Meaning in mathematics
–or– Belief as Irrefutability

Fritz Obermeyer

Department of Mathematics
Carnegie-Mellon University

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Goal Supply meaning for higher math
by defining heuristics to learn truth.

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Start with how skeptical computer scientists
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Generalize to how physicists/scientists imagine
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Seek heuristics for mathematical intuition.

Knowledge as sets of facts

Crow Arithmetic. (how many farmers are in the barn)

$$0 + 1 = 1$$

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$0 + 1 = 1 \qquad 1 + 1 = 2$

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$4 - 1 = 3 \quad 3 - 1 = 2 \quad 2 - 1 = 1 \quad 1 - 1 = 0$

this is static hard-wired knowledge

Learning as deduction

Presburger Arithmetic.

$$\frac{}{0 \neq x + 1} \qquad \frac{x + 1 = y + 1}{x = y} \qquad \frac{}{x + 0 = x}$$

$$\frac{}{(x + y) + 1 = x + (y + 1)} \qquad \frac{P(0) \qquad P(x) \implies P(x + 1)}{P(y)}$$

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A maximal deductive theory

Peano Arithmetic. (now with quantifiers)

...first order equational logic...

$$\frac{}{0 \neq \mathsf{x}+1}$$

$$\frac{\mathsf{x}+1=\mathsf{y}+1}{\mathsf{x}=\mathsf{y}} \qquad \frac{\mathsf{P}(0) \qquad \forall \mathsf{x}.\ \mathsf{P}(\mathsf{x}) \implies \mathsf{P}(\mathsf{x}+1)}{\forall \mathsf{y}.\ \mathsf{P}(\mathsf{y})}$$

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PA is analogy-complete among deductive systems...

Knowledge relates via analogy

Interpretation of rationals $\langle \mathbb{Q}, \leq, +, \times \rangle$ in PA.
(define addition, multiplication, division, then pairing)

$$\langle \mathsf{x}, \mathsf{y} \rangle = \mathsf{y} + (\mathsf{x} + \mathsf{y})(\mathsf{x} + \mathsf{y} + 1)/2$$

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$$\text{rational}(\langle x, y \rangle) \iff y \neq 0$$

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$$\begin{aligned} \text{less}(\langle w, x \rangle, \langle y, z \rangle) \iff & \quad \text{rational}(\langle w, x \rangle) \\ & \text{and } \text{rational}(\langle y, z \rangle) \\ & \text{and } wz \leq xy \end{aligned}$$

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$$\text{add}(\langle w, x \rangle, \langle y, z \rangle) = \langle wz + xy, xz \rangle$$

$$\text{mult}(\langle w, x \rangle, \langle y, z \rangle) = \langle wy, xz \rangle$$

How far can deduction get us?

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Far there are deduction systems into which all others can be interpreted.
(deduction = Σ^0_1,
and there are Σ^0_1-complete sets)

...but not so far...
(by Gödel's 1st incompleteness theorem)

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But as scientists, we can learn such facts using the scientific method.

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What is belief?

Meaning as falsifiability (a la Popper)

We believe what has not been falsified;

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(refutable-in-the-limit = Π^0_2)

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Very Far ...but first some theory...

Aside: descriptive complexity

(hierarchy picture)

Δ^0_1: decidable

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(x can make calls to h_1)

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Δ^0_ω: $d_\omega(x, n) = t_n(x)$

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Δ^0_ω: $d_\omega(x, n) = t_n(x)$
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Aside: descriptive complexity

(hierarchy picture)

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(x can make calls to h_1)
Π_2^0: $1 - t_2(x) \quad = $ "does program $x(h_1)$ not halt"
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Δ_ω^0: $d_\omega(x, n) = t_n(x)$
Σ_ω^0: $t_\omega(x) \quad = $ "does program $x(d_\omega)$ halt"
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Δ_1^1: "infinity"

Aside: descriptive complexity

(hierarchy picture)

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Δ^0_ω: $\mathsf{d}_\omega(\mathsf{x}, \mathsf{n}) = \mathsf{t}_\mathsf{n}(\mathsf{x})$
Σ^0_ω: $\mathsf{t}_\omega(\mathsf{x}) \;=$ "does program $\mathsf{x}(\mathsf{d}_\omega)$ halt"
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Δ^1_1: "infinity"
Π^1_1: $\mathsf{T}_1(\mathsf{x}) \;=$ "does $\mathsf{x}(\mathsf{s})$ halt on every stream s"
$\vdots$

How far can deduction get us?

Far there are deduction systems
into which all others can be interpreted.
(deductive = Σ^0_1,
and there are Σ^0_1-complete sets)

...but not so far...

- No decidable system can explain all other deduction systems. (Σ^0_1-complete is beyond Δ^0_1)
- Every analogy-complete deductive system expresses unprovable statements. (Σ^0_1 is not closed under complement)

How far can science get us?

Far there are refutation systems
into which all others can be interpreted.
(refutable = Π^1_1,
and there are Π^1_1-complete hypotheses)

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- No refutable theory can explain all other refutation systems. (Π^1_1-complete is beyond Δ^1_1)
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Aside: implications for physics

physically meaningful = falsifiable
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But there is no Δ^1_1-complete theory.

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or maybe: there is no coordinate-free GUT

What I am doing...

Asking ...

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How does step (1) work, in the Scientific Method? (making a guess)

from Proof systems to Belief systems

formalizing...

Proof Systems $\langle \mathbb{T},\ \mathsf{T}_0,\ +,\ \mathsf{con}\!:\!\Pi^0_1,\ \ \vdash\ \ :\!\Sigma^0_1 \rangle$

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...completion, limits, forcing...

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Theorem

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There is an ambiguous belief system
whose limits are uniformly Π^1_1*-complete.*

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Theorem

Step (1) of the scientific method is as hard as it gets (Δ^1_1-hard).

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Theorem

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Proof.

If we had a method of guessing, we could construct a limit with only Π^0_2-much more effort. $\square$

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Problem some assumptions only fail in their
lack of sensible complete extension