Math Studies **Exam 1** **J. Cummings and D. Handron**

This is an open book, open notes exam. You may consult any published references you like, provided you cite all sources you cite. You may not consult with your classmates. Any questions you may have should be directed to Dr. Cummings or Dr. Handron.

1. (Limit Comparison Test) Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. Show that if
$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$
where $0 < c < \infty$, then either both series converge or both series diverge.

2. If E is a nonempty subset of a metric space X, define the distance from $x \in X$ to E by
$$\rho_E(x) = \inf_{z\in E} d(x, z).$$
   (a) Show that $\rho_E(x) = 0$ if and only if $x \in \overline{E}$.
   (b) Also show that ρ_E is a uniformly continuous function on X , by showing that
$$|\rho_E(x) - \rho_E(y)| \leq d(x, y)$$
   for all $x \in X$ and $y \in Y$.
   (c) Now let K and F be disjoint sets in a metric space X. Suppose that K is compact and F is closed. Prove that there exists a $\delta > 0$ such that $d(p, q) > \delta$ if $p \in K$ and $q \in F$.
   (d) Show that the conclusion may fail for two disjoint closed sets if neither is compact.

   (cf. Rudin, p. 101 #20 and 21. You may want to use some of the hints there.)

3. Given metric spaces (X, d_X) and (Y, d_Y), a function $d_{X\times Y} : X \times Y \to \mathbb{R}$ may be defined by setting
$$d_{X\times Y}((x_1, y_1), (x_2, y_2)) = \left[d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2\right]^{1/2}$$
   (a) Show that $d_{X\times Y}$ is a *metric* on $X \times Y$.
   (b) Also, show that in the case where X is compact, $f : X \to Y$ is continuous if and only if the graph of f is a compact subset of $(X \times Y, d_{X\times Y})$.

4. (The Fundamental Group) If X is a metric space, continuous function $\alpha : [0, 1] \to X$ is called a *path* in X. A path α in X is called a *loop* if $\alpha(0) = \alpha(1)$.

   Two loops α and β are said to be *path homotopic* if
$$\alpha(0) = \alpha(1) = \beta(0) = \beta(1) = x,$$

and there is a continuous map $H : [0,1] \times [0,1] \to X$ satisfying

$$\begin{aligned} H(0,t) &= \alpha(t) \\ H(1,t) &= \beta(t) \\ H(s,0) = H(s,1) &= x \end{aligned}$$

In this case, the function H is called a *homotopy* and x is called the *basepoint* of the paths α and β. If α is path homotopic to β, we write $\alpha \simeq \beta$.

(a) Show that $\simeq$ is an equivalence relation. Denote the equivalence class of α by $[\alpha]$.

(b) If α and β share the same basepoint, their *product* is defined to be

$$(\alpha * \beta)(t) = \begin{cases} \alpha(2t) & 0 \le t \le \frac{1}{2} \\ \beta(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

Show that $\alpha * (\beta * \gamma)$ and $(\alpha * \beta) * \gamma$ are different paths in general, but that

$$[\alpha * (\beta * \gamma)] = [(\alpha * \beta) * \gamma].$$

(c) As we discussed in class the set $\{[\alpha] | \alpha \text{ is a path in } X\}$ with the operation $*$ is the fundamental group of X relative to x, denoted $\pi_1(X, x)$.

Suppose that X and Y are metric spaces, $x \in X$ and $y \in Y$. Show that a continuous map $f : X \to Y$ satisfying $f(x) = y$, induces a induces a group homomorphism $f_* : \pi_1(X, x) \to \pi_1(Y, y)$.

5. (a) Show that if a group G acts on a set X, then this action determines a group action of G on the power set $\mathcal{P}(X)$. Specifically, show that for $g \in G$ and $A \subseteq X$

$$gA = \{ga | a \in A\}$$

is a group action.

The group $\mathbb{Z}^2$ acts on the metric space $\mathbb{R}^2$ by

$$(m,n)(x,y) = (x+m, y+n).$$

(b) Suppose $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$. Show that if for some $(m,n) \in \mathbb{Z}^2$ $(m,n)\mathbf{x} \in N_\delta(\mathbf{y})$, then for every $(m', n') \in \mathbb{Z}^2$, there is an $(h,k) \in \mathbb{Z}^2$ such that

$$(m', n')\mathbf{x} \in N_\delta((h,k)\mathbf{y}).$$

(c) If $\mathbf{x} \in \mathbb{R}^2$, let $\mathfrak{O}_\mathbf{x}$ denote the orbit of $\mathbf{x}$ under the action of $\mathbb{Z}^2$. The collection of all orbits in $\mathbb{R}^2$ will be denoted $\mathbb{R}^2/\mathbb{Z}^2 = \{\mathfrak{O}_\mathbf{x} | \mathbf{x} \in X\}$. Define

$$\Delta(\mathfrak{O}_\mathbf{x}, \mathfrak{O}_\mathbf{y}) = \inf_{\mathbf{z} \in \mathfrak{O}_\mathbf{x}} d(\mathbf{z}, \mathbf{y})$$

Show this is a metric on $\mathbb{R}^2/\mathbb{Z}^2$.

6. Let G be a group and suppose that $H \leq G$, $N \lhd G$, $G = HN$, $H \cap N = \{e\}$. For each $h \in H$ define a map $\phi(h)$ with domain N by $\phi(h)(n) = n^h$.

   (a) Show that $\phi(h) \in Aut(N)$.

   (b) Show that ϕ is a HM from H to $Aut(N)$.

   (c) Show that every element of G has the form hn for a unique pair (h, n) with $h \in H$, $n \in N$.

   (d) Show that $h_1 n_1 h_2 n_2 = (h_1 \times h_2)(\phi(h_2^{-1})(n_1) \times n_2)$ where $h_i \in H$, $n_i \in N$.

7. (This is a kind of converse to the preceding question)

   Let H and N be arbitrary groups, let $\phi : H \to Aut(N)$ be a HM and let $G = \{(h, n) : h \in H, n \in N\}$. Define a binary operation $\times_G$ on G by the equation

$$(h_1, n_1) \times_G (h_2, n_2) = (h_1 \times_H h_2, \phi(h_2^{-1})(n_1) \times_N n_2).$$

   (a) Show that G is a group under $\times_G$.

   (b) Let $H^* = \{(h, e_N) : h \in H\}$ and $N^* = \{(e_H, n) : n \in N\}$.
   Show that $H^* \leq G$, $N^* \lhd G$, $H^* \cap N^* = \{e_G\}$, $G = H^* N^*$.

   (c) Compute the conjugate $(e, n)^{(h,e)}$.

8. Let G be a cyclic group of order n with a generator g. Given $a \in \mathbb{Z}$ let ϕ_a be the map from G to G given by $\phi_a(g^i) = g^{ia}$.

   (a) Show that the AMs of G are precisely the maps ϕ_a where $0 < a < n$ and $gcd(a, n) = 1$. You may assume the fact from elementary number theory that $gcd(a, n) = 1$ iff $ax + ny = 1$ for some integers x and y.

   (b) Suppose that n is an odd prime. Show that G has two AMs ϕ such that $\phi^2 = id$ and describe them.

9. Let p be an odd prime and fill in the details in the following analysis of groups of order $2p$. Suppose G has order $2p$.

   (a) G has a subgroup H of order 2 and a normal subgroup N of order p.

   (b) $G = HN$, $H \cap N = \{e\}$.

   (c) G is either cyclic or isomorphic to the group of symmetries of a regular p-gon.

10. Show that for any group N and any $\phi \in Aut(N)$ there is a group G and an element $g \in G$ such that $N \lhd G$, and $n^g = \phi(n)$ for all $n \in N$.