**21-465 Topology and Geometry** **Exam 1 Questions** **D. Handron**

The exam is open book and open notes. If you wish to use any other resource, you should check with me before doing so. Please feel free to ask me any questions you may have, particularly abaout confusing wording or typo's.

1. **Basic Concepts**

   (a) Let $X$ be a connected topological space and $Y$ a space with the discrete topology. Suppose that $f : X \to Y$ is continuous. Show that $f$ is a constant function.

   (b) **(29.3)** Let $X$ be a locally compact topological space, and $f : X \to Y$ a continuous map. Is $f(X)$ necessarily locally compact? What if $f$ is both continuous and open?

2. **Homotopies and the Fundamental Group**

   (a) **(52.3)** Let $X$ be a topological space, $A \subseteq X$ and $a_0 \in A$. Suppose that $r : X \to A$ is a retraction. Show that $r_* : \pi_1(X, a_0) \to \pi_1(A, a_0)$ is surjective.

   (b) Let $X$ be a simply connected topological space. Suppose $\alpha, \beta : [0, 1] \to X$ are paths in $X$ satisfying $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$. Show that $\alpha \simeq_p \beta$.

3. **Covering Spaces**

   (a) **(54.8)** Let $p : E \to B$ be a covering map, and suppose that $E$ is path connected and $B$ is simply connected. Show that $p$ is a homeomorphism.

   (b) **(53.3)** Let $B$ be a connected topological space and $p : E \to B$ a covering map. Show that if $p^{-1}(b_0)$ has $k$ elements for some $b_0 \in B$, then $p^{-1}(b)$ has $k$ elements for every $b \in B$. [In this case $p$ is said to be a *k-fold covering* of $B$.]

4. **Topological Groups**

   A *topological group* is a group that is also a $T_1$ topological space (i.e. a space satisfying the $T_1$ axiom) with the following properties: (1) The group operation gives a continuous map from $G \times G$ to $G$ and, (2) The map from $G$ to $G$ carrying each element to its inverse is continuous.

   (a) **(p. 145, #1)** Let $H$ be a set that has a group structure and a topological structure satisfying the $T_1$ axiom. Show that $H$ is a topological group if and only if the map $j : H \times H \to H$ given by $(a, b) \mapsto ab^{-1}$ is continuous.

   (b) **(53.7)** Let $G$ be a topological group with operation $\cdot$ and identity $e$. Let $\Omega(G, e)$ denote the set of all paths in $G$ from $e$ to $e$. For $f, g, \in \Omega(G, e)$ define a path $f \otimes g$ by the rule

   $$(f \otimes g)(s) = f(s) \cdot g(s).$$

      i. Show that this operation makes $\Omega(G, e)$ into a group, and induces a group operation on the set $\pi_1(G, e)$.

      ii. Show that the two group operations $*$ and $\otimes$ on $\pi_1(G, e)$ agree. [Hint: consider the product $(f * e_e) \otimes (e_e * g)$.]

      iii. Show that $\pi_1(G, e)$ is abelian.