21-241 **Matrix Algebra** **D. Handron**

Final Exam Review
May 4, 2004 **Name:**__________________________________

Closed book and notes; calculators not permitted.

1. **(20 points)** Let T be a linear transformation so that

$$T\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}0\\1\\0\end{bmatrix}, T\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\1\end{bmatrix}, T\begin{bmatrix}1\\0\\1\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix}$$

   (a) Find a matrix A so that $T(\mathbf{x}) = A\mathbf{x}$. (Hint: $A = [T(\mathbf{e}_1), T(\mathbf{e}_2), T(\mathbf{e}_3)]$).

   (b) Is the linear transformation T invertible? If yes, what is the matrix of T^{-1}, if no, why not?

2. **(20 points)** Let $M = \begin{bmatrix}1&0&2\\0&1&5\end{bmatrix}$.

   (a) Find a basis for $\text{Col}(M)$.

   (b) Find a basis for $\text{Nul}(M)$.

   (c) What is the dimension of $\text{Nul}(M^T)$? the dimension of $\text{Col}(M^T)$?

3. **(20 points)** Let

$$\mathbf{v}_1=\begin{bmatrix}4\\2\\1\\2\end{bmatrix}, \mathbf{v}_2=\begin{bmatrix}2\\4\\1\\4\end{bmatrix}, \mathbf{v}_3=\begin{bmatrix}1\\2\\0\\-4\end{bmatrix}.$$

   (a) Find an orthonormal basis for $\text{span}(\mathbf{v}_1, \mathbf{v}_2)$.

   (b) Find an orthonormal basis for $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

   (c) Find the QR factorization for $A = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$.

4. **(20 points)** Consider the matrix $A = \begin{bmatrix}1&k\\1&1\end{bmatrix}$.

   For what values of k is A invertible?

5. **(20 points)** Let A and B be $n \times n$ matrices, S is an invertible $n \times n$ matrix. $B = S^{-1}AS$.

   (a) Show that A and B have the same eigenvalues.

   (b) What is the relation between the eigenvectors of A and B?

6. **(20 points)**

   Sketch the curve defined by

$$Q(x_1, x_2) = 6x_1^2 + 4x_1x_2 + 3x_2^2 = 1$$

   by the following steps.

   (a) Rewrite the quadratic form $Q(x_1, x_2)$ as $Q(x_1, x_2) = \mathbf{x}^T A\mathbf{x}$.

   (b) Find out the principal axes and diagonalize A.

   (c) Give the formula of the curve in the coordinate system defined by the principal axes and then sketch the curve $Q(x_1, x_2) = 1$.

7. **(20 points)** Let

$$A = \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix}.$$

   (a) Find the singular values of A, σ_1 and σ_2 with $\sigma_1 \geq \sigma_2$.

   (b) Find an orthonormal basis $\mathbf{v}_1, \mathbf{v}_2$ for R^2 so that $\|A\mathbf{v}_1\| = \sigma_1$, $\|A\mathbf{v}_2\| = \sigma_2$ and $A\mathbf{v}_1 \cdot A\mathbf{v}_2 = 0$.

   (c) Show that for any unit vector $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2, (c_1^2 + c_2^2) = 1$,

$$\sigma_2 \leq \|A\mathbf{v}\| \leq \sigma_1.$$

8. **(20 points)** Let

$$A = \begin{bmatrix} 2 & 0 \\ 6 & -1 \end{bmatrix}$$

   (a) Find the eigenvalues and eigenvectors of A.

   (b) Plot several trajectories of the discrete dynamical system

$$\mathbf{x}_{k+1} = A\mathbf{x}_k$$

   (c) Let $\{\mathbf{x}_k\}$ be a solution of the difference equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$, $\mathbf{x}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Compute $\{\mathbf{x}_1\}$ and find a formula for $\{\mathbf{x}_k\}$.

9. **(20 points)**

Let $\mathcal{P}_4$ denote the linear space of all polynomials of degree ≤ 4. Which of the following subsets of $\mathcal{P}_4$ are subspaces of $\mathcal{P}_4$?

(a) $\{p(t) : \int_0^1 p(t)dt = 0\}$.

(b) $\{p(t) : p(-t) = p(t), \text{ for all } t\}$.

(c) $\{p(t) : p'(0) = 4\}$. ($'$ is the derivative.)

10. **(20 points)**

Let

$$\mathbf{u}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Note that $\mathbf{u}_1, \mathbf{u}_2$, and $\mathbf{u}_3$ form an orthonormal basis for R^3. Let

$$A = 3\mathbf{u}_1\mathbf{u}_1^T + 7\mathbf{u}_2\mathbf{u}_2^T + 9\mathbf{u}_3\mathbf{u}_3^T.$$

(a) Show that A is symmetric. (Hint: $(A+B)^T = A^T + B^T$)).

(b) Find $A\mathbf{u}_1$, $A\mathbf{u}_2$, and $A\mathbf{u}_3$.

(c) Diagonalize A. That is, write A as

$$A = S \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} S^{-1},$$

for some matrix S.

11. **(15 points)** Let

$$A = \begin{bmatrix} -1 & 1 & -8 \\ 2 & 0 & 6 \\ 2 & -1 & 11 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

Is $\mathbf{v}$ in $\text{im}(A)$?

12. **(15 points)** Let

$$A = \begin{bmatrix} 1 & -1 & -2 & -10 \\ 0 & 0 & 1 & 7 \\ -2 & 2 & 2 & 6 \end{bmatrix}$$

(a) Find a basis for the nullspace of A.

(b) Find the dimension of $\text{Nul}(A)$.

(c) Find the dimension of $\text{Col}(A)$.

13. **(20 points)** Let

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}. \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} 5 \\ 3 \\ 6 \end{bmatrix}.$$

(a) Find the QR factorization of $A = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3]$.

(b) What does $\det(A)$ tell you about the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$.

(c) In this case what does $\det(R)$ tell you about $\det(A)$? Why?

14. **(20 points)** Consider the matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) Find the eigenvalues of A.

(b) Find a basis for each eigenspace.

(c) What is $A \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$? What is $A^n \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$?

(d) Find an explicit formula for $\mathbf{x}_n = A^n \mathbf{x}(0)$, where $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

15. **(20 points)** Let $A = \begin{bmatrix} -3 & 7 \\ 7 & -3 \end{bmatrix}$. Find an orthogonal matrix S and diagonal matrix D such that $A = SDS^{-1}$.

16. **(10 points)** Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$, $\sigma_1 = 3$, and $\sigma_2 = 7$. Note that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthonormal set, as is $\{\mathbf{v}_1, \mathbf{v}_2\}$. Let $A = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T$.

(a) Show that $A^T A = \sigma_1^2 \mathbf{v}_1 \mathbf{v}_1^T + \sigma_2^2 \mathbf{v}_2 \mathbf{v}_2^T$.

(b) Show that $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of $A^T A$ with eigenvalues σ_1^2 and σ_2^2 respectively.

(c) Show that $\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1$, and $\mathbf{u}_2 = \frac{1}{\sigma_2} A\mathbf{v}_2$.

(d) Find the singular value decomposition $A = U\Sigma V^T$.