge \left(M\bar{x}(t) - M\bar{x}(t)\right)\right) + \int_{S(t)} \rho(x,t)\,(x - \bar{x}(t)) \wedge \left(w(t) \wedge (x - \bar{x}(t))\right) dx$$

$$= M\,\bar{x}(t) \wedge \bar{v}(t) + \int_{S(t)} \rho(x,t)\,(x - \bar{x}(t)) \wedge \left(w(t) \wedge (x - \bar{x}(t))\right) dx$$

Now we use $A \wedge (B \wedge A) = |A|^2 B - A(A \cdot B)$ to see

$$L(t) = M\bar{x}(t) \wedge \bar{v}(t) + \int_{S(t)} \rho(x,t)\left[|x - \bar{x}(t)|^2 w(t) - (x - \bar{x}(t))\, w(t) \cdot (x - \bar{x}(t))\right] dx$$

$$= M\bar{x}(t) \wedge \bar{v}(t) + J(t)\, w(t)$$

$$\Rightarrow \quad \boxed{L(t) = M\bar{x}(t) \wedge \bar{v}(t) + J(t)\, w(t)} \quad \text{where } L(t)$$

is the angular momentum about O. The inertia tensor is thus useful for more than just the rotational kinetic energy!

Note: if we define $\bar{L}(t)$ as the angular momentum about $\bar{x}(t)$, i.e.

$$\bar{L}(t) = \int_{S(t)} (x - \bar{x}(t)) \wedge p(x,t)\, dx,$$

then a computation similar to that above reveals that

$$\boxed{\bar{L}(t) = J(t)\, w(t)} \quad \text{and} \quad \boxed{L(t) = M\bar{x}(t) \wedge \bar{v}(t) + \bar{L}(t).}$$

Note that $\boxed{K_{rot}(t) = \frac{1}{2} \bar{L}(t) \cdot w(t).}$

§1.4 - Dynamics of $\bar{v}$, w

Now we can proceed to the dynamics. We start with the basic laws

$$\boxed{\begin{aligned} \frac{d}{dt} p(t) &= \mathcal{F}_{tot}(t) = \text{total force acting on } S(t) \\ \frac{d}{dt} L(t) &= \mathcal{T}_{tot}(t) = \text{total torque acting on } S(t), \\ &\qquad \text{as measured from } x=0 \end{aligned}}$$

The first is easily turned into an equation for $\bar{v}$ since M is const:

$$\boxed{M \frac{d}{dt} \bar{v}(t) = \frac{d}{dt}\left(M\bar{v}(t)\right) = \frac{d}{dt} p(t) = \mathcal{F}_{tot}(t).}$$

The second is a bit more delicate since J varies in time:

$$\frac{d}{dt} \bar{L}(t) = \frac{d}{dt}\left(J(t)\, w(t)\right) = \frac{d}{dt} J(t)\, w(t) + J(t) \frac{d}{dt} w(t), \quad \text{but}$$

$$J(t) = Q(t)\, J(0)\, Q^T(t) \Rightarrow \begin{aligned}[t] \dot{J} &= \dot{Q}\, J(0)\, Q^T + Q\, J(0)\, \dot{Q}^T \\ &= \dot{Q}Q^{-1}\, Q\, J(0)\, Q^T + Q\, J(0)\, Q^T\, Q\, \dot{Q}^T \\ &= \dot{Q}Q^{-1}\, J + J\,(\dot{Q}Q^{-1})^T \end{aligned}$$

$$\therefore \quad \begin{aligned}[t] \dot{J}(t)\, w(t) &= \dot{Q}(t)\, Q^{-1}(t)\, J(t)\, w(t) + J(t)\left(\dot{Q}(t)\, Q^{-1}(t)\right)^T w(t) \\ &= \dot{Q}(t)\, Q^{-1}(t)\, J(t)\, w(t) - J(t)\left(\dot{Q}(t)\, Q^{-1}(t)\right) w(t) \quad \text{(anti-sym)} \\ &= w(t) \wedge \left(J(t)\, w(t)\right) - J(t)\left(w(t) \wedge w(t)\right) \\ &= w(t) \wedge \left(J(t)\, w(t)\right) \end{aligned}$$

So, $\boxed{\frac{d}{dt} \bar{L}(t) = J(t) \frac{d}{dt} w(t) + w(t) \wedge J(t)\, w(t).}$

OTOH,

$$\boxed{\begin{aligned}\frac{d}{dt} M\bar{X}(t)\wedge\bar{V}(t) &= M\,\bar{V}(t)\wedge\bar{V}(t) + \bar{X}(t)\wedge M\frac{d}{dt}\bar{V}(t)\\ &= \bar{X}(t)\wedge\mathcal{F}_{tot}(t)\end{aligned}}$$

Hence

$$\boxed{\begin{aligned}\frac{d}{dt}L(t) &= \bar{X}(t)\wedge\mathcal{F}_{tot}(t) + J(t)\frac{dw(t)}{dt} + w(t)\wedge J(t)w(t)\\ &= \mathcal{T}_{tot}(t) \quad \text{as measured from } x=0.\end{aligned}}$$

Now let's expand on the forces and torques.

Since $S(t)$ is in contact with the fluid it experiences contact forces concentrated on $\partial S(t)$ in addition to the usual body forces caused by external forces.

$$\mathcal{F}_{tot}(t) = \mathcal{F}_{ctc}(t) + \mathcal{F}_{body}(t)$$

The contact force density is given via the fluid stress tensor $pI - \mu\,\mathbb{D}u = pI - \mu(\nabla u + \nabla u^T)$, i.e. the density is

$(pI - \mu\,\mathbb{D}u)\nu$, where ν is the <u>inner</u> pointing unit normal to $\partial S(t)$

Then the total contact force acting on $S(t)$ is

$$\mathcal{F}_{ctc}(t) = \int_{\partial S(t)} (pI - \mu\,\mathbb{D}u)\nu$$

The body force comes from the external force density $f(x,t)$ via $\rho(x,t)f(x,t)$. Then the total body force acting on $S(t)$ is

$$\mathcal{F}_{body}(t) = \int_{S(t)} \rho(x,t)\,f(x,t)\,dx$$

So,

$$\boxed{\mathcal{F}_{tot}(t) = \int_{\partial S(t)} (pI - \mu\,\mathbb{D}u)\nu + \int_{S(t)} \rho(x,t)\,f(x,t)\,dx}$$

with ν inner-pointing.

For the torques we have contact ones and body ones in the same way: $\mathcal{T}_{tot}(t) = \mathcal{T}_{ctc}(t) + \mathcal{T}_{body}(t)$, where both are measured w.r.t. $x=0$.

Here

$$\mathcal{T}_{ctc}(t) = \int_{\partial S(t)} x \wedge \left[(pI - \mu \mathbb{D}u)\nu\right]$$

$$= \int_{\partial S(t)} \bar{x}(t) \wedge \left[(pI - \mu \mathbb{D}u)\nu\right] + \int_{\partial S(t)} (x - \bar{x}(t)) \wedge \left[(pI - \mu \mathbb{D}u)\nu\right]$$

$$= \bar{x}(t) \wedge \mathcal{F}_{ctc}(t) + \int_{\partial S(t)} (x - \bar{x}(t)) \wedge \left[(pI - \mu \mathbb{D}u)\nu\right]$$

and

$$\mathcal{T}_{bdy}(t) = \int_{S(t)} x \wedge \rho(x,t) f(x,t) = \int_{S(t)} \bar{x}(t) \wedge \rho(x,t) f(x,t) + (x - \bar{x}(t)) \wedge \rho(x,t) f(x,t)$$

$$= \bar{x}(t) \wedge \mathcal{F}_{bdy}(t) + \int_{S(t)} (x - \bar{x}(t)) \wedge \rho(x,t) f(x,t)\, dx$$

$\therefore$

$$\boxed{\begin{aligned} \mathcal{T}_{tot}(t) = {} & \bar{x}(t) \wedge \mathcal{F}_{tot}(t) + \int_{\partial S(t)} (x - \bar{x}(t)) \wedge \left[(p(x,t) - \mu \mathbb{D}u(x,t))\nu(x,t)\right] d\sigma(x) \\ & + \int_{S(t)} (x - \bar{x}(t)) \wedge \rho(x,t) f(x,t)\, dx \end{aligned}}$$

Now we can combine all of the above to write down the dynamics of $\bar{v}(t)$, $w(t)$.

$$\boxed{\begin{aligned} M \frac{d}{dt} \bar{v}(t) &= \int_{\partial S(t)} (p(x,t) I - \mu \mathbb{D}u(x,t))\nu(x,t)\, d\sigma(x) + \int_{S(t)} \rho(x,t) f(x,t)\, dx \\ J(t) \frac{dw(t)}{dt} &= J(t) w(t) \wedge w(t) + \int_{\partial S(t)} (x - \bar{x}(t)) \wedge \left[(p(x,t) - \mu \mathbb{D}u(x,t))\nu(x,t)\right] d\sigma(x) \\ & \quad + \int_{S(t)} (x - \bar{x}(t)) \wedge \rho(x,t) f(x,t)\, dx \qquad \text{with } \nu \text{ \underline{inner}-pointing} \end{aligned}}$$

§1.5 – Synthesis: Equations of motion

Now we assume that $S^i(t) = \eta^i(S^i(0), t)$, $i = 1, \dots, n$, where $\eta^i(y,t) = \bar{x}^i(t) + Q^i(t)(y - \bar{x}^i(0))$,

$$v^i(x,t) = \bar{v}^i(t) + w^i(t) \wedge (x - \bar{x}^i(t)), \quad M^i = \int_{S^i(t)} \rho^i, \quad \text{etc.}$$

Write $\Omega(t) = \Omega \setminus \bigcup_{i=1}^{n} S^i(t)$. Then the fluid density $\rho(\cdot,t) : \Omega(t) \to (0,\infty)$ and velocity $u(\cdot,t) : \Omega(t) \to \mathbb{R}^3$ satisfy

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 & \text{in } \Omega(t) \\ \rho(\partial_t u + u \cdot \nabla u) + \operatorname{div}(pI - \mu \mathbb{D}u) = \rho f & \text{in } \Omega(t) \\ \operatorname{div} u = 0 & \text{in } \Omega(t) \end{cases}$$

$$\begin{cases} u = 0 & \text{on } \partial\Omega \\ u = v^i & \text{on } \partial S^i(t) \end{cases}$$

The $\bar{v}^i$, w^i satisfy

$$\begin{cases} M^i \frac{d}{dt} \bar{v}^i(t) = \int_{\partial S^i(t)} (p(x,t) I - \mu \mathbb{D}u(x,t)) \nu(x,t) \, d\sigma(x) + \int_{S^i(t)} \rho^i(x,t) f(x,t) \, dx \\ J^i(t) \frac{dw^i(t)}{dt} = J^i(t) w^i(t) \wedge w^i(t) + \int_{\partial S^i(t)} (x - \bar{x}^i(t)) \wedge \left[(p(x,t) - \mu \mathbb{D}u(x,t)) \nu(x,t) \right] d\sigma(x) \\ \qquad + \int_{S^i(t)} (x - \bar{x}^i(t)) \wedge \rho^i(x,t) f(x,t) \, dx \end{cases}$$

with ν inner-pointing on $\partial S^i(t)$.

We also specify initial data $u_0 : \Omega(0) \to \mathbb{R}^3$ and $\bar{v}^i(0)$, $\bar{w}^i(0) \in \mathbb{R}^3$, $i = 1, \dots, n$.

§2 – The energy law and the weak formulation

§2.1 The integral relation

As per usual, we will arrive at the weak formulation by first assuming a classical solution exists and then deriving an integral relation that it must satisfy.

In this section we'll write $T = pI - \mu \mathbb{D}u$ for the stress tensor.

Suppose w is another function satisfying $\operatorname{div} w = 0$ in $\Omega(t)$ and $\mathbb{D}w = 0$ in $S^i(t)$, $i = 1, \dots, n$, and $w = 0$ on $\partial\Omega$.

Then, upon multiplying by w and integrating, we find

$$\int_{\Omega(t)} \rho(\partial_t u + u\cdot\nabla u)\cdot w + \underbrace{\int_{\Omega(t)} \operatorname{div} T \cdot w}_{:=I} = \int_{\Omega(t)} \rho f \cdot w$$

pointing out of $\Omega(t)$ — pting into S^i

$$I = \int_{\Omega(t)} -T : \nabla w + \int_{\partial\Omega(t)} T\nu \cdot w = \int_{\Omega(t)} \frac{\mu}{2} \mathbb{D}u : \mathbb{D}w + \sum_i \int_{\partial S^i(t)} T\nu\cdot w$$

$$:= \int_{\Omega(t)} \frac{\mu}{2} \mathbb{D}u : \mathbb{D}w + A.$$

Now, since $\mathbb{D}w = 0$ in $S^i(t)$, it holds that $w(x,t) = \alpha^i(t) + \beta^i(t)\wedge(x - \bar{x}(t))$ $\forall x \in S^i(t)$ for $\alpha,\beta \in \mathbb{R}^3$ (the kernel of $\mathbb{D}$ is translations and rotations). Then this allows us to write

$$A = \sum_i \left(\alpha^i(t)\cdot \int_{\partial S^i(t)} T\nu + \beta^i(t)\cdot\int_{\partial S^i(t)} (x-\bar{x}^i)\wedge T\nu\right)$$

$$= \sum_i \alpha^i(t)\cdot\left[M^i \frac{d}{dt}\bar{V}^i - \int_{S^i(t)} \rho^i f\right] + \beta^i(t)\cdot\left[\frac{d}{dt} L^i(t) - \int_{S^i(t)} (x-\bar{x}^i)\wedge \rho^i f\right]$$

$$= \sum_i M^i \alpha^i \frac{d}{dt}\bar{V}^i + \beta^i \frac{d}{dt}(J^i w) - \int_{S^i(t)} \rho^i f\cdot[\alpha^i + \beta^i\wedge(x-\bar{x}^i)]$$

$$= \sum_i M^i\alpha^i \frac{d}{dt}\bar{V}^i + \beta^i\frac{d}{dt}(J^i w) - \int_{S^i(t)} \rho^i f\cdot w.$$

Hence,

$$I = \int_{\Omega(t)} \frac{\mu}{2}\mathbb{D}u:\mathbb{D}w + \sum_i\left[M^i\alpha^i\cdot\frac{d}{dt}\bar{V}^i + \beta^i\frac{d}{dt}(J^i w) - \int_{S^i(t)} \rho^i f\cdot w\right].$$

OTOH,

$$\int_{S^i(t)} \rho^i \partial_t V^i\cdot w^i = \int_{S^i(t)} \rho^i\left[\dot{\bar{V}}^i + \dot{\omega}^i\wedge(x-\bar{x}^i) - \omega^i\wedge\bar{V}^i\right]\cdot\left[\alpha^i + \beta^i\wedge(x-\bar{x}^i)\right]$$

$$= M\alpha^i\cdot\frac{d}{dt}\bar{V}^i + \dot{\bar{V}}^i\cdot\beta^i\wedge\int_{S^i(t)}\rho^i(x-\bar{x}^i) + \alpha^i\cdot\dot{\omega}^i\wedge\int_{S^i(t)}\rho^i(x-\bar{x}^i)$$

$$+\int_{S^i(t)}\rho^i\left[\dot{\omega}^i\wedge(x-\bar{x}^i)\right]\cdot\left[\beta^i\wedge(x-\bar{x}^i)\right] - M\alpha^i\cdot\omega^i\wedge\bar{V}^i$$

$$-(\omega^i\wedge\bar{V}^i)\cdot\beta^i\wedge\int\rho^i(x-\bar{x}^i)$$

$$= M\alpha^i\cdot\frac{d}{dt}\bar{V}^i - M\alpha^i\cdot\omega^i\wedge\bar{V}^i + J^i\dot{\omega}^i\cdot\beta^i + \dot{J}^i w^i\cdot\beta^i - \dot{J}^i w^i\cdot\beta^i$$

$$= M\alpha^i\cdot\frac{d}{dt}\bar{V}^i - M\alpha^i\cdot\omega^i\wedge\bar{V}^i + \beta^i\cdot\frac{d}{dt}J^i w^i - (w^i\wedge J w^i)\cdot\beta^i$$

$$= M\alpha^i\frac{d}{dt}\bar{V}^i + \beta^i\cdot\frac{d}{dt}J^i w^i - M w^i\cdot\bar{V}^i\wedge\alpha^i - w^i\cdot(J w^i\wedge\beta^i).$$

Now we compute in $S^i(t)$ (drop superscripts for the moment)

$$v\cdot\nabla v = \nabla v\, v^T = \nabla(w\wedge x)\cdot v^T = w\wedge v$$

$$\Rightarrow \int_S \rho\, v\cdot\nabla v\cdot w = \int_S \rho\, w\wedge v\cdot w = \int_S \rho\, w\cdot w\wedge v = \int_S \rho\, w\cdot v\wedge w$$

$$= w\cdot\int_S \rho\, v\wedge w$$

$$= w\cdot\int \rho\left(\bar v + w\wedge(x-\bar x)\right)\wedge\left(\alpha+\beta\wedge(x-\bar x)\right)$$

$$= w\cdot\int \rho\left[\bar v\wedge\alpha + \bar v\wedge\left(\beta\wedge(x-\bar x)\right) - \alpha\wedge\left(w\wedge(x-\bar x)\right) + \left(w\wedge(x-\bar x)\right)\wedge\left(\beta\wedge(x-\bar x)\right)\right]$$

$$= w\cdot M\bar v\wedge\alpha + 0 + 0 + w\cdot\int\rho\left(w\wedge(x-\bar x)\right)\wedge\left(\beta\wedge(x-\bar x)\right)$$

We have the identities for w, β, z

$$(w\wedge z)\wedge(\beta\wedge z) = -\beta\wedge\left(z\wedge(w\wedge z)\right) - (\beta\cdot z)(w\wedge z)$$

$$|z|^2 w\wedge\beta - (z\cdot w)\, z\wedge\beta = \left[z\wedge(w\wedge z)\right]\wedge\beta = -\beta\wedge\left[z\wedge(w\wedge z)\right].$$

These then allow us to compute

$$w\cdot\int\rho\left(w\wedge(x-\bar x)\right)\wedge\left(\beta\wedge(x-\bar x)\right) = w\cdot\int\rho\left[|x-\bar x|^2 w\wedge\beta - \left[(x-\bar x)\cdot w\right](x-\bar x)\wedge w\right]$$

$$- w\cdot\int\rho\,\left(\beta\cdot(x-\bar x)\right)\left(w\wedge(x-\bar x)\right)$$

$$= w\cdot(Jw\wedge\beta) - 0 \qquad = w\cdot Jw\wedge\beta.$$

Plugging back in above, we get

$$\int_S \rho\, v\cdot\nabla v\cdot w = w\cdot\left(M\bar v\wedge\alpha + Jw\wedge\beta\right).$$

Hence $\displaystyle\int_{S^i(t)} \rho^i\left(\partial_t v^i + v^i\cdot\nabla v^i\right)\cdot w = M\alpha^i\cdot\frac{d}{dt}\bar v^i + \beta^i\cdot\frac{d}{dt}J^i w^i,$

so we have that

$$I = \int_{\Omega(t)} \frac{\mu}{2}\,\mathbb{D}u:\mathbb{D}w + \sum_i\left[\int_{S^i(t)} \rho^i\left(\partial_t v^i + v^i\cdot\nabla v^i\right)\cdot w - \int_{S^i(t)} \rho^i f\cdot w\right].$$

Finally, we have the integral relation

$$\boxed{\begin{aligned}&\int_{\Omega(t)} \rho(\partial_t u + u\cdot\nabla u)\cdot w + \sum_i \int_{S^i(t)} \rho^i(\partial_t v^i + v^i\cdot\nabla v^i)\cdot w + \int_{\Omega(t)} \frac{\mu}{2} \mathbb{D}u : \mathbb{D}w \\ &= \int_{\Omega(t)} \rho f\cdot w + \sum_i \int_{S^i(t)} \rho^i f\cdot w.\end{aligned}}$$

It's convenient to unify notation for ρ, u. To this end we redefine (by abuse of notation)

$$u(x,t) = \begin{cases} u(x,t), & x\in\Omega(t) = \Omega \setminus \bigcup_{i=1}^{n} S^i(t) \\ v^i(x,t), & x\in S^i(t) \end{cases}$$

$$\rho(x,t) = \begin{cases} \rho(x,t), & x\in\Omega(t) \\ \rho^i(x,t), & x\in S^i(t) \end{cases}.$$

Then, since $\mathbb{D}w = 0$ in $S^i(t)$, we can rewrite the integral relation as

$$\boxed{\int_\Omega \rho(\partial_t u + u\cdot\nabla u)\cdot w + \int_\Omega \frac{\mu}{2}\mathbb{D}u : \mathbb{D}w = \int_\Omega \rho f\cdot w.}$$

§ 2.2 – The energy equation

Note that u itself is a valid choice for w above. This yields

$$\int_{\Omega(t)} \rho(\partial_t u + u\cdot\nabla u)\cdot u + \sum_i \int_{S^i(t)} \rho^i(\partial_t v^i + v^i\cdot\nabla v^i)\cdot v^i + \int_{\Omega(t)} \frac{\mu}{2}\mathbb{D}u : \mathbb{D}u$$

$$= \int_\Omega \rho f\cdot u.$$

Write LHS $= \mathrm{I} + \mathrm{II} + \mathrm{III}$. Then since $\partial_t\rho + \mathrm{div}(\rho u)$ in $\Omega(t)$,

$$\frac{d}{dt}\int_{\Omega(t)} \rho\frac{|u|^2}{2} = \int_{\Omega(t)} \partial_t\left(\rho\frac{|u|^2}{2}\right) + \mathrm{div}\left(\rho\frac{|u|^2}{2}u\right)$$

$$= \int_{\Omega(t)} (\partial_t\rho + \mathrm{div}(\rho u))\frac{|u|^2}{2} + \rho(\partial_t u + u\cdot\nabla u)\cdot u$$

$$= 0 + I = I.$$

Similarly, since $\partial_t \rho^i + \operatorname{div}(\rho^i v^i) = 0$ in $S^i(t)$,

$$II = \frac{d}{dt} \sum_i \int_{S^i(t)} \frac{\rho^i |v^i|^2}{2}.$$

Since $\mathbb{D} v^i = 0$, we can write $III = \int_{\Omega(t)} \frac{\mu}{2} \mathbb{D}u : \mathbb{D}u = \int_\Omega \frac{\mu}{2} |\mathbb{D}u|^2.$

$\therefore$

$$\boxed{\frac{d}{dt} \int_\Omega \frac{\rho |u|^2}{2} + \int_\Omega \frac{\mu}{2} |\mathbb{D}u|^2 = \int_\Omega \rho f \cdot u.}$$

The nice thing about this equation is that it exactly agrees with what we would get without any solids.

In the original notation, the energy law reads

$$\boxed{\begin{aligned} &\frac{d}{dt}\left[\int_{\Omega(t)} \rho \frac{|u|^2}{2} + \sum_i \int_{S^i(t)} \frac{\rho^i |v^i|^2}{2} \right] + \int_{\Omega(t)} \frac{\mu}{2} |\mathbb{D}u|^2 \\ &= \int_{\Omega(t)} \rho f \cdot u + \sum_i \int_{S^i(t)} \rho^i f \cdot v^i. \end{aligned}}$$

§2.3 The weak / variational formulation

Recall the integral relation

$$\int_\Omega \rho (\partial_t u + u \cdot \nabla u) \cdot w + \int_\Omega \frac{\mu}{2} \mathbb{D}u : \mathbb{D}w = \int_\Omega \rho f \cdot w,$$

which holds for $w \in \{ v \in H_0^1 \mid \operatorname{div} w = 0,\ \mathbb{D}w = 0 \text{ on } S^i(t)\ i = 1, \dots, n \}$.

To get rid of the time derivative, let's integrate in time and IBP, assuming $w(\cdot,t)$ is as above $\forall t\in[0,T]$:

$$\int_0^T\int_\Omega \rho(\partial_t u + u\cdot\nabla u)\cdot w = \int_0^T\int_{\Omega(t)} \rho(\partial_t u\cdot u\cdot\nabla u)\cdot w + \sum_{i=1}^{n}\int_{S^i(t)} \rho(\partial_t u\cdot u\cdot\nabla u)\cdot w$$

Note that

$$\frac{d}{dt}\int_{\Omega(t)} \rho u\cdot w = \int_{\Omega(t)} \partial_t(\rho u\cdot w) + \operatorname{div}(\rho u\cdot w\, u) = \int_{\Omega(t)} (\partial_t\rho + \operatorname{div}(\rho u))(u\cdot w)$$
$$+ \int_{\Omega(t)} \rho(\partial_t u + u\cdot\nabla u)\cdot w + \rho u\cdot\partial_t w + \rho u\otimes u : \frac{\mathbb{D}w}{2}$$
$$= \int_{\Omega(t)} \rho(\partial_t u + u\cdot\nabla u) w + \rho u\cdot\partial_t w + \rho u\otimes u : \frac{\mathbb{D}w}{2},$$

and similarly $\displaystyle \frac{d}{dt}\int_{S^i(t)} \rho u\cdot w = \int_{S^i(t)} \rho(\partial_t u + u\cdot\nabla u) + \rho u\cdot\partial_t w + \underbrace{\rho u\otimes u : \frac{\mathbb{D}w}{2}}_{=0}.$

So,

$$\int_0^T\int_\Omega \rho(\partial_t u + u\cdot\nabla u)\cdot w = \int_0^T\left[\frac{d}{dt}\int_\Omega \rho u\cdot w - \int_\Omega \rho u\cdot\partial_t w + \rho u\otimes u : \frac{\mathbb{D}w}{2}\right].$$

If we further assume $w(\cdot,0) = w(\cdot,T) = 0$, then we arrive at

$$\boxed{\int_0^T\int_\Omega -\rho u\,\partial_t w - \rho u\otimes u : \frac{\mathbb{D}w}{2} + \frac{\mu}{2}\mathbb{D}u : \mathbb{D}w = \int_0^T\int_\Omega \rho f\cdot w.}$$

So, to use this for a weak formulation we assume we know $\{S^i(0), \eta^i\}_{i=1}^n$ with $S^i(t) = \eta^i(S^i(0), 0)$ and let

$$Q^s = \{(x,t)\in\Omega\times(0,T) \mid x\in \bigcup_{i=1}^n S^i(t)\}$$
$$Q^f = \Omega\times(0,T)\setminus\Omega^s,$$

and

$$\mathcal{T}(Q^s) = \{\varphi\in C_c^\infty(\Omega\times(0,T)) \mid \operatorname{div}\varphi = 0 \text{ on } \Omega\times(0,T) \text{ and }$$
$$\mathbb{D}\varphi = 0 \text{ on a nbd. of } Q^s\}.$$

NOTE: The space of test functions depends on the soln itself!

We couple the motion of the solids to the fluid by assuming that

$$\eta^i(y,t) = \bar{X}^i(t) + Q^i(t)(y - \bar{x}^i(0)),$$

where $\frac{d}{dt}\bar{x}^i(t) = \bar{V}^i(t)$, $\frac{d}{dt}Q(t)Q^{-1}(t) = w(t)\wedge\cdot$

and imposing the condition

$$\boxed{u(x,t) = \bar{V}^i(t) + w^i(t)\wedge(x - \bar{x}(t)) \quad \text{for a.e. } x \in S^i(t).}$$

For simplicity let's assume the fluid is of constant density ρ_f. This way we don't have to worry about formulating a weak version of the continuity eqn. We could do this, though (cf. Feireisl). Let's also assume $S^i(0)$ satisfy the following:

1) $S^i(0)$ are compact and connected
2) $S^i(0)$ is the closure of its interior, which is non-empty
3) $|S^i(0) \setminus \text{int}(S^i(0))| = 0$.

Also, let $\boxed{u_0 \in {}_\sigma L^2(\Omega) = cl_{L^2}\{u \in C_c^\infty \mid \text{div}\, u = 0\}}$. ($cl_{L^2} = L^2$ closure)

We can now state the def of variational soln.

$\boxed{\text{Definition:}}$ Let $\Omega \subseteq \mathbb{R}^3$ be bounded and open, $T > 0$. We say u, $\{S^i(0), \eta^i\}_{i=1}^n$ are a variational solution on $\Omega \times (0,T)$ if the following hold.

1) $u \in L^\infty((0,T); L^2(\Omega)) \cap L^2((0,T); {}_\sigma H_0^1(\Omega))$ where ${}_\sigma H_0^1(\Omega)$ is the closure of $\{\varphi \in C_c^\infty(\Omega) \mid \text{div}\,\varphi = 0\}$ in the norm $\int_\Omega |\nabla u|^2$.

2) u satisfies

$$\int_0^T\int_\Omega -\rho u\, \partial_t w - \rho u \otimes u : \frac{\mathbb{D}w}{2} + \frac{\mu}{2}\mathbb{D}u : \mathbb{D}w = \int_0^T\int_\Omega \rho f \cdot w$$

for every $w \in \mathcal{T}(Q^s)$, where Q^s is determined by $\{S^i(0), \eta^i\}_{i=1}^n$.

3) The energy inequality

$$\int_\Omega \frac{\rho}{2}|u|^2(t)\,dx + \int_0^t\int_\Omega \frac{\mu}{2}|\mathbb{D}u|^2\,dx\,ds \le \int_\Omega \frac{\rho(\cdot,0)|u_0|^2}{2}dx + \int_0^t\int_\Omega \rho f \cdot u$$

holds for a.e. $t \in (0,T)$.

4) The mapping η^i are affine isometries and $\{S^i(0), \eta^i\}_{i=1}^n$ are compatible with u in the sense that

$$\eta^i(y,t) = \bar{x}^i(t) + Q^i(t)(y - \bar{x}(0)),$$

with $\bar{x}^i(t) \in \Omega$, $Q^i(t) \in SO(3)$ abs. cont and

$$u(x,t) = \bar{V}^i(t) + w^i(t) \wedge (x - \bar{x}(t)) \quad \text{for } x \in S^i(t),$$

where

$$\begin{cases} \frac{d}{dt} x^i(t) = \bar{V}^i(t), & \frac{d}{dt} Q(t) Q^{-1}(t) = w(t) \wedge \cdot \\ x^i(0) = \int_{S^i(0)} \rho^i(y) y \, dy, & Q(0) = I \end{cases}$$

5) $$\rho(x,t) = \begin{cases} \rho_f & \text{if } x \in \Omega \setminus \bigcup_{i=1}^n S^i(t) \\ \rho^i(x,t) & \text{if } x \in S^i(t), \quad i = 1, \dots, n \end{cases}$$

where $\rho^i(\eta^i(y,t),t) = \rho_0^i(y) \quad \forall\, y \in S^i(0)$, i.e.

$$\rho^i(x,t) = \rho_0^i\left(\bar{x}^i(0) + (Q^i(t))^{-1}(x - \bar{x}(t))\right).$$

6) The function $h_\psi(t) = \int_\Omega \rho u \cdot \psi$ is continuous on $[0,T]$

for all $\psi \in \{ v \in C^0((0,T); L^2(\Omega)) \mid \operatorname{div} v = 0,\ \mathbb{D}\psi = 0$ on a nbd of $\bigcup_{i=1}^n S^i(t) \}$.

Also, $h_\psi(0) = \int_\Omega \rho(x,0)\, u_0(x)\, \psi(x,0)\, dx$.

§3 – Existence of solutions

Our aim is to sketch the proof of the following theorem, which is a special case of the result proved in Feireisl (J.E.E. 2003).

- **Theorem**: Let $\Omega \subset \mathbb{R}^3$ be bounded and open, $T > 0$. Assume $S^i(0)$, $i = 1, \dots, n$ satisfy the assumptions in §2.3, and $S^i(0) \cap S^j(0) = \emptyset$ $i \neq j$, $S^i(0) \cap (\mathbb{R}^3 \setminus \Omega) = \emptyset$ $i = 1, \dots, n$. Let $\rho_0^i : S^i(0) \to (0, \infty)$ be measurable and bounded above and below. Let $u_0 \in \sigma L^2(\Omega)$. Let $f \in L^\infty(\Omega \times (0,T))$.

  Then there exists a variational weak solution on $\Omega \times (0,T)$.

§3.1 – Continuation and existence when the bodies are smooth

The key result in the paper deals with a way of continuing a variational soln.

• Lemma (Continuation): Let r^s denote the radius of the largest ball that is contained in the interior of any $S^i(0)$. Suppose $u, \{S^i(0), \eta^i\}_{i=1}^n$ are a soln on $\Omega \times (0,T_1)$. Then the following hold.

1) The isometries η^i are Lipschitz in T, and
$\|\eta^i(t_2,\cdot) - \eta^i(t_1,\cdot)\|_{C^0(B;\mathbb{R}^3)} \le C(B)|t_1 - t_2|$ for bounded $B \subseteq \mathbb{R}^3$.

Here $C(B)$ depends on B, r^s, u_0, f, T.

2) $\exists$ $u(\cdot,T_1)$ in ${}_\sigma L^2(\Omega)$ s.t.
$$\lim_{t\to T_1} \int_\Omega \varphi \rho u(\cdot,t)\, dx = \int_\Omega \varphi \rho u(\cdot,T_1)$$
$\forall \varphi \in \{u \in L^\infty_c \mid \operatorname{div} u = 0,\ \mathbb{D}u = 0 \text{ on nbd of } \bigcup_{i=1}^n S^i(T_1)\}$.
Also, $\int_\Omega \frac{1}{2} \rho(x,T_1)|u(x,T^1)|^2\, dx \le \liminf_{t\to T_1} \int_\Omega \frac{1}{2} \rho |u|^2 (t)$

3) [Continuation] If $u, \{S^i(T_1), \eta^i\}_{i=1}^n$ are solns on $\Omega \times (T_1,T_2)$ with data $u(\cdot,T_1)$, then $u, \{S^i(0), \eta^i\}_{i=1}^n$ are solns on $\Omega \times (0,T_2)$.

Remark: The point is that one can choose end-pt data for the velocity, $u(\cdot,T_1)$, so that another soln starting from here can be glued onto the original soln.

Sketch of proof:

(1) Follows from the energy estimate since η^i is abs cont, and $|\bar{V}^i|^2 + |w^i|^2 \lesssim \frac{M^i}{2}|\bar{V}^i|^2 + \frac{1}{2} J^i w^i \cdot w^i = \int_{S^i(t)} \frac{\rho}{2} |u|^2$.

(2) The intergral formula allows us to show that

$$\mathcal{L}\varphi := \lim_{t\to T_1} \int_\Omega \varphi \rho u(t) \quad \text{for } \varphi \text{ as above is}$$

a bounded linear functional with
$$|\mathcal{L}\varphi|^2 \le 2\left(\int_\Omega \rho(x,T_1)|\varphi(x)|^2 dx\right)\left(\liminf_{t\to T_1} \int_\Omega \frac{1}{2}\rho |u|^2\right).$$
Then we use Hahn-Banach to extend $\mathcal{L}$ to be a bounded functional on ${}_\sigma L^2$ w/ inner-product $\langle u,v\rangle_{{}_\sigma L^2} = \int_\Omega \rho(x,T_1)\, u(x)\cdot v(x)\, dx$. Then Riesz representation gives us $u(\cdot,T_1) \in {}_\sigma L^2$ s.t. $\mathcal{L}\varphi = \langle u(\cdot,T_1), \varphi\rangle_{{}_\sigma L^2}$, which satisfies what we want.

(3) This then follows from the properties of $u(\cdot, T_1)$.

$\square$

Next is the existence result for smooth solids.

- Proposition (smooth existence): Suppose Ω, $S^i(0)$ are all C^∞. Let $u_0 \in \sigma L^2(\Omega)$, $f \in L^\infty(\Omega \times (0,\infty))$. Then $\exists$ a variational soln on a maximal time interval $[0, T_{max})$, $T_{max} > 0$. Moreover, if $T_{max} < \infty$ then

$$d(\{S^i(t)\}) := \min\left\{ \min_{i \neq j} \mathrm{dist}(S^i(t), S^j(t)),\ \min_i \mathrm{dist}(S^i(t), \Omega^c)\right\} = 0.$$

Sketch of proof:
The proof goes in two steps.

1) Solns exist on $(0, T_0)$ with $T_0 > 0$, T_0 determined by r^s, $\{\rho_0^i\}$, f, $d(\{S^i(0)\})$, the latter being required to be > 0.

Feireisl omits a proof of this, instead claiming it can be carried out as in the work of
Desjardin-Esteban (CPDE 2000)
Gunzburger-Lee-Seregin (JMFM 2000)
San Martin-Starovoitov-Tucsnak (ARMA 2002).

See these for details.

2) Using the continuation lemma, one can continue these solns until $d(\{S^i(T_{max})\}) = 0$, i.e. the first collision time.

$\square$

§3.2 – Existence up to collision for general S^i

Now Feireisl gets rid of smoothness.

- Proposition (local existence): Let $\Omega \subseteq \mathbb{R}^3$ be bounded, open, $S^i(0)$ as in §2.3 with $d(\{S^i(0)\}) > 0$. Let $u_0 \in \sigma L^2$, $f \in L^\infty$. Then $\exists$ a variational solution on a maximal time interval $(0, T_{max})$, where $T_{max} > 0$. If $T_{max} < \infty$ then $d(\{S^i(T_{max})\}) = 0$.

Sketch of proof:

The idea is to fit a sequence of smooth domains in Ω, i.e. $\Omega_k \subseteq \Omega$, and a sequence of smooth solids outside, $S^i_k(0) \supseteq S^i(0)$ $i=1,\dots,n$. Then the smooth existence proposition produces solutions $u_k, \{S^i_k(0), \eta^i_k\}_{i=1}^n$ on $\Omega_k \times (0, T_k)$.

Then the smooth approximations are chosen s.t. $\Omega_k \to \Omega$, $S^i_k(0) \to S^i(0)$ in an appropriate sense (local uniform convergence of a certain signed distance function) which allows us to prove that
$$T_k \geq T_0 \quad \forall\ k \text{ large enough.}$$

Then the energy inequality estimates and an auxilary argument using the continuity equation yield the convergence results
$$\begin{cases} \rho_k \to \rho & \text{in } C([0,T_0]; L^1(\Omega)) \\ u_k \rightharpoonup u & \text{weakly in } L^2((0,T_0); H^1_0) \\ u_k \overset{*}{\rightharpoonup} u & \text{weakly-}* \text{ in } L^\infty((0,T_0); \sigma L^2). \end{cases}$$

The next step is to show the limit is a variational solution. This is rather involved, so we'll skip it here. It hinges on solving some auxiliary elliptic problems that are sensitive to the roughness of domains. See Feireisl for the details.

$\square$

§3.3 - Proof of the theorem

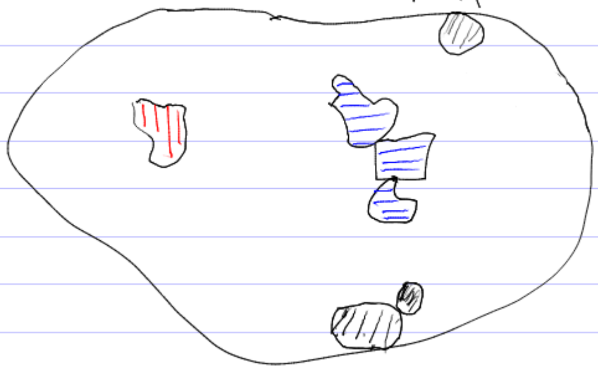
Step 1 - Local soln.

Use local existence prop. to get a solution on $\Omega \times (0, T_1)$ with $T_1 > 0$ maximal. If $T \leq T_1$ then we're done, so assume $T_1 < T$, in which case $d(\{S^i(T_1)\}) = 0$.

Step 2 - Merging

Write $M_{00} = \{ k \in \{1,\dots,n\} \mid \mathrm{dist}(S^k(T_1), \Omega^c) = 0 \}$ for the indices of the objects that hit the boundary, $M_0 = \{ k \mid \mathrm{dist}(S^i(T_1), S^k(T_1)) = 0 \text{ for } i \in M_0 \}$ for the indices of the objects that either hit the boundary or cluster with the objects that hit the boundary.

Then partition $\{1,\dots,m\}\setminus M_0$ into $\bigsqcup_{j=1} M_j$, a disjoint union, so that $\operatorname{dist}\left(S^i(T_1), \bigcup_{k\in M_j\setminus\{i\}} S^k(T_1)\right)=0$ for $i\in M_j$ and $\operatorname{dist}\left(S^i(T_1), \bigcup_{k\notin M_j} S^k(T_1)\right)>0$. That is, group according to collision:



Black (///) $= M_0$
Red (||||) $= M_1$
Blue ($\equiv$) $= M_2$.

Note: $m<n$, since there must be mergings.

Define $\Omega_2=\Omega\setminus\bigcup_{j\in M_0} S^j(T_1)$, and for $i=1,\dots,m$ let $S_2^i(T_1)=\bigcup_{j\in M_i} S^j(T_1)$. That is, delete the parts that hit the boundary from the domain and call the merged collections $S_2^i(T_1)$.

Step 3 — Extension

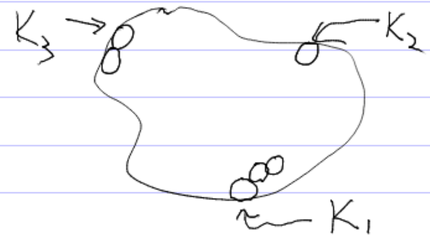
The collection $\{S_2^i(T_1)\}_{i=1}^m$ satisfies the hypotheses of the local existence proposition in the domain Ω_2. Hence we can apply it to get a soln $u_2, \{S_2^i(T_1), \eta_2^i\}_{i=1}^m$ defined on a maximal interval (T_1, T_2), $T_2>T_1$, with initial data

$$\begin{cases} \rho^i(\cdot,T_1) \text{ defined in the obvious way, } i=1,\dots,m \\ u(\cdot,T_1) \\ S_2^i(T_1). \end{cases}$$

We claim that $u_2, \{S^i(T_1), \eta^i\}_{i=1}^n$, with $\eta^j=\eta_2^i$ for all $j\in M_i$, $i=1,\dots,m$, constitutes a variational solution on $\Omega\times(T_1,T_2)$. This is possible since the definition says nothing about the sets remaining disjoint.

To prove the claim it suffices to show that $\varphi\in\{\varphi\in C_c^\infty(\Omega\times(T_1,T_2)) \mid \operatorname{div}\varphi=0 \text{ on } \Omega\times(T_1,T_2) \text{ and } \mathbb{D}\varphi=0 \text{ on a nbd. of } Q^s\}$
$\Rightarrow \varphi\in C_c^\infty(\Omega_2\times(T_1,T_2))$. To prove this

we first partition $M_0 = \coprod_{i=1}^{\ell} K_\ell$ exactly as we partitioned the non-boundary-intersecting solids above.



Now note that $D\varphi = 0$ on a nbd. of Q^s

$\Rightarrow$ $D\varphi(\cdot, t) = 0$ in a nbd. of $\mathbb{R}^3 \setminus \Omega_2$, and in particular in a nbd. of $\bigcup_{j \in K_i} S^j(T_1)$ for each $i = 1, \dots, \ell$.

$\Rightarrow$ $\varphi(x, t) = \alpha^i(t) + \beta^i(t) \wedge x$ $\forall x$ in a nbd. of of $\bigcup_{j \in K_i} S^j(T_1)$, $i = 1, \dots, \ell$ for some $\alpha^i(t), \beta^i(t) \in \mathbb{R}^3$.

Each such neighborhood necessarily intersects Ω^c, where $\varphi(\cdot, 0) = 0$ $\Rightarrow$ $\alpha^i = \beta^i = 0$ in the nbd. So, $\varphi(x, t) = 0$ $\forall x$ in a nbd. of of $\bigcup_{j \in K_i} S^j(T_1)$, $i = 1, \dots, \ell$.

This holds $\forall t \in (T_1, T_2)$, so $\varphi \in C_c^\infty(\Omega_2 \times (T_1, T_2))$, which proves the claim.

Step 4 – Continuation

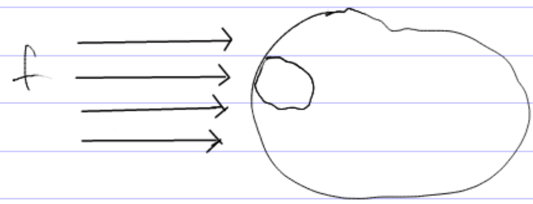
Now we just use the continuation lemma to glue the solns together to get one on $(0, T_2)$.

If $T_2 > T$, then we're done. Otherwise $d(\{S^i(T_2)\}) = 0$, and we iterate. Since every collision strictly decreases the number of solids, we will eventually pass T after finitely many iterations.

$\square$

Remarks:

1) After each collison the solid pieces merge and stay stuck together forever. This is not so physically reasonable since this holds regardless of the force f. For instance one would expect strong f to unstick the solids from the boodary.



2) Eventually either the collisions stop or the merging process eliminates all of the solids in the interior. In this may we can say the maximal existence time is $T = +\infty$, so the solutions are global.

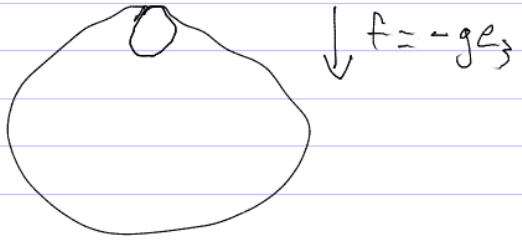
§ 4 – Much ado about nothing?

Feireisl's paper can be viewed as a guide for what to do _if_ collisions occur in order to still produce a global-in-time solution. As remarked above, the resolution leads to some behavior that is rather questionable from a physical point of view. This begs the question:

Do collisions actually occur?

The complete answer in general remains unknown, but some partial results exist which indicate that the answer is **no.**

Before listing these, let's mention the contradictory evidence. To Feireisl's credit, his paper is not addressing a vacuous technical problem. Indeed, in the precursor to the JEE paper, "On the motion of ridid bodies in a viscous compressible fluid," (ARMA, 2003) he explicitly constructs a solution to the compressible problem in which a dense ball sticks to the roof of a cavity regardless of the force of gravity.



Next, let's mention "Behavior of a rigid body in an incompressible viscous fluid near the boundary" by Starovoitov (In Free Boundary Problems, 2004), which provides information on collisions in terms of the integrability of $\mathbb{D}u$ and the regularity of the solids. Among the results we find three nice things:

1) If the bodies are $C^{1,1}$, then if a collision occurs, the bodies must have zero velocity. This is for weak solutions.
2) If the bodies are $C^{1,1}$ and one has a strong solution, then collision is impossible.
3) In 2D, one can construct a forcing $f \in L^2((0,T); H^{-1})$ so that a disk inside another will actually collide w/ the boundary (weak solns.).

OTOH, "Lack of collision between solid bodies in a 2D incompressible viscous flow" by Hillairet (CPDE, 2007) together with the thesis of Hesla, "Collisions of smooth bodies in viscous fluids," show that a disk inside another won't collide if there is no forcing or if $f \in L^2((0,T); L^2)$. The same holds holds for a disk above a flat plane in both 2D and 3D (the latter is in Hillairet-Takahashi (Siam JMA, 2009)).

One final point: Since one expects bodies to collide and to bounce, it's reasonable to think that there is something wrong with the model. Some people believe the problem lies in the no slip condition

$$\begin{cases} u = 0 & \text{on } \partial\Omega \\ u = v^i & \text{on } \partial S^i(t). \end{cases}$$

The argument goes that these should be replaced by Navier slip conditions:

$$\begin{cases} u \cdot \nu = 0, & u \wedge \nu = -\beta_{\partial\Omega} (\mathbb{D}u\,\nu) \wedge \nu \quad \text{on } \partial\Omega \\ (u - v^i) \cdot \nu = 0, & (u - v^i) \wedge \nu = -\beta_i (\mathbb{D}u\,\nu) \wedge \nu \quad \text{on } \partial S_i(t) \end{cases}$$

Where the β_i, $\beta_{\partial\Omega}$ are the slip parameters. Weak solns were constructed by Gerard-Varet – Hillairet (arXiv: 1207.0469) and were shown to admit finite time collision by the same authors in ESIAM Math. Model. Numer. Anal., 2012.