ITERATED FORCING AND THE CONTINUUM HYPOTHESIS

Appalachian Set Theory Workshop, May 29–30, 2009
Lectures by Todd Eisworth and Justin Tatch Moore
Notes taken by David Milovich

Remark. The notes which follow reflect the content of a two day tutorial which took place at the Fields Institute on 5/29 and 5/30 in 2009. Most of the content has existed in the literature for some time (primarily in the original edition of [10]) but has proved difficult to read and digest for various reasons. The only new material contained in these lectures concerns the notion of a *fusion scheme* presented in Sections 6 and 7 and even this has more to do with style than with mathematics. Our presentation of the iteration theorems follows [4]. The *k-iterability condition* is a natural extrapolation of what appears in [4] and [5], where the iteration theorem for the $\aleph_0$-iterability condition is presented (with a weakening of $< \omega_1$-properness). The formulation of complete properness is taken from [8]. We stress, however, these definitions and theorems are really technical and/or pedagogical modifications of the theorems and definitions of Shelah presented in [10]. Those interested in further reading on the topic of the workshop should consult: [1], [4], [5], [8], [10], and [12]. We would like to thank the anonymous referee for their careful reading and suggesting a number of improvements.

1. Introduction

The focus of the following lectures is on forcing axioms in the presence of the Continuum Hypothesis. Not long after Solovay and Tennenbaum's proof that Souslin's Hypothesis was relatively consistent [11], Jensen showed that Souslin's Hypothesis is relatively consistent with CH (see [3]). While Martin's Maximum provides a provably optimal consistent forcing axiom [6], it is still not clear whether there is an optimal forcing axiom which is consistent with CH.[1] Over the last three decades, Shelah and others developed a number of sufficient conditions for establishing that consequences of forcing axioms are consistent with CH. The purpose of these lectures is to present these conditions in a form which strikes some balance between utility and ease of understanding.

We will begin by stating an open problem which seems to require new ideas and at the same time serves to illustrate what can be accomplished through existing methods. If X and Y are countable subsets of ω_1 which are closed in their suprema, then we say that X *measures* Y if there is an $\alpha_0 < \alpha = \sup X$ such that $X \cap (\alpha_0, \alpha)$ is contained in or disjoint from Y. *Measuring* is the assertion that whenever $\langle D_\alpha : \alpha \in \omega_1\rangle$ is a sequence with D_α a closed subset of α for each $\alpha \in \omega_1$, there is a club $E \subseteq \omega_1$ such that $E \cap \alpha$ measures D_α whenever α is a limit point of E.

---

[1] In September 2009 Aspero, Larson, and Moore announced that there are two Π_2 sentences ψ_1 and ψ_2 in the language of $(H(\omega_2), \in, \omega_1)$ such that it is forcible that $(H(\omega_2), \in, \omega_1)$ satisfy $\psi_i \wedge$CH for $i = 1$ or 2, but such that $\psi_1 \wedge \psi_2$ implies $\neg$CH. This essentially rules out the possibility of a provably optimal forcing axiom which is consistent with CH.

Question 1.1. Is *measuring* consistent with CH?

It is easy to show that $\diamondsuit$ implies that measuring fails. We will also see that there is a canonical partial order for forcing an instance of measuring without adding reals. By the book keeping arguments of [11], the question reduces to showing that an iteration of these partial orders does not add new reals. Dealing with this difficulty — determining when an iteration of forcings does not add reals — will be the central theme throughout these lectures.

In order to demonstrate the type of problem which arises here, let us consider another combinatorial principle. Recall that a *ladder system* is a sequence $\langle C_\alpha : \alpha \in \lim(\omega_1)\rangle$ such that for each $\alpha \in \lim(\omega_1)$, $C_\alpha \subseteq \alpha$ is cofinal and has ordertype ω. Let (U) be the assertion that if $\vec{C}$ is a ladder system and $g : \omega_1 \to 2$, then there is an $f : \omega_1 \to 2$ such that for each $\alpha \in \lim(\omega_1)$,

$$f \restriction C_\alpha \equiv_* g(\alpha).$$

Here $\equiv_* g(\alpha)$ means "takes the constant value $g(\alpha)$ except on a finite set." We will see that there is a partial order which forces an instance of (U) and which does not add new reals. Still, Devlin and Shelah [2] have shown that (U) implies $2^\omega = 2^{\omega_1}$ and in particular that CH fails. To see this, fix a bijection $h : \omega \to \omega \times \omega$ such that $i, j \leq n$ whenever $h(n+1) = (i, j)$. For each $g : \omega_1 \to 2$ construct a sequence of functions $f_n : \omega_1 \to 2$ such that $f_0 = g$ and

$$f_{n+1} \restriction C_\alpha \equiv_* f_i(\alpha + j)$$

whenever $\alpha \geq \omega$ is a limit and $h(n+1) = (i, j)$. Given f_k $(k \leq n)$, f_{n+1} exists by applying (U) to the coloring $\alpha \mapsto f_i(\alpha + j)$ where $h(n+1) = (i, j)$. Now observe that for each limit $\alpha \geq \omega$, $\langle f_n \restriction \alpha : n \in \omega\rangle$ uniquely determines $\langle f_n \restriction \alpha + \omega : n \in \omega\rangle$. Hence, by the transfinite recursion theorem, $\langle f_n \restriction \omega : n \in \omega\rangle$ uniquely determines $\langle f_n : n \in \omega\rangle$ and in particular uniquely determines $g = f_0$. Hence $2^\omega = 2^{\omega_1}$.

In fact Devlin and Shelah showed that the following weak form of $\diamondsuit$ is equivalent to $2^\omega < 2^{\omega_1}$ [2]:

> For all $F\colon 2^{<\omega_1} \to 2$ there exists a $g\colon \omega_1 \to 2$ such that for all $f\colon \omega_1 \to 2$ the set $\{\delta < \omega_1 : F(f \restriction \delta) \neq g(\delta)\}$ is stationary.

Here F can be viewed as a method for coding and *weak diamond* can be viewed as asserting that each of these coding methods fails to code at least some element of 2^{ω_1}. In the example just discussed, $F(f \restriction \delta) = i$ if $f \restriction C_\delta \equiv_* i$ (with $F(f \restriction \delta)$ defined arbitrarily if $f \restriction C_\delta$ is not eventually constant).

2. Proper forcing

Before proceeding it will be useful to review so terminology associated to proper forcing. First, let us agree to the following conventions about forcing. A forcing Q is a partial order with the following additional properties:

- Q is separative, *i.e.*, if $q \not\leq p$, then $\exists r \leq q(r \perp p)$.
- Q has a maximal element.

The first condition is out of convenience for our general discussion of forcing and iterations. There is no loss of generality since we may always replace a given partial order (or even quasi-order) with its separative quotient and the underlying set with its cardinality. Similarly, we may always adjoin a maximal element to the partial order if it is not present. This will frequently be done without further mention. We will also use the following notation:

- If P is a forcing, $G \subseteq P$ is V-generic, and x is in $V[G]$, then $\dot{x}$ denotes a name such that $\dot{x}[G] = x$.
- For each x in V, $\check{x}$ is the canonical name for x.

The choice of names in the first convention is not canonical but will be taken to be when possible (for instance the generic filter does have a canonical name, as do new sets which are explicitly constructed from it).

Recall that if P is a forcing and $\dot{Q}$ is a P-name for a forcing, then the *two-step iteration* is defined by $P*\dot{Q} = \{p*\dot{q} : p \in P,\ \Vdash \dot{q} \in \dot{Q}\}$ and declaring that $p*\dot{q} \leq r*\dot{s}$ if and only if $p \leq r$ and $p \Vdash \dot{q} \leq \dot{s}$.

Regarding elementary submodels, we will adopt the following conventions:

(1) χ always denotes a regular cardinal sufficiently large for the argument at hand;
(2) $H(\chi)$ denotes the sets hereditarily of size less than χ;
(3) $<_\chi$ is a well-ordering of $H(\chi)$.

When discussing forcing at an abstract level, we all always assume that the underlying set of a given forcing is a cardinal. If P is a notion of forcing and $\chi \geq \left(2^{|P|}\right)^+$, then all statements of interest about P are absolute between V and $H(\chi)$ (the reason for our assumption on the underlying set is that then P is necessarily in $H(\chi)$). Recall that if X is an uncountable set, then a *club* in $[X]^\omega$ is a set E of the form $\{Z \in [X]^\omega : f''Z^{<\omega} \subseteq Z\}$ for some $f : X^{<\omega} \to X$. A subset $S \subseteq [X]^\omega$ is *stationary* if it intersects every club in $[X]^\omega$. The countable elementary submodels of $H(\chi)$ form a closed unbounded subset of $[H(\chi)]^\omega$.

Definition 2.1. Our usual situation is that P is a notion of forcing, $P \in N \prec H(\chi)$, and N is countable. We won't explicitly mention all the parameters included in N (*e.g.*, $P, \leq_P, \mathbb{1}_P \in N \prec \langle H(\chi), \in, <_\chi\rangle$). Rather than say all this, we will just say, "*let $P \in N$ as usual*" or "*let N be a suitable model for P.*" Unless explicitly stated otherwise, all elementary submodels are countable.

Definition 2.2. Given $P \in N$ as usual, set $N^P = \{\dot{\tau} : \dot{\tau} \in N \wedge \dot{\tau} \text{ is a } P\text{-name}\}$. If $G \subseteq P$ is generic, then $N[G] \prec H(\chi)[G]$ where $N[G] = \{\dot{\tau}[G] : \dot{\tau} \in N^P\}$.

Is $N[G]$ as above a generic extension of N? This is where properness comes in.

- A condition $q \in P$ is (N,P)-*generic* if $q \Vdash N \cap \dot{G}_P \cap D \neq \varnothing$ for every dense $D \subseteq P$ for which $D \in N$.
- P is *proper* if whenever $P \in N$ as usual and $p \in N \cap P$, there is an (N,P)-generic $q \leq p$.

Remark. This definition is robust with respect to demanding that $\chi = \left(2^{|P|}\right)^+$, that χ be arbitrarily large, that additional parameters be added to $H(\chi)$, and so forth.

Definition 2.3. "$N \cap D$ is predense below q" means that every extension of q has an extension below something in $N \cap D$.

Proposition 2.4. *Suppose $P \in N$ as usual. The following are equivalent.*

(1) *q is (N,P)-generic.*
(2) *$N \cap D$ is predense below q for all dense $D \subseteq P$ from N.*
(3) *If $\mathcal{A}$ is a maximal antichain and $\mathcal{A} \in N$, then $q \Vdash \mathcal{A} \cap N \cap \dot{G}_P \neq \varnothing$.*

The following is a key property of proper forcing. In fact, together with the preservation of properness under countable support iterations, it gives the essential properties of properness.

Proposition 2.5. *If P is proper and $G \subseteq P$ is generic, then every countable set of ordinals in $V[G]$ is covered by a countable set from V. In particular:*

- *P does not collapse ω_1;*
- *P adds a countable sequence of elements of V if and only if it adds a new real.*

Proof. Suppose p is in P and forces $\dot{A}$ is a countable set of ordinals. Let $\dot{\alpha}_n$ be forced by p to be the n^{th} element of $\dot{A}$, $N \prec H(\chi)$ be a suitable model with $p, P, \{\dot{\alpha}_n : n \in \omega\} \in N$, and $q \leq p$ be (N, P)-generic. We claim that $q \Vdash \dot{\alpha}_n \in \check{N}$ for all n. To see this, observe that $D_n = \{r \in P : r \text{ decides a value for } \dot{\alpha}_n\}$, which is dense in P. It follows that $q \Vdash N \cap D_n \cap \dot{G}_P \neq \varnothing$. Let $G \subseteq P$ be generic with $q \in G$. Working in $V[G]$, we have that $\dot{\alpha}_n[G]$ is an ordinal and some $r \in N \cap D_n \cap G$ decides the value of α_n. We recover the ordinal in N from r and $\dot{\alpha}_n$ by elementarity. □

Theorem 2.6. *P is proper if and only if forcing with P preserves stationary subsets of $[X]^\omega$ for any uncountable set X.*

The following are easy observations which give important examples of proper forcings. Recall that a forcing P has the *c.c.c.* if every antichain in P is countable. A forcing P is countably closed if every countable descending sequence in P has a lower bound.

Proposition 2.7. *Every partial order with the c.c.c. is proper.*

Proof. Let P be a c.c.c. forcing and let $P \in N$ be as usual. Every condition is (N, P)-generic because if $\mathcal{A}$ is a maximal antichain from N, then $\mathcal{A} \subseteq N$. □

Proposition 2.8. *Every countably closed forcing is proper.*

Proof. Let P be a countably closed forcing and let $P \in N$ be as usual with $p \in N \cap P$. Build $\langle p_n \rangle_{n<\omega}$ such that $p_0 = p$, $p_{n+1} \leq p_n$, and $p_{n+1} \in D_n \cap N$. Let $q \leq p_n$ for all n. Then q is (N, P)-generic. □

As simple as it is, this last construction provides the template for the constructions to come. In general if a proper forcing does not add new reals, it need not be the case that an arbitrary sequence of conditions has a lower bound. If some additional care is taken in constructing the sequence, however, one can often arrange that the resulting sequence is bounded.

Let P be proper and not add reals. Let N be as usual. Let p be (N, P)-generic. Let $\langle D_n \rangle_{n<\omega}$ enumerate the dense subsets of P from N. Let $\langle d^n_m \rangle_{m<\omega}$ enumerate $D_n \cap N$. We have that $p \Vdash \forall n \ \exists m \ d^n_m \in \dot{G}_P$ because p is (N, P)-generic. Let $\dot{f}$ be a P-name such that $p \Vdash \forall n \ d^n_{\dot{f}(n)} \in \dot{G}_P$. Since P doesn't add reals, we can find $g \in \omega^\omega$ and $q \leq p$ such that $q \Vdash \check{g} = \dot{f}$. For each n, $q \Vdash d^n_{g(n)} \in \dot{G}_P$; hence, $q \leq d^n_{g(n)}$. If this were not the case, q would have an extension $r \perp d^n_{g(n)}$, which would imply that every generic filter G with $r \in G$ would have incompatible elements, which is absurd. So, q is (N, P)-generic in a strong sense: whenever $D \in N$ is dense in P, there is a $d \in N \cap D$ such that $q \leq d$. This motivates the following definition.

- q is *totally (N, P)-generic* if q extends an element of $N \cap D$ for any dense $D \subseteq P$ with $D \in N$.
- P is *totally proper* if whenever N, P are as usual, any $p \in N \cap P$ has a totally (N, P)-generic extension.

Of course being totally (N, P)-generic is equivalent to being a lower bound for a (N, P)-generic filter.

Proposition 2.9. *P is totally proper if and only if P is proper and adds no new reals.*

An important point which we will come to momentarily is that, even in a totally proper forcing P, conditions which are (N, P)-generic need not be totally (N, P)-generic. It is true, however, that every (N, P)-generic condition in a totally proper forcing can be extended to a totally (N, P)-generic condition.

3. Two-step iterations

If P is proper and $\Vdash_P \dot{Q}$ is proper, then $P * \dot{Q}$ is proper: P preserves stationary subsets of $[\lambda]^\omega$ and then Q preserves them, so $P * \dot{Q}$ preserves them. Similarly, if P is totally proper and $\Vdash_P \dot{Q}$ is totally proper, then $P * \dot{Q}$ is totally proper.

Understanding preservation of properties such as properness and total properness in transfinite iterations is more subtle and ultimately requires a finer and more localized analysis of two step iterations. To illustrate this, suppose that P is proper and that P forces $\dot{Q}$ is proper. Let N be as usual with $P * \dot{Q} \in N$. It can be shown that $p * \dot{q}$ is $(N, P * \dot{Q})$-generic if and only if p is (N, P)-generic and $p \Vdash \dot{q}$ is $(N[\dot{G}_P], \dot{Q})$-generic.

This refinement fails for total properness and this is ultimately the source of all of the difficulties which we will encounter in these lectures. There are $N, P, \dot{Q}, p, \dot{q}$ such that P is totally proper, $\Vdash_P \dot{Q}$ is totally proper, p is totally (N, P)-generic, and $p \Vdash \dot{q}$ is totally $(N[\dot{G}_P], \dot{Q})$-generic, but $p * \dot{q}$ is *not* totally $(N, P * \dot{Q})$-generic.

This is best illustrated in an example.

Example 3.1. Let $\vec{C} = \langle C_s : s \in \lim(\omega_1)\rangle$ be a ladder system. Let $g : \omega_1 \to \{0, 1\}$. Does there exist $f : \omega_1 \to \{0, 1\}$ such that for all $\delta \in \lim(\omega_1)$, $f \restriction C_\delta$ is eventually constant with value $g(\delta)$? Generally the answer is 'no' if, for instance, $\diamondsuit$ holds. That is, for a given ladder system one can use a $\diamondsuit$-sequence to predict possible uniformizing functions f and build the desired coloring g. We'll force the answer to be yes, for a given g, without adding new reals. Define P_g to be the collection of all countable approximations to the desired uniformizing function f. Specifically, $\mathrm{dom}(p) = \delta$ for some $\delta < \omega_1$, and if $\alpha \leq \delta$ is a limit ordinal, then $p \restriction C_\alpha$ is eventually constant with value $g(\alpha)$.

Proposition 3.2. *P_g is totally proper and forces the existence of a uniformizing function f for the coloring g.*

The following three lemmas constitute the essence of the proof. Moreover, the role of each is quite typical in arguments of this sort.

Lemma 3.3. *For each $\alpha < \omega_1$, the set of conditions p for which $\alpha \in \mathrm{dom}(p)$ is dense.*

Proof. Suppose that the lemma holds for all $\beta < \alpha$. Suppose $p \in P_g$ and $\alpha \notin \mathrm{dom}(p)$. Let $\mathrm{dom}(p) = \beta + 1$ for some $\beta < \alpha$. If $\alpha = \gamma + 1$, then extend p to

q with $\gamma \in \mathrm{dom}(q)$; extend q to $q \cup \{\langle \alpha, 0\rangle\}$. So, we may assume α is a limit ordinal. Let $\langle \gamma_n \rangle_{n<\omega}$ be strictly increasing with limit α and satisfy $\gamma_0 = \beta$. Build $\langle p_n \rangle_{n<\omega}$ such that $p_0 = p$, $p_{n+1} \leq p_n$, $\gamma_n \in \mathrm{dom}(p_n)$, and $p_{n+1}(\delta) = F(\alpha)$ for all $\delta \in \mathrm{dom}(p_{n+1} \setminus p_n) \cap C_\alpha$. The last requirement can be met because P_g is closed with respect to finite modification. Finally, let $q = \bigcup_{n<\omega} p_n \cup \{\langle \alpha, 0\rangle\}$. □

Lemma 3.4. *Suppose that p_n $(n \in \omega)$ is a strictly descending sequence in P_g. The following are equivalent:*

(1) *p_n $(n \in \omega)$ has a lower bound in P_g.*
(2) *$\cup p_n$ is in P_g.*
(3) *if $\alpha = \mathrm{dom}(\bigcup_n p_n)$, then there is an $\alpha_0 < \alpha$ such that $p_n(\xi) = g(\alpha)$ whenever ξ is in $C_\alpha \cap \mathrm{dom}(p_n)$ with $\alpha_0 < \xi$.*

Proof. Follows from the definitions. □

Proof. (of Proposition 3.2) Let N be as usual with $P_g, g, \text{etc.} \in N$. Let $p \in P_g \cap N$. Let N_k $(k \in \omega)$ be an $\in$-chain of countable elementary submodels of $H(\omega_2)$ such that g and p are in N_0 and $N \cap H(\omega_2) = \bigcup_k N_k$. Let $\langle D_k \rangle_{k<\omega}$ enumerate the dense subsets of P from N such that D_k is in N_k. We build $\langle p_k \rangle_{k<\omega}$ such that $p = p_0$, $p_k \geq p_{k+1} \in N_k \cap D_k$, and, for any $\alpha \in C_\delta \cap \mathrm{dom}(p_{k+1} \setminus p_k)$ where $\delta = N \cap \omega_1$, we have $p_{k+1}(\alpha) = g(\delta)$ if $\delta \in S$.

To see that this can be done, suppose we have p_k and look at $N_k \cap C_\delta \setminus \mathrm{dom}(p_k)$. It's finite and hence an element of N_k. Inside N_k, extend p_k to a condition r such that $C_\delta \cap N_k \subseteq \mathrm{dom}(r)$. Modify r on $C_\delta \cap N_k \setminus \mathrm{dom}(p_n)$ to agree with $g(\delta)$. This modification is finite and hence r remains both in N_k and P_g. Now extend r to $p_{k+1} \in N_k \cap D_k$. p_{k+1} is as desired. By Lemma 3.4, p_k $(k \in \omega)$ has a lower bound as desired. □

Example 3.5. Set $P = \langle 2^{<\omega_1}, \supseteq \rangle$ and let $\dot{g}$ be the generic function coded by P. Define $\dot{Q} = P_{\dot{g}}$ and let N be as usual with $P, \dot{Q} \in N$. Let p be (N, P)-generic with $\mathrm{dom}(p) = N \cap \omega_1$. We claim there is no $\dot{q}$ such that $p * \dot{q}$ is totally $(N, P * \dot{Q})$-generic. To see this, let $\delta = N \cap \omega_1$ and let $\langle \alpha_n \rangle_{n<\omega}$ enumerate C_δ. Let

$$D_n = \{r * \dot{s} \in P * \dot{Q} : r * \dot{s} \text{ decides the value of } \dot{f}(\alpha_n)\}$$

where $\dot{f}$ is the function added by $\dot{Q}$. If some $p * \dot{q}$ could decide every $\dot{f}(\alpha_n)$, then it would also decide $\dot{g}(\delta)$ to be ϵ, which is absurd, for we can extend p to force $\dot{g}(\delta)$ to be $1 - \epsilon$.

This problem can be remedied by requiring p to be totally generic over models above N as well.

Proposition 3.6. *Suppose that P is totally proper and $\dot{Q}$ is a P-name for a totally proper forcing. Let $N_0 \in N_1$ be as usual with $P * \dot{Q} \in N_0$. If p is totally (N_i, P)-generic for $i = 0, 1$, then there is a $\dot{q}$ such that $p * \dot{q}$ is totally $(N_0, P * \dot{Q})$-generic.*

Proof. Set $G_i = \{r \in N_i \cap P : p \leq r\}$. We have the following facts.

(1) $G_0 = N_0 \cap G_1$.
(2) $G_0 \in N_1$.
(3) G_0 has a lower bound $p' \in G_1$.

To see that this last fact is true, observe that the set of conditions which decide a particular value for $N_0 \cap \dot{G}_P$ is dense in P and is in N_1. Choose $p' \in N_1 \cap G_1$ which decides $N_0 \cap \dot{G}_P$—it is decided to be G_0. Since P is separative, p' must extend every element of G_0.

Now fix $\dot{q} \in N_1 \cap \dot{Q}$ such that $\Vdash_P \dot{q}$ is totally $(N_0[\dot{G}_P], Q)$-generic and fix a dense $D \subseteq P * \dot{Q}$ such that $D \in N_0$. There is a P-name D/G_P in N_0 for $\{\dot{s}[\dot{G}_P] : \exists r \in \dot{G}_P \;\; r * \dot{s} \in D\}$, which is dense in Q. Since p is totally (N_1, P)-generic, it forces that $\dot{q}$ decides $(D/G_P) \cap N_0$. Also, p' decides $\dot{G}_P \cap N_0$. Therefore, $p * \dot{q}$ decides $(\dot{G}_P * \dot{G}_Q) \cap D \cap N_0$. $\square$

It is important to note, however, that $p * \dot{q}$ is not $(N_1, P * \dot{Q})$-generic. For longer iterations, we expect to need more models above N_0 than just N_1 and for transfinite iterations we expect to need an infinite tower of models above N_0. This creates a new challenge. In order to describe it, we will need a definition.

Definition 3.7. If χ is a regular uncountable cardinal, a *suitable tower of models* is a $\subseteq$-continuous sequence $\mathcal{N} = \langle N_\xi : \xi < \alpha \rangle$ of countable elementary submodels of $H(\chi)$ such that if $\xi < \alpha$, $\langle N_\eta : \eta \leq \xi \rangle$ is in $N_{\xi+1}$. We will say that $\mathcal{N}$ is suitable for P if N_0 is suitable for P. We will abuse notation and write $P \in \mathcal{N}$ to mean $P \in N_0$. In what follows, $\mathcal{N}$ will always refer to a suitable tower of models.

If $P \in \mathcal{N}$ are as usual, then a condition p is $(\mathcal{N}, P)$-generic if it is (N, P)-generic for each N in $\mathcal{N}$. For finite towers of models $\mathcal{N}$, the existence of $(\mathcal{N}, P)$-generic conditions below elements of $N_0 \cap P$ is already guaranteed by the properness of P. For infinite towers of models, however, this is no longer the case and, stated for towers of height α, this yields the definition of α-properness. We will write *$(< \omega_1)$-proper* to mean *α-proper for every $\alpha < \omega_1$*.

Another important point in the formulation of Proposition 3.6 is that p is required to be *totally* (N_1, P)-generic. If Q satisfies a strengthening of properness, which we will term *complete properness*, then this requirement can be relaxed to p being (N_1, P)-generic. A precise definition of *complete properness* will be given in Section 7.

Proposition 3.8. *Suppose P is totally proper and*

$$\Vdash_P \dot{Q} \text{ is completely proper.}$$

Given $N_0 \in N_1 \in N_2$ as usual and p that is (N_2, P)-generic, (N_1, P)-generic, and totally (N_0, P)-generic, there is a $\dot{q}$ such that $p * \dot{q}$ is totally $(N_0, P * \dot{Q})$-generic.

To illustrate the significance of this change, let us return to Example 3.5. Replace $2^{<\omega_1}$ in the definition of P with its regular open algebra. This forcing generates the same generic extensions as $2^{<\omega_1}$, but also has a well defined join operation. If $N_0 \in N_1 \in N_2$ are suitable models for P as usual and $p \in P$ is totally (N_0, P)-generic and (N_i, P)-generic for $i = 1, 2$, it is possible that p does not decide $\dot{g}(\delta)$ where $\delta = N_0 \cap \omega_1$. To see this, let p_0 and p_1 be totally (N_i, P)-generic for $i = 0, 1, 2$ with $p_0 \restriction \delta = p_1 \restriction \delta$ and $p_0(\delta) \neq p_1(\delta)$. Then $p = p_0 \vee p_1$ is as desired.

The point is that the previous arguments still show there is no $\dot{q}$ such that $p * \dot{q}$ is totally $(N_0, P * \dot{Q})$-generic. Complete properness is in fact designed to avoid this sort of situation which we know from the demonstration in the introduction represents a fundamental obstruction to an iteration theorem for total properness.

We will see that transfinite iterations of forcings which are completely proper and α-proper for every $\alpha < \omega_1$ do not add new reals. We will also see that the degree to which α-properness is needed in this theorem is somewhat of a mystery.

4. Countable support iterations

Definition 4.1. A *countable support (c.s.) iteration* $\mathbb{P} = \langle P_\alpha, \dot{Q}_\alpha \rangle_{\alpha<\varepsilon}$ satisfies the following conditions (which also define P_ε).

- $P_0 = \varnothing$.
- For all $\alpha \le \varepsilon$, elements of P_α are functions with domain α.
- For all $\alpha < \varepsilon$ $P_{\alpha+1}$ is forcing equivalent to $P_\alpha * \dot{Q}_\alpha$ as witnessed by a coordinate preserving function.
- If $\alpha \le \varepsilon$ is a limit ordinal, then P_α is the set of countably supported functions f for which $f \restriction \beta \in P_\beta$ for all $\beta < \alpha$.

If ε is a limit ordinal, then for any generic $G \subseteq P_\varepsilon$, $p \in G$ if and only if $p \restriction \alpha \in G_\alpha$ for all $\alpha < \varepsilon$ where $G_\alpha = \{f \restriction \alpha : f \in G\}$.

Theorem 4.2. *Let $\mathbb{P} = \langle P_\alpha, \dot{Q}_\alpha \rangle_{\alpha<\varepsilon} \in N$ be as usual, $\mathbb{P}$ be a countable support iteration of proper forcings, $\alpha \in N \cap \varepsilon$, q be (N, P_α)-generic, $q \Vdash \dot{p} \in N \cap P_\varepsilon$, and $q \Vdash \dot{p} \restriction \alpha \in \dot{G}_\alpha$. Then there is a $q^\dagger \in P_\varepsilon$ such that $q^\dagger$ is (N, P_ε)-generic, $q^\dagger \restriction \alpha = q$, and $q^\dagger \Vdash \dot{p} \in \dot{G}_\varepsilon$.*

Lemma 4.3. *Suppose P is proper, $\Vdash_P \dot{Q}$ is proper, N is as usual, $P * \dot{Q} \in N$, p is (N, P)-generic, $\dot{\tau}$ is a P-name for a condition in $P * \dot{Q}$ whose first component is forced by p to be in $\dot{G}_P$, and $\dot{\tau} \in N$. There is a $\dot{q}$ such that $p * \dot{q}$ is $(N, P * \dot{Q})$-generic and $p * \dot{q} \Vdash \dot{\tau} \in \dot{G}_{P*\dot{Q}}$.*

Proof of theorem. Proceed by induction on ε. The lemma is trivially true if $\varepsilon = 0$ and follows from Lemma 4.3 if ε is a successor ordinal. Now assume ε is a limit ordinal and let $\langle \alpha_n \rangle_{n<\omega}$ be strictly increasing, cofinal in $N \cap \varepsilon$, and such that $\alpha_0 = \alpha$. Let $\langle D_n \rangle_{n<\omega}$ enumerate the dense subsets of P_ε from N. Define $\langle q_n, \dot{p}_n \rangle_{n<\omega}$ such that:

(1) $\dot{p}_0 = \dot{p}$ and $q_0 = q$
(2) q_n is (N, P_{α_n})-generic
(3) $q_n \Vdash \dot{p}_{n+1}$ is a P_{α_n}-name for a condition in $P_\varepsilon \cap N$
(4) $q_n \Vdash \dot{p}_n \ge \dot{p}_{n+1} \in D_n$
(5) $m < n \Rightarrow q_n \restriction \alpha_m = q_m$
(6) $q_n \Vdash \dot{p}_n \restriction \alpha_n \in \dot{G}_{\alpha_n}$

Given $\dot{p}_n, q_n$, let $G_n \subseteq P_{\alpha_n}$ be generic with $q_n \in G_n$, so that $\dot{p}_n$ is interpreted as p_n with $p_n \restriction \alpha_n \in G_n$. The set of restrictions to α_n of conditions in D_n which extend p_n is dense below $p_n \restriction \alpha_n$ in P_{α_n}. This set is also in $N[G_n]$, so there exists $p_{n+1} \le p_n$ such that $p_{n+1} \in N \cap D_n$ and $p_{n+1} \restriction \alpha_n \in G_n$. Let $\dot{p}_{n+1}$ be a P_{α_n}-name forced by q_n to have the above properties of p_{n+1}. Now apply our induction hypothesis to $\mathbb{P} \restriction \alpha_{n+1}, \alpha_n, q_n, \dot{p}_{n+1} \restriction \alpha_{n+1}$ to get $q_{n+1} \in P_{\alpha_{n+1}}$ such that (2), (5), and (6) hold.

Let $q^\dagger = \bigcup_{n<\omega} q_n$ inside of P_ε. Strictly speaking, this union has domain $\sup(N \cap \varepsilon)$, which may be less than ε, but we extend the domain to all of ε without increasing the support. It suffices to show that $q^\dagger \Vdash \dot{p}_n \in \dot{G}_{P_\varepsilon}$ for all n. We do this by proving that $q^\dagger \Vdash \dot{p}_n \restriction \alpha_m \in \dot{G}_{P_{\alpha_m}}$ for all m. Let $G \subseteq P_\varepsilon$ be generic with $q^\dagger \in G$. It follows that $q_m \in G \restriction \alpha_m$ for each m. Given n, we know that $p_n \restriction \alpha_n \in G \restriction \alpha_n$

by our construction. If $m > n$, we know that $p_m \restriction \alpha_m \in G \restriction \alpha_m$. Since $p_m \leq p_n$, we have $p_m \restriction \alpha_m \leq p_n \restriction \alpha_m$. Hence, $p_n \restriction \alpha_m \in G \restriction \alpha_m$; hence, $p_n \in G_{P_\varepsilon}$. Thus, $q^\dagger \Vdash \dot{p}_n \in \dot{G}_{P_\varepsilon}$. □

Recall the following definition:

Definition 4.4. Q is α-proper if every $p \in Q$ extends to an $(\mathcal{N}, Q)$-generic condition, for any tower of models $\mathcal{N}$ of length α where $p \in N_0 \cap Q$ and $Q \in N_0$.

Note that Q is totally proper and α-proper if and only if Q is "totally α-proper" (i.e. suitable towers of models admit sufficiently many conditions which are totally generic for all of their models).

Theorem 4.5. [10] *A countable support iteration of forcings which are α-proper for all $\alpha < \omega_1$ and $\mathbb{D}$-complete for a simple 2-completeness system $\mathbb{D}$ is totally proper.*

Remark. We will only prove this theorem for $\aleph_1$-completeness systems. The corresponding iteration theorem is already sufficient to handle most applications of this theorem and the proof is considerably easier to digest.

Definition 4.6. Given P totally proper and $P \in N$ as usual, let

$$\mathrm{Gen}(N, P) = \{G^* \subseteq N \cap P : G^* \text{ is an } N\text{-generic filter of } N \cap P\},$$
$$\mathrm{Gen}^+(N, P) = \{G^* \in \mathrm{Gen}(N, P) : G^* \text{ has a lower bound}\}, \text{ and}$$
$$\mathrm{Gen}(N, P, p) = \{G^* \in \mathrm{Gen}(N, P) : p \in G^*\}.$$

Given any $\dot{\tau}_0, \ldots, \dot{\tau}_m \in N^P$, $G^* \in \mathrm{Gen}(N, P)$, and Φ a formula, G^* decides if $\Phi(\dot{\tau}_0, \ldots, \dot{\tau}_m)$ holds in the generic extension $N[\dot{G}_P]$. G^* gives us a snapshot of $N[\dot{G}_P]$. In particular, G^* tells us a lot about $\dot{Q} \cap N[\dot{G}_P]$ where $\dot{Q} \in N^P$ is a name for a notion of forcing. G^* also tells us about $\dot{D}$ where $\dot{D} \in N^P$ is a name for a dense subset of $\dot{Q}$.

Definition 4.7. Suppose $G^* \in \mathrm{Gen}(N, P)$ and $\dot{Q}$ is a P-name from N for a totally proper notion of forcing. A sequence $\langle \dot{q}_n \rangle_{n<\omega}$ of names from N^P is an $(N[G^*], \dot{Q})$*-generic sequence* if

- $\Vdash_P \dot{q}_n \in \dot{Q}$,
- $N[G^*] \models \dot{q}_{n+1} \leq \dot{q}_n$ (*i.e.*, $\exists p \in G^* \ p \Vdash \dot{q}_{n+1} \leq \dot{q}_n$), and
- if $\dot{D} \in N^P$ is a name for a dense set in $\dot{Q}$, there exists m such that $N[G^*] \models \dot{q}_m \in \dot{D}$.

Any lower bound for G^* forces that the sequence of interpretations $\langle \dot{q}_n[\dot{G}_P] \rangle_{n<\omega}$ generates an element of $\mathrm{Gen}(N[\dot{G}_P], \dot{Q})$. Notice that G^* doesn't tell us whether $\langle \dot{q}_n[\dot{G}_P] \rangle_{n<\omega}$ has a lower bound. However, if $G^* \in \mathrm{Gen}(M, P)$ where $N \in M$, then G^* can determine if an $(N[G^* \cap N], \dot{Q})$-generic sequence in M has a lower bound.

Definition 4.8. Given a totally proper forcing P, $\Vdash_P \dot{Q}$ is totally proper, and $k \in \{2, 3, 4, \ldots, \omega, \omega_1\}$, we say that $\dot{Q}$ satisfies the *k-iterability condition over P* if the hypotheses below always imply the conclusion below.

Hypotheses:

- $N_0 \in N_1$ are as usual.
- $G^* \in \mathrm{Gen}(N_0, P) \cap N_1$.
- $l^* < 1 + k$.
- $\forall l < l^* \ G^l \in \mathrm{Gen}(N_1, P)$.

- $\forall l < l^* \;\; G^l \cap N_0 = G^*$.
- $\dot{q}$ is a P-name from N_0 for a condition in $\dot{Q}$.

Conclusion: There exists a $(N_0[G^*], \dot{Q})$-generic sequence $\langle \dot{q}_n \rangle_{n<\omega}$ such that $\Vdash_P \dot{q}_0 \leq \dot{q}$ and, whenever some $p \in P$ forces that $\exists l < l^* \;\; N_1 \cap \dot{G}_P = G^l$, p also forces that $\langle \dot{q}_n \rangle_{n<\omega}$ has a lower bound. The *iterability condition* means the 2-*iterability condition.*

Main Theorem. *If $\mathbb{P} = \langle P_\alpha, \dot{Q}_\alpha \rangle_{\alpha<\varepsilon}$ is a CS iteration of totally proper forcings, then P_ε is totally proper provided that:*

(1) $\Vdash_{P_\alpha} \dot{Q}_\alpha$ *is* $<\!\omega_1$*-proper.*
(2) $\dot{Q}_\alpha$ *satisfies the iterability condition over* P_α*.*

The connection to complete properness (which we have yet to define) is the following proposition.

Proposition 4.9. *If P is totally proper and P forces $\dot{Q}$ is a k-completely proper forcing, then $P * \dot{Q}$ satisfies the k-iterability condition.*

The proof which will be presented is only for the ω_1-iterability condition. While an assumption like the iterability condition is clearly necessary in light of the fact that statements such as (U) are not consistent with CH, the role of α-properness is less clear. This condition can be weakened [5] or replaced by a seemingly unrelated condition [10, XVIII]. It can not be dropped altogether in light of an example of Shelah which we will discuss in Section 9 below.

5. Limitations of $(<\omega_1)$-proper forcing

$(<\omega_1)$-properness is not merely a technical assumption satisfied by all forcings of interest. Recall the following definition.

Definition 5.1. [9] $\clubsuit$ asserts that there exists a ladder system $\vec{C}$ such that for all $X \in [\omega_1]^{\aleph_1}$ there exists δ such that $C_\delta \subseteq X$. *Club guessing (on ω_1)*, denoted by $\clubsuit_C$, asserts the same as $\clubsuit$ except that X is required to be a club.

$\clubsuit$ ($\clubsuit_C$) is equivalent to $\clubsuit$ (respectively $\clubsuit_C$) with the demand that there exist stationarily many δ as above. $\Diamond$ is equivalent to the conjunction of $\clubsuit$ and CH and it is well known and easily demonstrated that MA_{ω_1} implies that $\clubsuit$ fails. On the other hand, $\clubsuit_C$ is a much weaker assumption. It is well known that if Q is a c.c.c. forcing, then every club in a generic extension by Q contains a club from the ground model. In particular, $\clubsuit_C$ is preserved by c.c.c. forcing. Since it is possible to force $\mathrm{MA} + \neg\mathrm{CH}$ with a c.c.c. forcing, $\clubsuit_C$ is consistent with MA_{ω_1}.

The following proposition of Shelah connects this with $(<\omega_1)$-properness.

Proposition 5.2. *Suppose that $\langle C_\alpha : \alpha \in \lim(\omega_1) \rangle$ is a $\clubsuit_C$-sequence and Q is an ω-proper forcing. Forcing with Q preserves that $\langle C_\alpha : \alpha \in \lim(\omega_1) \rangle$ is a $\clubsuit_C$-sequence.*

Proof. Suppose that $\dot{E}$ is a Q-name for a club, q is in Q, and let $\langle N_\xi : \xi < \omega_1 \rangle$ be a suitable tower of models with q, $\dot{E}$, and Q in N_0. Define $D = \{\xi : N_\xi \cap \omega_1 = \xi\}$ and let α be such that $C_\alpha \subseteq D$. Since Q is ω-proper, there is a $\bar{q} \leq q$ which is (N_ξ, Q)-generic for all ξ in C_α. This implies that $\bar{q}$ forces ξ is in $\dot{E}$ for all ξ in C_α and hence that $\check{C}_\alpha \subseteq \dot{E}$. $\square$

A similar argument shows that if measuring fails and Q is an ω-proper forcing, then measuring fails after forcing with Q. Notice that *measuring* can be viewed as the ultimate failure of club guessing. If $\vec{D}$ is a ladder system and E is a club, then for club-many $\delta \in E$ we have $\mathrm{otp}(\delta \cap E) = \delta$, so $E \cap (\delta_0, \delta) \subseteq D_\delta$ is impossible for all $\delta_0 < \delta$, and hence measuring implies that a tail of E misses D_δ. Thus, measuring implies that there is a club $F \subseteq E$ not guessed by $\vec{D}$. This remains true even for sequences where the map $\alpha \mapsto \mathrm{otp}(D_\alpha)$ is regressive.

6. Methods for verifying $(< \omega_1)$-properness

We will now turn to a framework for proving that forcings satisfy conditions such as total $(< \omega_1)$-properness and complete properness. The purpose is not to reformulate these conditions or simplify their statement. Rather it is to provide a sufficient criteria — analogous to the existence of an Axiom A structure — which is both easy to verify and sufficiently general to accommodate the important classes of examples of totally proper forcings.

Definition 6.1. Let Q be a fixed forcing notion with order $\leq$. A *fusion scheme* on Q is an indexed family of partial orders $\leq_\sigma$ $(\sigma \in X^{<\omega})$ such that the following conditions are satisfied:

1. $\leq_\emptyset$ is $\leq$ and if $\sigma \subseteq \tau$, $q \leq_\tau p$ implies $q \leq_\sigma p$;
2. Player II has a winning strategy in $G(Q, \vec{\leq}, M)$ whenever M is a suitable model for Q and $\vec{\leq}$.

The game $G(Q, \vec{\leq}, M)$ is defined as follows. In the n^{th} inning, Player I plays q_n in $Q \cap M$ and Player II responds by playing σ_n in $X^{<\omega}$. The players are required to play so that $q_{n+1} \leq_{\sigma_n} q_n$ and $\sigma_n \subseteq \sigma_{n+1}$; the first player to break one of these rules loses. If q_n $(n \in \omega)$ is the result of a play of the game in which the players followed the rules, then Player II wins if either $\{q_n : n \in \omega\}$ does not generate an (M, Q)-generic filter or else there is a $\bar{q}$ with $\bar{q} \leq q_n$ for all $n \in \omega$.

This definition becomes of interest only when additional conditions are placed on the scheme. Before proceeding, let us see how Example 3.1 fits into this framework.

Let $\langle C_\alpha : \alpha \in \lim(\omega_1)\rangle$ be a ladder system and let $g : \omega_1 \to 2$ be a function. Let P_g be the collection of all countable partial uniformizing functions with the order $\leq$ of extension. Set $X = \lim(\omega_1)$ and if σ is in $X^{<\omega}$, define $q \leq_\sigma p$ iff whenever ξ is in $\mathrm{dom}(q) \setminus \mathrm{dom}(p)$ and $i < |\sigma|$ is minimal such that ξ is in $C_{\sigma(i)}$, $q(\xi) = g(\sigma(i))$. Player II's strategy is to play $\langle M \cap \omega_1 \rangle$ in every round of $G(Q, \vec{\leq}, M)$. It should be clear that this defines a strategy for Player II which is winning from every initial position.

Fix a fusion scheme $\leq_\sigma$ $(\sigma \in X^{<\omega})$ on a forcing Q.

Definition 6.2. The fusion scheme satisfies (TP) if whenever M is a suitable model for the fusion scheme, p is in $Q \cap M$, $D \subseteq Q$ is dense and in M, and σ is in $X^{<\omega}$, there is a q in $D \cap M$ such that $q \leq_\sigma p$.

We have already shown that the fusion scheme defined above on P_g satisfies (TP).

Theorem 6.3. *If a forcing Q admits a fusion scheme satisfying (TP), then Q is totally proper.*

Proof. Let $\leq_\sigma$ $(\sigma \in X^{<\omega})$ be a fusion scheme on Q which satisfies (TP). Suppose that M is a suitable model for the fusion scheme and let q be in M. Let D_n $(n \in \omega)$ enumerate the dense subsets of Q which are in M. Player I plays by the following strategy. To begin the game, Player I plays q_0. If in the n^{th} inning q_n and σ_n were played, Player I picks a q_{n+1} in $D_n \cap M$ which satisfies $q_{n+1} \leq_{\sigma_n} q_n$. This is possible by our assumption that the scheme satisfies (TP). Now let q_n $(n \in \omega)$ be the result of a play of this strategy against Player II's winning strategy. Player I has arranged that $\{q_n : n \in \omega\}$ generates a (M, Q)-generic filter and hence there must be a $\bar{q}$ in Q such that $\bar{q} \leq q_n$ for all n. Such a $\bar{q}$ is totally (M, Q)-generic. □

Next we will consider a condition on a fusion scheme which can be used to verify α-properness for each $\alpha < \omega_1$. First we will need a preliminary definition.

Definition 6.4. If Q is a forcing equipped with a fusion scheme $\leq_\sigma$ $(\sigma \in X^{<\omega})$ and $\langle q_n : n < \omega \rangle$ is a $\leq$-descending sequence in Q, then we say that $\bar{q}$ is a *conservative lower bound* for $\langle q_n : n < \omega \rangle$ if whenever $\sigma \in X^{<\omega}$ is such that $\langle q_n : n < \omega \rangle$ is eventually $\leq_\sigma$-descending, $\bar{q} \leq_\sigma q_n$ for all but finitely many n. Here $\langle q_n : n < \omega \rangle$ is *eventually $\leq$-descending* if there is an m such that $\langle q_n : m < n < \omega \rangle$ is $\leq$-descending.

Returning to Example 3.1, $\bar{q} = \bigcup_n q_n$ is a conservative lower bound for $\langle q_n : n \in \omega \rangle$ provided that $\langle q_n : n < \omega \rangle$ has a lower bound in P_g. This is in fact often the case in practice.

Definition 6.5. A fusion scheme satisfies (A) if the following conditions are met:

1. for any countable $Q_0 \subseteq Q$, $\{\leq_\sigma \restriction Q_0 : \sigma \in X^{<\omega}\}$ is countable;
2. every bounded $\leq$-descending sequence in Q has a conservative lower bound;
3. Player II has a winning strategy in $G(Q, \vec{\leq}, M)$ starting from any initial position.[2]

Notice that it is in fact trivial to verify that P_g with its fusion scheme satisfies (A). The point of this definition and the following theorem is that (A) is usually trivial to verify for a given fusion scheme which satisfies it. This should be contrasted by the often tedious direct verification of α-properness by induction on α (see the proof of Lemma 5.11 of [8]).

Theorem 6.6. *If a forcing Q admits a fusion scheme which satisfies (TP) and (A), then Q is totally α-proper for all $\alpha < \omega_1$.*

Proof. Fix a forcing Q and fusion scheme $\leq_\sigma$ $(\sigma \in X^{<\omega})$ on Q which satisfies (TP) and (A). Let $\langle M_\xi : \xi < \alpha \rangle$ be a tower of models of ordertype α which is suitable for Q and $\leq_\sigma$ $(\sigma \in X^{<\omega})$. As usual, we may assume that α is a limit ordinal. It will be convenient to adopt the convention that $M_\alpha = \bigcup_{\xi<\alpha} M_\xi$ and $M_{\alpha+1}$ is $H(\theta)$. We will verify the following by induction on $\zeta \leq \alpha$:

> If $\xi \leq \zeta$, q is in $Q \cap M_\xi$ and is totally (M_η, Q)-generic for all $\eta < \xi$, and σ is in $X^{<\omega}$, then there is a $\bar{q} \leq_\sigma q$ such that $\bar{q}$ is in $M_{\zeta+1}$ such that $\bar{q}$ is totally (M_η, Q)-generic for all $\eta \leq \zeta$.

[2] I.e. given any partial play of the game, there is a strategy for Player II such that any completion of game play in which II follows this strategy in the remainder of the game results Player II winning.

Let $\xi < \zeta$ be given and assume the induction hypothesis holds for all smaller values of ζ. Since our fusion scheme satisfies (A), $\{\leq_\tau \restriction M_\zeta : \tau \in X^{<\omega}\}$ is countable and hence contained in $M_{\zeta+1}$. Therefore we can choose a $\bar{\sigma}$ in $M_{\zeta+1} \cap X^{<\omega}$ such that $\leq_\sigma \restriction M_\zeta = \leq_{\bar{\sigma}} \restriction M_\zeta$.

If $\zeta = \zeta_0 + 1$ for some ζ_0, then we can first apply the induction hypothesis to find a $q' \leq_\sigma q$ in M_{ζ_0+1} which is totally (M_η, Q)-generic for all $\eta \leq \zeta_0$. We now need to find a $\bar{q}$ in $M_{\zeta+1}$ such that $\bar{q} \leq_\sigma q'$ and $\bar{q}$ is totally-(M_ζ, Q)-generic. To do this, we run the proof of Theorem 6.3 for q' inside $M_{\zeta+1}$ except that we begin with the following position in $G(Q, \vec{\leq}, M_\zeta)$: $\langle q', \bar{\sigma}, q' \rangle$. What results will be a sequence $\langle q_n : n \in \omega \rangle$ in $M_{\zeta+1}$ which is $\leq_{\bar{\sigma}}$-descending, $\leq$-bounded, and which generates a (M_ζ, Q)-generic filter. By our assumption and elementarity of $M_{\zeta+1}$, there is a $\bar{q}$ which a conservative lower bound for $\langle q_n : n \in \omega \rangle$. Consequently $\bar{q} \leq_\sigma q'$ and $\bar{q}$ is totally (M_ζ, Q)-generic.

Next suppose that ζ is a limit and let $\langle \zeta_n : n < \omega \rangle$ be a sequence converging to ζ which is in $M_{\zeta+1}$ and such that $\zeta_0 = \xi$. By our assumption of suitability, $\langle M_{\zeta_n} : n < \omega \rangle$ is in $M_{\zeta+1}$. We will play $G(Q, \vec{\leq}, M_\zeta)$, this time starting from the position $\langle q, \bar{\sigma}, q \rangle$. Let $\sigma_0 = \bar{\sigma}$ and $q_0 = q$. We now describe a strategy for Player I to use in the remainder of the game. If q_n and σ_n were played in the previous round, Player I plays q_{n+1} in $M_{\zeta_{n+1}+1}$ such that $q_{n+1} \leq_{\sigma_n} q_n$ and q_{n+1} is totally (M_η, Q)-generic for all $\eta \leq \zeta_{n+1}$. In the end, Player I has arranged that $\langle q_n : n \in \omega \rangle$ will generate a (M_ζ, Q)-generic filter since any dense subset of Q in M_ζ will be in some M_{ζ_n}. Let $\langle q_n : n < \omega \rangle$ be the result of a play of the game which is in $M_{\zeta+1}$ and in which Player I played this strategy but Player II won. Such a play exists by our assumptions and by elementarity of $M_{\zeta+1}$. Let $\bar{q}$ be a conservative lower bound for $\langle q_n : n < \omega \rangle$ which is in $M_{\zeta+1}$. It follows that $\bar{q} \leq_\sigma q$ and $\bar{q}$ is (M_η, Q)-generic for all $\eta \leq \zeta$. □

7. Complete Properness

Now we will turn to the definition of *complete properness* and some analogous approaches to verifying it. First we will need some preliminary definitions. If M and N are sets, then $M \to N$ will symbolize an elementary embedding ϵ of $(M, \in)$ into $(N, \in)$ such that ϵ is in N and N satisfies that M is countable (i.e. N contains an injection of M into ω). If X is an element of M, we will use X^N to denote $\epsilon(X)$. If G is a subset of M which is not an element, we will use G^N to denote the point-wise image of G under ϵ. If there is no cause for confusion, the superscript will sometimes be omitted to simplify notation.

Definition 7.1. If Q is a forcing, then a *Q-diagram* is a collection of the form $M \to N_i$ $(i \in \lambda)$ where M is a suitable model for Q.

Definition 7.2. If Q is a forcing and M is a suitable model and $M \to N$, then we say that a (M, Q)-generic filter $G \subseteq Q \cap M$ is *$\overrightarrow{MN}$-prebounded* if whenever $N \to N^*$ and G is in N^*, then $N^* \models G^{N^*}$ is bounded.

Definition 7.3. Q is *λ-completely proper* if whenever $M \to N_i$ $(i < \kappa)$ is a Q-diagram for some $\kappa < 1 + \lambda$, there is a (M, Q)-generic filter G which is $\overrightarrow{MN_i}$-prebounded for all $i < \kappa$. We will write *completely proper* to mean 2-completely proper.

To see the relevance of this definition, let us return to our example of the uniformizing forcing P_g for a ladder system coloring g of $\vec{C}$. If $M \to N_i$ $(i < 2)$ is a P_g-diagram, then in general it is possible that

$$g^{N_0}(M \cap \omega_1) \neq g^{N_1}(M \cap \omega_1).$$

If this occurs and $C_\delta^{N_0} \cap C_\delta^{N_1}$ is infinite, then there is no (M, P_g)-generic G which is $\overrightarrow{MN_i}$-prebounded for both i.

In general it seems difficult to prove that a specific forcing such as P_g fails to be completely proper. For instance it is not clear at all whether every forcing P_g fails to be completely proper. Usually there is a hypothetical scenario suggesting the forcing is not completely proper — such as what we have just demonstrated for P_g — and there is a corresponding proof (via [2]) that iterations of forcings of this type can add new reals. We only know that some member of the given class of forcings (which shows up in a forcing extension) fails to be completely proper. This is not well understood. A (seemingly) isolated exception to this will be discussed in Section 9 below.

The connection of this notion to the λ-iterability condition is made clear by the following proposition.

Proposition 7.4. *Suppose that P is totally proper and P forces that $\dot{Q}$ is λ-completely proper. Then $P * \dot{Q}$ satisfies the λ-iterability condition.*

Proof. Let $M \in N$ be a pair of suitable models for $P * \dot{Q}$, $G \subseteq Q \cap M$ be (M, Q)-generic, $\kappa < \lambda$, and p_i $(i < \kappa)$ be totally (N, Q)-generic conditions which are lower bounds for G. Set $\bar{N}$ equal to the transitive collapse of N and let G^i denote the image of $\{p \in P \cap N : p_i \leq p\}$ under the collapsing map. Let $\hat{G}$ be a V-generic filter containing G. In $V[\hat{G}]$, we now have a Q-diagram $M[G] \to \bar{N}[G^i]$ $(i < \kappa)$ and therefore there is an $H \subseteq Q \cap M[G]$ which is $\overrightarrow{M[G]\bar{N}[G^i]}$-prebounded. It suffices to show that each p_i forces that H has a lower bound. To see this, let $\hat{G}^i$ be a V-generic filter containing p_i. Let N^* be a suitable model containing N as an element such that $N^* \cap \hat{G}^i$ is (N^*, P)-generic, N and G^i are in $N^*[\hat{G}^i]$. Since the inclusion map of $N[\hat{G}^i]$ into $N^*[\hat{G}^i]$ induces an embedding $\bar{N}[G^i] \to N^*[\hat{G}^i]$, it follows that $N^*[\hat{G}^i]$ must satisfy that H has a lower bound. Since $\hat{G}^i$ was arbitrary subject to containing p_i, it must be that p_i forces that H has a lower bound. □

While we will not formulate the notion of being $\mathbb{D}$-*complete*, the connection to the present terminology is provided by the following proposition.

Proposition 7.5. [8] *A λ-completely proper forcing is $\mathbb{D}$-complete with respect to a (specific) simple completeness system $\mathbb{D}$.*

In fact this proposition has a partial converse; see [8, 4.14].

Now we will return to our discussion of fusion schemes. It is not clear how to formulate a single condition to verify complete properness which works for all of our examples. Still there are two easy adaptations of (TP) which handle the main examples and even if these are not sufficient in a future application, it seems that a simple adaptation may work.

Remark. It should be noted that employing fusion schemes to verify complete properness is usually only warranted if one additionally wishes to verify that the forcing is $(<\omega_1)$-proper. In cases where this is not appropriate or necessary, a more direct approach is likely more efficient.

Definition 7.6. If $k<\omega$, we say a fusion scheme $\leq_\sigma$ $(\sigma \in X^{<\omega})$ satisfies (CP_k) if whenever $M \to N_i$ $(i<k)$ is a Q-diagram, p is in $Q\cap M$, $D\subseteq Q$ is dense and in M, and σ_i is in X^{N_i}, there is a q in $D\cap M$ such that for all $i<k$,

$$N_i \models q^{N_i} \leq_{\sigma_i} p^{N_i}.$$

(CP) will be used to abbreviate (CP_2). A fusion scheme satisfies $(\mathrm{CP}_{\aleph_1})$ if it satisfies (CP_k) for all k.

Remark. While it would perhaps seem more natural to define $(\mathrm{CP}_{\aleph_0})$ to mean $\forall k(\mathrm{CP}_k)$, the above definition is chosen so that there is a correspondence between (CP_λ) and λ-complete properness. See Theorem 7.8 below.

Definition 7.7. A fusion scheme satisfies (CP'_λ) if whenever $M\to N_i$ $(i<\kappa)$ is a Q-diagram for $\kappa<1+\lambda$, Player II has a strategy in $G(Q,\vec{\leq},M)$ so that if q_n $(n\in\omega)$ is the result of a legal play of the game by this strategy and q_n $(n<\omega)$ generates an (M,Q)-generic filter G, then G is $\overrightarrow{MN_i}$-prebounded for all $i<\kappa$.

Notice that if $M\in N$ are suitable models for Q and $M\to N$ denotes the identity map, then any $\overrightarrow{MN}$-prebounded filter in $Q\cap M$ is actually bounded (since we may take N^* a suitable model with $G,N\in N^*$ and $N\to N^*$ being the identity map). Hence (CP'_1) is a more restrictive property than (TP).

Theorem 7.8. *If Q admits a fusion scheme satisfying either* (CP_λ) *or* (CP'_λ)*, then Q is λ-completely proper.*

Proof. That (CP'_λ) implies λ-complete properness is a trivial modification of the proof of Theorem 6.3. The argument that (CP_k) implies k-complete properness of Q for $k<\omega$ is similar except that Player I plays their 'book keeping' strategy (in V) against a team of k many Player II's, each playing their winning strategy in a model N_i for $i<k$. Player I's ability to follow this strategy is made possible by (CP_k). Let q_n $(n\in\omega)$ be a resulting play of Player I and G be the filter it generates. Notice that Player I has arranged that G is (M,Q)-generic. We now must show that G is $\overrightarrow{MN_i}$-prebounded for each $i<k$. To this end let k be fixed and let $N_i\to N^*$ be such that G is in N^*.

Claim. *There is a sequence q'_n,σ_n $(n\in\omega)$ in N^* which is the result of a play by Player I against Player II playing their winning strategy in N_i such that*

$$G=\{p\in Q\cap M : \exists n(q'_n\leq p)\}$$

Proof. Let p_n $(n\in\omega)$ be an enumeration of G which is in N^*. In N^*, let T be the tree of all partial plays τ of $G(Q,\vec{\leq},M)$ in which Player II's winning strategy is followed and Player I plays elements of G. We order $\tau \lhd \tau'$ if there is a play by Player I in τ' which extends p_n where $n=|\tau|$. Clearly T has an infinite branch if and only if the conclusion of the claim holds and from the outside, T has an infinite branch as witnessed by q_n $(n<\omega)$. Since N^* is well founded and satisfies a sufficient fragment of ZFC, N^* must also satisfy that T has an infinite branch since otherwise N^* would contain a strictly decreasing function from T into its (well founded) ordinals. □

Since the team member for Player II who was in N_i used a winning strategy (from the vantage point of N_i) and since N_i is elementarily embedded into N^*, N^* satisfies G is bounded.

To handle the case $\lambda = \omega_1$, suppose that $M \to N_i$ $(i < \omega)$ is a Q-diagram. As before, Player I plays in V against a team of Player II's playing from the models N_i with their winning strategies. The difference is that the Player II playing from N_i begins playing only in round i. The rest of the argument is as before. $\square$

Now we will consider two more examples of forcings.

7.1. **Destroying $\clubsuit$-sequences.** First, let us consider a forcing that destroys instances of $\clubsuit$.

Definition 7.9. Suppose $\vec{C}$ is a ladder system. Define $Q_{\vec{C}}$ to be the collection of all countable subsets q of ω_1 such that if $\delta \leq \sup(q)$ is a limit ordinal, then $C_\delta \not\subseteq q$. Q is ordered by reverse end extension.

Theorem 7.10. *$Q_{\vec{C}}$ admits a fusion scheme which satisfies* $(\mathrm{CP}_{\aleph_1})$ *and (A). In particular $Q_{\vec{C}}$ is both completely proper and $(<\omega_1)$-proper.*

Remark. The following proof actually shows that the forcing to add an uncountable subset of ω_1 which is almost disjoint from every ladder is both completely proper and $(<\omega_1)$-proper.

Proof. Let $X = \lim(\omega_1)$ and, for $\sigma \in X^{<\omega}$, define $q \leq_\sigma p$ if $q \setminus p \cap C_{\sigma(i)}$ is empty for all $i < |\sigma|$. To see that this defines a fusion scheme, let M be a suitable model for Q. Player II plays $C_{M \cap \omega_1}$ in the first round of the game and arbitrarily after that. Suppose that q_n $(n \in \omega)$ is a play by Player I in which Player II followed this strategy and suppose that q_n $(n \in \omega)$ generates an (M, Q)-generic filter. Define $\bar{q} = \bigcup_n q_n$. It is trivial to verify that $\{q \in Q : \sup(q) > \alpha\}$ is dense for all $\alpha < \omega_1$ and hence $\sup(\bar{q}) = \delta$ where $\delta = M \cap \omega_1$. Since all proper initial parts of $\bar{q}$ are in Q, it is sufficient to check that C_δ is not contained in $\bar{q}$. But this follows from the fact that $C_\delta \cap (\bar{q} \setminus q_0) = \emptyset$, since $q_{n+1} \leq_{\langle \delta \rangle} q_n$ for all n.

Notice that this argument shows that Player II has a winning strategy in this game starting from any partial play of the game. Also, for each $\delta < \omega_1$, $\{C_\alpha \cap \delta : \alpha \in \omega_1\}$ is countable. Finally, $\langle q_n : n < \omega \rangle \mapsto \bigcup_n q_n$ defines a conservative lower bound when $\langle q_n : n < \omega \rangle$ is descending and bounded. It follows that the fusion scheme satisfies (A).

We now claim that this fusion scheme satisfies $(\mathrm{CP}_{\aleph_1})$. To see this, suppose that $M \to N_i$ $(i < k)$ is a Q-diagram, q is in $Q \cap M$, $D \subseteq Q$ is dense and in M, and $N_i \models \sigma_i \in \omega_1^{<\omega}$, Define

$$C = \bigcup_{i<k} \bigcup_{j<|\sigma_i|} C^{N_i}_{\sigma(j)}$$

and observe that the ordertype of C (and hence of the closure of C) is less than ω^2. Let $N \prec H(\aleph_2)$ be in M such that everything relevant is in N and $N \cap \omega_1$ is not in the closure of C. Let $\xi = \sup(C \cap N) + 1$ and define $q' = q \cup \{\xi\}$. q' is in N and therefore there is a $\bar{q} \leq q'$ in $D \cap N$. Notice that $q' \leq_{\sigma_i} q$ and, since $\bar{q} \setminus q'$ is contained in N and bounded below by ξ. $\square$

Now consider the variation $Q'_{\vec{C}}$ of $Q_{\vec{C}}$ in which the conditions are required to be closed sets. While the above proof shows that $Q'_{\vec{C}}$ is completely proper, we have already seen that $Q'_{\vec{C}}$ will almost never be ω-proper. The reader should convince themselves that the analogous fusion scheme fails to have conservative lower bounds.

7.2. **The forcing to measure a sequence of closed sets.** In section we will consider the forcing associated with *measuring*.

Definition 7.11. Let $\langle D_\alpha : \alpha < \omega_1 \rangle$ be a sequence such that for each $\alpha < \omega_1$, D_α is a closed subset of α. Define $Q_{\vec{D}}$ to be the set of all pairs $q = (x_q, E_q)$ such that:

(1) x_q is a countable closed set;
(2) $E_q \subseteq \omega_1$ is a club with $\max(x_q) < \min(E_q)$;
(3) if $\alpha \leq \max(x_q)$, then there is an $\alpha_0 < \alpha$ such that $x_q \cap (\alpha_0, \alpha)$ is contained in or disjoint from D_α.

The order on $Q_{\vec{D}}$ is defined by $q \leq p$ if and only if x_p is an initial part of x_q, $E_q \subseteq E_p$, and $x_q \setminus x_p \subseteq E_p$.

Theorem 7.12. *The forcing Q for measuring a sequence $\langle D_\alpha : \alpha < \omega_1 \rangle$ admits a fusion scheme satisfying* ($\mathrm{CP}'_{\aleph_1}$)*. In particular, it is completely proper.*

Remark. By remarks above, $Q_{\vec{D}}$ is not ω-proper unless $\langle D \rangle$ is already measured. In this case, $Q_{\vec{D}}$ is countably closed.

Proof. If $U \subseteq \omega_1$ is a countable open set, define $q \leq_U p$ if either $(x_q \setminus x_p) \cap \sup(U) \subseteq U$ or else $E_q \cap U = \emptyset$. If σ is a finite sequence of countable open subsets of ω_1, then $q \leq_\sigma p$ means that for each $i < |\sigma|$, $q \leq_U p$ where $U = \bigcap_{j \leq i} \sigma(j)$.

Let $M \to N_i$ ($i \in \omega$) be a Q diagram and let $\mathcal{U}$ be the collection of all open M-stationary subsets of $\delta = M \cap \omega_1$ which are in N_i for some $i < \omega$. Construct a $\subseteq$-decreasing sequence U_k ($k < \omega$) of elements of $\mathcal{U}$ such that if V is an open subset of $M \cap \omega_1$ in N_i for some i, then there is a $k < \omega$ such that U_k is either contained in or disjoint from V. It suffices to show that if Player II plays U_k in round i of $G(Q, \vec{\leq}, M)$, then this defines a strategy as required by ($\mathrm{CP}'_{\aleph_1}$). To see this, suppose that $G \subseteq M \cap Q$ is an (M, Q)-generic filter resulting from a play of the game by this strategy and $i < \omega$ is given.

Define $V = \delta \setminus D_\delta^{N_i}$ and let k be such that $U_k \subseteq V$ or $V \cap U_k$ is empty. By following the above strategy, Player II forces Player I to play so that $x_l \setminus x_k \subseteq U_k$ for all $l > k$. It follows that G is $\overrightarrow{MN_i}$-prebounded. $\square$

7.3. **Adding subtrees to Aronszajn trees.** Recall that an *Aronszajn tree (A-tree)* is an uncountable tree in which all levels and chains are countable. A Souslin tree is an A-tree in which, moreover, all antichains are countable. In this section, we will consider a forcing which adds a generic subtree to a given A-tree T. This subtree will have the property that the minimal elements of its complement will form an uncountable antichain and hence witness that T is not Souslin in the generic extension.

We will first fix some notation. In order to simplify matters later on, assume without loss of generality that the α^{th}-level of T consists of functions from α into ω and that T is ordered by extension. This equips T with a canonical lexicographic order. Let $T^{[n]}$ denote the collection of all tuples $\langle t_i : i < n \rangle$ which all come from some level of T and which are listed in non decreasing $\leq_{\text{lex}}$-order. For each n, $T^{[n]}$ is an A-tree when ordered by coordinate-wise extension and these trees will collectively be referred to as the *finite powers of* T. If u is in $T^{[n]}$ and $A \subseteq T$, then we will write $u \subseteq A$ to mean that the range of u is contained in A. If t is in T and $\alpha < \omega_1$, then $t \restriction \alpha$ is the element s of T of height α with $s \leq t$ if t has height at least α and $t \restriction \alpha = t$ otherwise. Similarly, if u is in $T^{[n]}$ for some n, then $u \restriction \alpha$ is defined by coordinate-wise restriction (which agrees with the definition of restriction in $T^{[n]}$).

Definition 7.13. Define Q_T to be the set of all $q = (x_q, \mathcal{U}_q)$ for which:

- x_q is a subtree of T which has a last level α_q.
- $\mathcal{U}_q$ is a countable collection of *pruned subtrees*[3] of some finite power of T.
- for all $U \in \mathcal{U}_q$ there is a u in U_{α_q} with $u \subseteq x_q$.

Q is ordered by declaring $q \leq p$ to mean that $x_q \supseteq x_p$ and $\mathcal{U}_q \supseteq \mathcal{U}_p$.

Theorem 7.14. *Q_T admits a fusion scheme satisfying (A) and* $(\mathrm{CP}_{\aleph_1})$*. In particular, Q_T is completely proper and $(< \omega_1)$-proper.*

By applying the Main Theoremand standard chain condition and book keeping arguments, we obtain the following corollary.

Corollary 7.15. *Souslin's hypothesis is consistent with* CH.

Remark. Observe that each $U \in \mathcal{U}_p \cap \mathcal{P}(T)$ "promises" that the generic tree $\dot{S} = \bigcup_{q \in \dot{G}_Q} x_q$ added by Q_T will intersect U uncountably often. Q_T is a simplification of a forcing of Shelah [10] that specializes A-trees without adding reals. We will show that Q_T adds an uncountable antichain to T. It is not clear whether Q_T necessarily specializes T.

Proof. Define a fusion scheme on $Q = Q_T$ as follows. Set $X = T$ and if σ is in $X^{<\omega}$, define $r \leq_\sigma q$ to mean that $r \leq q$ and for all $i < |\sigma|$, if $\sigma(i) \restriction \alpha_q \in x_q$ then $\sigma(i) \restriction \alpha_r \in x_r$. We leave it as an exercise to the reader to verify that this defines a fusion scheme and that in fact Player II has a winning strategy in $G(Q, \vec{\lesssim}, M)$ from every initial position whenever M is a suitable model for Q.

To see that this fusion scheme satisfies (A), observe that if M is a suitable model and $\delta = M \cap \omega_1$, then

$$\leq_\sigma \restriction (Q \cap M) = \leq_{\sigma \restriction \delta} \restriction (Q \cap M).$$

Since T is Aronszajn, it follows that $\{\leq_\sigma \restriction (Q \cap M) : \sigma \in X^{<\omega}\}$ is countable. To see that Q has conservative lower bounds, let q_n $(n < \omega)$ be a descending sequence which has a lower bound. Set

$$\bar{\alpha} = \sup_n \alpha_{q_n} \qquad \mathcal{U}_{\bar{q}} = \bigcup_n \mathcal{U}_{q_n} \qquad x = \bigcup_n x_{q_n}$$

Define $x_{\bar{q}}$ to be the union of x together with all t in $T_{\bar{\alpha}}$ such that $t \restriction \xi$ is in x for all $\xi < \bar{\alpha}$. It is straightforward to verify that $\bar{q} = (x_{\bar{q}}, \mathcal{U}_{\bar{q}})$ is a conservative lower bound for q_n $(n < \omega)$.

Lemma 7.16. *If M is as usual, $q \in Q \cap M$, $D \subseteq Q$ is dense, $D \in M$, and $\sigma \in T^{<\omega}$, then there exists $r \leq_\sigma q$ such that $r \in D \cap M$. In particular, the fusion scheme define above satisfies* $(\mathrm{CP}_{\aleph_1})$*.*

[3]A *subtree* of a tree is an initial part (*i.e.*, downward closed subset) of that tree. A tree is *pruned* if every element has uncountably many extensions.

Proof. We will begin by arguing that the second part of the lemma follows from the first. Observe that if $M \to N_i$ $(i < k)$ is a Q-diagram and for each $i < k$ $N_i \models \sigma_i \in T^{<\omega}$, then there is a model M', suitable (enough) for Q and containing q and D. Let $\delta = M' \cap \omega_1$. Even though, for a given $i < k$, σ_i is typically not in N_j for $i \neq k$, $\sigma_i \restriction \delta$ is in each M for all $i < k$. Therefore we can find a single σ in M which corresponds to the concatenation of these restrictions. Hence if we verify the first sentence in the lemma for σ and $M' \to M$, we have verified the instance of $(\mathrm{CP}_{\aleph_1})$ for $M \to N_i$ $(i < k)$, σ_i $(i < k)$, q, and D. Hence we may now focus on the first sentence in the lemma.

Seeking a contradiction, suppose the first sentence of the lemma is false. Without loss of generality, σ consists of entries t such that $\mathrm{ht}(t) \leq \delta = M \cap \omega_1$ and $t \restriction \alpha_q \in x_q$. Since T_δ is countable, there is an $h \colon \omega_1 \to T^{<\omega}$ such that $h(\alpha) \in T^{<\omega}_\alpha$ and $h(\delta)$ is the entries of σ of height δ (listed in the same order). We may assume $h \in M$ because $h(\delta)$ is in the Skolem hull $M \cup \delta$. Let Ξ be the set of $\xi < \omega_1$ for which $h(\xi) \restriction \alpha_q = h(\delta) \restriction \alpha_q$ and, if $r \leq q$ with $r \in D$ and $\alpha_r < \xi$, then $r \not\leq_{h(\xi)} q$.

Claim. *$\delta \in \Xi$, so Ξ is uncountable.*

Proof. Seeking a contradiction, suppose r witnesses that $\delta \notin \Xi$. Fix $\beta \in M \cap [\alpha_r, \omega_1)$. Observe that $r \leq_{h(\delta)} q$ if and only if $r \leq_{h(\delta)\restriction\beta} q$. By elementarity, there is an $r' \in M \cap D$ such that $\alpha_{r'} = \alpha_r$ and $r' \leq_{h(\delta)\restriction\beta} q$. Thus, $r' \leq_{h(\delta)} q$, in contradiction with our assumption that the lemma fails. □

Returning to the proof of the lemma, let $U \subseteq T^{[n]}$ be the set of u for which uncountably $\xi \in \Xi$ satisfy $u \leq h(\xi)$. It follows that U is pruned and $q' = \langle x_q, \mathcal{U}_q \cup \{U\}\rangle$ is a condition in M. Now q has no extension in $D \cap M$, which is absurd. □

We leave the following as an exercise (see Lemma 5.7 of [8]).

Exercise 7.17. Prove that for every β, the set $\{q \in Q : \alpha_q \geq \beta\}$ is dense and that if $q \in Q$, then there is an $r \leq q$ such that $\forall s \in x_q\ \exists t \in T_{\alpha_r}\ s \leq t \notin x_r$. This proves that $\dot{S}$ does not contain any cone of T (*i.e.*, a set of the form $\{t \in T : t \geq s\}$). Prove that this implies that $\dot{A}$, a name for the set of minimal elements of $\check{T} \setminus \dot{S}$, is forced to be an uncountable antichain of $\check{T}$.

This finishes the proof of the theorem. □

8. Proof of the Main Theorem

Definition 8.1. Given an iteration $\langle P_\alpha, \dot{Q}_\alpha\rangle_{\alpha<\varepsilon}$, $\alpha < \varepsilon$, $p \in P_\alpha$, and $q \in P_\varepsilon$, we say that q is a *completion* of p if $q \restriction \alpha = p$.

The following argument, which we will call the Three Models Argument, is the key component of the proof of the Main Theorem. Suppose:

- $\dot{Q}$ satisfies the ω_1-iterability condition over a totally proper forcing P;
- $N_0 \in N_1 \in N_2$ are as usual with $P, \dot{Q} \in N_0$;
- $G^* \in \mathrm{Gen}(N_0, P) \cap N_1$;
- p is (N_2, P)-generic, p is (N_1, P)-generic, p is totally (N_0, P)-generic, and p is a lower bound of G^*.

It follows that $p \Vdash N_1 \cap \dot{G}_P \in N_2$ because $p \Vdash N_2[\dot{G}_P] \cap V = N_2$. Let $\langle G^l\rangle_{l<\omega}$ enumerate $N_2 \cap \mathrm{Gen}(N_1, P)$. Every p satisfying the above assumptions forces that $N_1 \cap \dot{G}_P \in \{G^l : l < \omega\}$. This is by elementarity of N_2 and because P adds no new countable subsets to V. Hence, by the iterability condition, there is a $(N_0[G^*], \dot{Q})$-generic sequence $\langle \dot{q}_n \rangle_{n<\omega}$ such that any p satisfying the above assumptions forces that $\langle \dot{q}_n \rangle_{n<\omega}$ has a lower bound. Therefore, there is a P-name $\dot{s}$ such that any such p forces $\dot{s}$ to be a lower bound of $\langle \dot{q}_n \rangle_{n<\omega}$. Hence, for any such p, $p * \dot{s}$ is totally $(N, P * \dot{Q})$-generic. Note that $\{r * \dot{t} : r \in G^*, N_0[G^*] \models \exists n \ \dot{t} \geq \dot{q}_n\} \in \mathrm{Gen}^+(N_0, P * \dot{Q})$. Thus, we can complete G^* to a filter $G^\dagger \in \mathrm{Gen}^+(N_0, P * \dot{Q})$ such that any p satisfying the above assumptions can be completed to a lower bound $p * \dot{s}$ of $G^\dagger$.

The Main Theorem is a corollary of the following claim.

Claim. *Given $\mathbb{P} = \langle P_\alpha, \dot{Q}_\alpha \rangle_{\alpha<\varepsilon}$ satisfying the hypotheses of the Main Theorem, the following hypotheses imply the following conclusion.*

Hypotheses:

- *N is as usual with $\mathbb{P} \in N$.*
- *$\alpha \in \varepsilon \cap N$.*
- *$\mathcal{N} = \langle N_\xi : \xi \leq 2\varepsilon^* \rangle$ is a tower of models with $N_0 = N$ where $\forall \beta \ \beta^* = \mathrm{otp}(N \cap \beta)$.*
- *$G^* \in \mathrm{Gen}(N_0, P_\alpha) \cap N_{2\alpha^*+1}$.*
- *$p \in N \cap P_\varepsilon$.*
- *$p \restriction \alpha \in G^*$.*

Conclusion: There is a $G^\dagger \in \mathrm{Gen}(N_0, P_\varepsilon, p)$ such that any lower bound for G^* that is (N_ξ, P_α)-generic for all $\xi \in (2\alpha^*, 2\varepsilon^*]$ can be completed to a lower bound for $G^\dagger$.

Proof. Proceed by induction on ε. First, consider the case $\varepsilon = \gamma + 1$. We have $\varepsilon^* = \gamma^* + 1$ because $\mathbb{P} \in N \Rightarrow \varepsilon \in N \Rightarrow \gamma \in N$. Let $\alpha, \mathcal{N}, G^*, p$ be as given. Step inside $N_{2\gamma^*+1}$. Apply our induction hypothesis inside this model to $\alpha, \mathcal{N} \restriction (2\gamma^* + 1), G^*, p \restriction \gamma$, assuming $\alpha < \gamma^*$—the case $\alpha = \gamma^*$ is marginally simpler. This application gives us $G' \in \mathrm{Gen}(N_0, P_\gamma) \cap N_{2\gamma*+1}$ such that any lower bound for G^* that is (N_ξ, P_α)-generic for all $\xi \in (2\alpha^*, 2\gamma^*]$ can be completed to a lower bound for G'. Note that $N_0 \in N_{2\gamma^*+1} \in N_{2\varepsilon^*}$ and $G' \in \mathrm{Gen}(N_0, P_\gamma) \cap N_{2\gamma^*+1}$, so by the Three Models Argument, we can extend G' to a filter $G^\dagger \in \mathrm{Gen}(N_0, P_\varepsilon)$ such that any lower bound for G' that is also $(N_{2\gamma^*+1}, P_\gamma)$-generic and $(N_{2\varepsilon^*}, P_\gamma)$-generic can be completed to a lower bound for $G^\dagger$.

Subclaim. *Suppose q is a lower bound for G^* and is (N_ξ, P_α)-generic for all $\xi \in (2\alpha^*, 2\varepsilon^*]$. It follows that there is a P_α-name $\dot{r}$ such that q forces each of the following.*

- *$\dot{r} \in N_{2\gamma^*+1} \cap P_\gamma$.*
- *$\dot{r}$ is a lower bound for G'.*
- *$\dot{r} \restriction \alpha \in \dot{G}_{P_\alpha}$.*

Proof. Let $G \subseteq P_\alpha$ be generic with $q \in G$. We then have $N_0 \cap G = G^*$ and $G \cap N_\xi \in \mathrm{Gen}^+(N_\xi, P_\alpha) \cap N_{\xi+1}$ for all $\xi \in (2\alpha^*, 2\varepsilon^*)$. Step inside $N_{2\gamma^*+1}$. By elementarity of $N_{2\gamma^*+1}$, $N_{2\gamma^*} \cap G$ has a lower bound s. Such an s is totally (N_ξ, P_α)-generic for all $\xi \in (2\alpha^*, 2\gamma^*]$. We can then complete s to a lower bound $r \in P_\gamma$ for G'. Let $\dot{r}$ be a name for r, forced by q to have our desired properties. $\square$

We now have $P_\alpha \ni q \Vdash \dot{r} \in P_\gamma \cap N_{2\gamma^*+1}$ and $q \Vdash \dot{r} \restriction \alpha \in \dot{G}_{P_\alpha}$. By properness, we can complete q to a condition $q' \in P_\gamma$ such that $q' \Vdash \dot{r} \in \dot{G}_{P_\gamma}$ and q' is

$(N_{2\gamma^*+1}, P_\gamma)$-generic and $(N_{2\varepsilon^*}, P_\gamma)$-generic. Therefore, to finish the successor case, it suffices to prove that q' is a lower bound for G'. Seeking a contradiction, suppose it is not. Let $G \subseteq P_\gamma$ be generic with $q' \in G$. Since $q' \Vdash \dot{r} \in \dot{G}_{P_\gamma}$, we have $q' \leq \dot{r}[G]$. Since $q' \Vdash \dot{r} \leq G'$, we also have $\dot{r}[G] \leq G'$, so $q' \leq G'$, in contradiction with our assumption that $q' \not\leq G'$.

Next, consider the case where ε is a limit. Choose $\langle \alpha_n \rangle_{n<\omega}$ strictly increasing and cofinal in $N \cap \varepsilon$ with $\alpha_0 = \alpha$. Let $\langle D_n \rangle_{n<\omega}$ enumerate the dense subsets of P_ε from N. Build $\langle p_n, G_n \rangle_{n<\omega}$ such that:

- $p_0 = p$ and $G_0 = G^*$.
- $p_n \geq p_{n+1} \in N_0 \cap D_n$.
- $G_n \in \mathrm{Gen}(N_0, P_{\alpha_n}, p_n \restriction \alpha_n) \cap N_{2\alpha_n^*+1}$.
- $p_{n+1} \restriction \alpha_n \in G_n$.
- Any lower bound for G_n that is (N_ξ, P_{α_n})-generic for all $\xi \in (2\alpha_n^*, 2\alpha_{n+1}^*]$ can be completed to a lower bound for G_{n+1}.

How? For $n = 0$, there's nothing to do. Assume we have p_n and G_n. We can then find $p_{n+1} \in N_0 \cap D_n$ such that $p_{n+1} \leq p_n$ and $p_{n+1} \restriction \alpha_n \in G_n$. Why? The set of conditions in P_{α_n} that can be completed to an extension of p_n in D_n is dense below $p_n \restriction \alpha_n$, which is in G_n. Moreover, this dense set is in N_0. Since $G_n \in \mathrm{Gen}(N_0, P_{\alpha_n}, p_n \restriction \alpha_n)$, we can find p_{n+1} as desired.

Apply our induction hypothesis to

$$\mathbb{P} \restriction \alpha_{n+1}, p_{n+1} \restriction \alpha_{n+1}, \alpha_n, G_n, \mathcal{N} \restriction (2\alpha_{n+1}^* + 1)$$

inside the model $N_{2\alpha_{n+1}^*+1}$. This application gives us a filter

$$G_{n+1} \in \mathrm{Gen}(N_0, P_{\alpha_{n+1}}, p_{n+1} \restriction \alpha_{n+1}) \cap N_{2\alpha_{n+1}^*+1}$$

such that any condition which is a lower bound for G_n and is (N_ξ, P_{α_n})-generic for all $\xi \in (2\alpha_n^*, 2\alpha_{n+1}^*]$ can be completed to a lower bound for G_{n+1}, as desired.

Let $G^\dagger = \{r \in P_\varepsilon \cap N : \exists n \ r \geq p_n\}$. It follows that $G^\dagger \in \mathrm{Gen}(N_0, P_\varepsilon, p)$. Let q be a lower bound for G^* that is (N_ξ, P_α)-generic for all $\xi \in (2\alpha^*, 2\varepsilon^*]$. Define by induction a sequence $\langle q_n \rangle_{n<\omega}$ such that:

- $q_0 = q$.
- $q_n \in P_{\alpha_n}$.
- $q_{n+1} \restriction \alpha_n = q_n$.
- q_n is a lower bound for G_n.
- q_n is (N_ξ, P_{α_n})-generic for all $\xi \in (2\alpha_n^*, 2\varepsilon^*]$.

Given q_n, let us find q_{n+1} as follows. Arguing as in the successor case, there is a P_{α_n}-name $\dot{r}$ such that q_n forces each of the following:

- $\dot{r} \in N_{2\alpha_{n+1}^*+1} \cap P_{\alpha_{n+1}}$.
- $\dot{r}$ is a lower bound for G_{n+1}.
- $\dot{r} \restriction \alpha_n \in \dot{G}_{P_{\alpha_n}}$.

By $<\omega_1$-properness, we can complete q_n to a condition $q_{n+1} \in P_{\alpha_{n+1}}$ such that $q_{n+1} \Vdash \dot{r} \in \dot{G}_{P_{\alpha_{n+1}}}$ and q_{n+1} is $(N_\xi, P_{\alpha_{n+1}})$-generic for all $\xi \in (2\alpha_n^*, 2\varepsilon^*]$. Arguing as in the successor case, q_{n+1} is a lower bound for G_{n+1}.

Set $q^\dagger = \bigcup_{n<\omega} q_n$ in P_ε. (Technically, this union has domain $\sup(N \cap \varepsilon)$, which may be less than ε, but we extend the domain to all of ε without increasing the support.) It suffices to show that $q^\dagger \leq G^\dagger$, so it suffices to show that $q^\dagger \leq p_m$ for all m. Seeking a contradiction, suppose $q^\dagger \not\leq p_m$. It follows that for some $n \geq m$

we have $q_n = q^\dagger \restriction \alpha_n \not\leq p_m \restriction \alpha_n$. Hence, $q_n \not\leq p_{n+1} \restriction \alpha_n \in G_n$; hence, $q_n \not\leq G_n$, in contradiction with how $\langle q_n \rangle_{n<\omega}$ was constructed. $\square$

9. The role of $(< \omega_1)$-properness

The point of this section is to present a proof of the following theorem of Shelah which shows that the hypothesis of $(< \omega_1)$-properness can not be dropped entirely from Main Theorem.

Theorem 9.1. [10, XVIII.1.1] *Assume $V = L$. There is a countable support iteration $\langle P_\xi; \dot{Q}_\xi : \xi < \omega^2 \rangle$ of forcings such that:*

1. *$\dot{Q}_\xi$ is forced to be an $\aleph_1$-completely proper forcing;*
2. *$\dot{Q}_\xi$ is forced to have cardinality $\aleph_1$;*
3. *P_{ω^2} introduces a new subset of ω.*

Remark. In fact P_{ω^2} introduces a club which does not contain any infinite subset from V. It is not clear whether this is true in general.

Our presentation of Theorem 9.1 is somewhat different than in [10] in that we define the partial order explicitly and then work to prove its properties. It seems that the differences with the construction in [10] are, however, superficial.

Let $\mathcal{D}$ consist of all countable subsets of ω_1 which are closed in their supremum. Fix a surjection $\phi : \omega \to H(\omega)$ which is in L and satisfies that $\phi^{-1}(X)$ is infinite for all X in $H(\omega)$. In all of the discussion which follows, we need to assume a minimum of CH. Fix a bijection $\mathrm{ind} : \omega_1 \to \mathcal{D}$. Additional assumptions on ind will be stated as they are needed.

Fix a ladder sequence $\vec{C} = \langle C_\alpha : \alpha \in \lim(\omega_1) \rangle$ for the moment.

Definition 9.2. If q is in $\mathcal{D}$, we say q is *self coding* with respect to $\vec{C}$ if and only if whenever ν is a limit point of q, there is a well ordering $\lhd$ of ω and an $X \subseteq \omega$ such that

$$(\omega, \lhd, X) \simeq (\nu, \in, q \cap \nu)$$

and for all $m < \omega$ there is a $\nu_m < \nu$ such that whenever ξ is in $q \cap (\nu_m, \nu)$ there is a $n > m$ such that

$$\phi(|C_\nu \cap \xi|) = (n, \lhd \restriction n, X \cap n).$$

The following fact states a key property of this definition.

Fact 9.3. *If p and q are self coding with respect to ladder systems $\vec{C}^p$ and $\vec{C}^q$, respectively, $C^p_\delta = C^q_\delta$, and $p \cap \delta \neq q \cap \delta$, then $p \cap q$ is not cofinal in δ.*

Proof. Suppose that $p \cap q \cap \delta$ is cofinal in δ for some limit ordinal δ and $C^p_\delta = C^q_\delta$. If ξ_i $(i < \omega)$ is contained in $p \cap q$ and cofinal in δ, then the structures

$$\phi(|C^p_\delta \cap \xi_i|) = (n_i, \lhd_i, X_i)$$

must converge in the sense that for any m, $n_i > m$ for all but finitely many i the sequence $\{(m, \lhd_i \restriction m, X_i \cap m)\}_i$ is eventually constant. Let $\lhd$ be a relation on ω and X be a subset of ω such that for any m, for all but finitely many i, $\lhd \restriction m = \lhd_i \restriction m$ and $X \cap m = X_i \cap m$. Now $(\omega, \lhd, X)$ is isomorphic to $(\delta, \in, p \cap \delta)$ and $(\delta, \in, q \cap \delta)$. Since the only automorphism of $(\delta, \in)$ is the identity, it must be that $p \cap \delta = q \cap \delta$. $\square$

Definition 9.4. If $E \subseteq \omega_1$ is a club and q is in $\mathcal{D}$, then we say that q is *E-fast* if whenever ν is a limit point of q,

$$\min(E \setminus (\nu + 1)) < \operatorname{ind}(q \cap \nu) < \min(q \setminus (\nu + 1))$$

(here we define the latter inequality to be vacuous if $\nu = \sup(q)$).

The following fact is our motivation for this definition.

Fact 9.5. *Suppose that E_i $(i < \omega)$ is a sequence of clubs such that for every $i < j$, all initial parts of E_j are E_i-fast. Then whenever δ is in $\cap\{E_i : i < \omega\}$*

$$\sup\{\operatorname{ind}(E_i \cap \delta) : i < \omega\} = \min(\cap\{E_i : i < \omega\} \setminus (\delta + 1))$$

(note that this ordinal is a limit point of E_i for all $i < \omega$).

Definition 9.6. Define $Q_{\vec{C},E}$ to be the collection of all elements of $\mathcal{D}$ which are E-fast and self coding with respect to $\vec{C}$. $Q_{\vec{C},E}$ is viewed as a forcing notion with the order of end extension.

In general we do not expect $Q_{\vec{C},E}$ to preserve ω_1. Note, however, that it is trivial that

$$\{q \in Q_{\vec{C},E} : \exists\beta > \alpha(\beta \in q)\}$$

is dense for all $\alpha < \omega_1$ and hence every condition in $Q_{\vec{C},E}$ forces that the generic self coding set is cofinal in ω_1. Recall that a ladder system $\langle C_\alpha : \alpha < \omega_1\rangle$ is a *strong club guessing sequence* if whenever $E \subseteq \omega_1$ is a club, $\{\delta < \omega_1 : C_\delta \subseteq_* E\}$ contains a club.

Theorem 9.7. *Suppose that $\langle C_\alpha : \alpha < \omega_1\rangle$ is a strong club guessing sequence and E is a club. Then $Q_{\vec{C},E}$ is proper.*

Proof. Let $Q = Q_{\vec{C},E}$ for brevity. We will actually prove something more precise. Let $\vec{C}$ be an arbitrary ladder sequence and let S consist of all countable elementary submodels M of $H(\omega_2)$ such that if $\delta = M \cap \omega_1$ and $E \subseteq \omega_1$ is a club in M, $C_\delta \setminus E$ is finite. We will prove that if $S' \subseteq S$ is stationary, then forcing with Q preserves S'.

Let M be as usual such that $M \cap H(\omega_2)$ is in S, let q be in $Q \cap M$. Fix an enumeration D_i $(i < \omega)$ of the dense subsets of Q which are in M. Let ζ_i $(i < \omega)$ enumerate the ordinals which are at most $\zeta = \min(E \setminus (\delta + 1))$. Fix a bijection $\pi : \omega \to \delta$ such that $|C_\delta \cap \pi(k)| \leq k$ and define $i \lhd j$ if $\pi(i) < \pi(j)$.

Construct a descending sequence q_k $(k < \omega)$ in $Q \cap M$ by induction. Start by putting $q_0 = q$. Now suppose that q_k has been constructed. Define $X_k = \pi^{-1}(q_k)$ and $n_k = |C_\delta \cap \sup(q_k)|$. By our choice of ϕ, there are infinitely many i such that $\phi(i) = (n_k, \lhd \restriction n_k, X_k \cap n_k)$. Our assumptions on S imply that for all but finitely many i, there is a countable elementary submodel N of $H(\omega_2)$ such that q_k and D_k are in N and $|C_\delta \cap N| = i$. Therefore it is possible to find such an N with

$$|C_\delta \cap \nu| > n_k$$

$$\phi(|C_\delta \cap \nu|) = (n_k, \lhd \restriction n_k, X_k \cap n_k)$$

where $\nu = N \cap \omega_1$. Let $q_k' = q_k \cup \{\sup(q_k), \xi\}$ where $\xi < \nu$ is such that $\sup(q_k) < \xi$ and $C_\delta \cap \nu = C_\delta \cap \xi$. Finally, let q_{k+1} be an extension of q_k' in N such that q_{k+1} is in D_k and if $\bar{q}$ is in Q with $\operatorname{ind}(\bar{q}) = \zeta_k$, then q_{k+1} is not an initial part of $\bar{q}$. The key point here is that if η is in $q_{k+1} \setminus q_k$, then $C_\delta \cap \eta = C_\delta \cap \nu$. Furthermore, if $\bar{q}$ is any extension of q_{k+1} in M, $\pi^{-1}(\bar{q}) \cap n_k = \pi^{-1}(q_k) \cap n_k$. Finish the construction by letting $\bar{q} = \cup\{q_k : k < \omega\}$.

Since $\{p \in Q : \sup(p) > \alpha\}$ is dense in Q and in M for all $\alpha < \delta$, we will necessarily have that $n_k \to \infty$. Also we have arranged that if ξ is in $q_{k+1} \setminus q_k$, then

$$\phi(|C_\delta \cap \xi|) = (n_k, \triangleleft \restriction n_k, \pi^{-1}(\bar{q}) \cap n_k).$$

It follows that $\bar{q}$ is self coding with respect to $\vec{C}$. Furthermore, we have arranged that $\min(E \setminus (\delta+1)) < \operatorname{ind}(\bar{q})$ which, together with the fact that $\bar{q} \cap \xi$ is a condition in Q for all $\xi < \delta$, implies that $\bar{q}$ is E-fast. Hence $\bar{q}$ is in Q and we have clearly arranged that $\bar{q}$ is (M, Q)-generic. $\square$

In order to prove that the forcing $Q_{\vec{C},E}$ is completely proper, we need to know that $\vec{C}$ satisfies the following strong condition for some $A \subseteq \omega_1$:

$(*)_A$**:** The following hold:
1. $L[A]$ contains E and $\vec{C}$;
2. For every limit ordinal δ, $L[A \cap \delta]$ satisfies δ is countable;
3. $L[A \cap \delta]$ satisfies C_δ is the $<_{L[A\cap\delta]}$-least ladder in δ such that whenever $L_\alpha[A \cap \delta]$ satisfies "δ is ω_1 and every two closed unbounded subsets of δ intersect", C_δ is almost contained in every closed unbounded subset of δ in $L_\alpha[A \cap \delta]$.

The most stringent requirement we will need is on the function ind in proving the $\aleph_1$-complete properness of $Q_{\vec{C},E}$:

$(**)$**:** $\operatorname{ind}(q) = \xi$ if and only if q is the ξ^{th}-least element of $\mathcal{D}$ in the $\triangleleft_L$-ordering.

Proposition 9.8. *($V = L$) If $\vec{C}$ satisfies $(*)_A$ for some $A \subseteq \omega_1$ and* ind *satisfies (**), then $Q_{\vec{C},E}$ is $\aleph_1$-completely proper.*

Proof. Suppose that $M \to N_i$ $(i < \omega)$ is a Q-diagram and that $q \in Q \cap M$. Let ζ_i denote $\min(E^{N_i} \setminus (\delta + 1))$. While ζ_i depends on i, we can take the supremum ζ of this sequence. Working as in Proposition 9.7, we can build a sequence q_k $(k < \omega)$ of extensions of q such that, setting $\bar{q} = \cup\{q_k : k < \omega\}$, we have $\zeta < \operatorname{ind}(\bar{q})$ and $\bar{q}$ is self coding with respect to $\vec{C}$. Notice that if $N \to N'$ and the filter G generated by q_k $(k < \omega)$ is in N', then so is $\bar{q}$. By absoluteness of L_α, $\operatorname{ind}^{N'}$ is a restriction of ind. In particular, N' satisfies $\bar{q}$ is $E^{N'}$-fast. Furthermore, while $\vec{C}^{N'}$ may not be an initial segment of $\vec{C}$, $\vec{C}^{N'} \restriction (\delta+1) = \vec{C} \restriction (\delta+1)$ by absoluteness of $L_\alpha[A \cap \delta]$ for $\alpha > \delta$ and this is the only portion of $\vec{C}^{N'}$ relevant in determining whether $\bar{q}$ is self coding with respect to $\vec{C}^{N'}$. Thus $\bar{q}$ is in $Q^{N'}$ and clearly N' satisfies $\bar{q}$ is a lower bound for G. $\square$

Remark. This is clearly against the spirit of complete properness. We do not expect in general that if, e.g., $\vec{C}$ is a ladder system in some suitable model M and $M \to N_i$ $(i < 2)$, then N_0 and N_1 should agree about C_δ. In fact this sort of behavior can be ruled out if, for instance, there is a measurable cardinal.

Proposition 9.9. *($V = L$) Suppose that $\langle P_\xi; \dot{Q}_\xi : \xi < \omega^2\rangle$ is an iteration of forcings such that for all $\xi < \omega^2$:*
1. *$E_0 = \lim(\omega_1)$;*
2. *$\dot{Q}_\xi = Q_{\vec{C}^\xi, \dot{E}^\xi}$;*
3. *$\dot{E}_{\xi+1}$ is the $P_{\xi+1}$ name for the union of the generic filter for $\dot{Q}_\xi$;*

(4) *if η is a limit ordinal, then $\dot{E}_\eta$ is the P_η-name for $\cap\{\dot{E}_\xi : \xi < \eta\}$;*
(5) *$\vec{C}^\xi$ is the P_ξ-name for the ladder system satisfying (*) with respect to some A coding $\langle(\vec{C}^\eta, E^\eta) : \eta < \xi\rangle$ in some canonical way;*
(6) *$\dot{Q}_{\vec{C}^\xi, E_\xi}$ is computed using a fixed function* ind *in L.*

Then P_{ω^2} introduces a new real.

Proof. In fact we will show that if

$$\alpha_0 = \min(\cap\{E_\xi : \xi < \omega^2\})$$

then $\langle E_\xi \cap \alpha_0 : \xi < \omega^2\rangle$ is not in L. We will assume for contradiction that this is not the case and prove that $\langle E_\xi : \xi < \omega^2\rangle$ is in L. Observe first that $\dot{Q}_\xi$ is a P_ξ-name for a subset of $\mathcal{D}$. In order to make statements in the forcing language easier to read, we will suppress "checks" on the names for ground model elements of $\mathcal{D}$. Define sequences α_ζ $(\zeta < \omega_1)$ and $q_{\xi,\zeta}$ $(\xi < \omega^2; \zeta \in \lim(\omega_1) \cup \{0\})$ by recursion as follows:

$$q_{\xi,0} = E_\xi \cap \alpha_0$$
$$\alpha_{\zeta+k+1} = \sup\{\mathrm{ind}(q_{\omega\cdot k+i,\zeta}) : i < \omega\}$$

If $\zeta > 0$ then:

$$\alpha_\zeta = \sup_{\zeta'<\zeta} \alpha_{\zeta'}.$$

The next claim is used to handle the recursive definition of $q_{\xi,\zeta}$ for limit ordinals ζ.

Claim. *For each $\xi < \omega^2$ and $\zeta \in \lim(\omega_1)$, there is a unique element $q_{\xi,\zeta}$ of $\mathcal{D}$ such that:*
(1) *$q_{\xi,\zeta}$ is a cofinal subset of α_ζ;*
(2) *either:*
  (a) *ζ is a limit of limit ordinals and $\{\alpha_{\zeta'} : \zeta' \in \zeta \cap \lim(\omega_1)\} \subseteq q_{\xi,\zeta}$ or*
  (b) *$\zeta = \zeta_0 + \omega$ and there is a k_0 such that for all $k > k_0$, α_{ζ_0+k} is in $q_{\xi,\zeta}$;*
(3) *$\langle q_{\eta,\zeta} : \eta < \xi\rangle$ forces $q_{\xi,\zeta}$ is in $\dot{Q}_\xi$.*

Moreover $q_{\xi,\zeta} = E_\xi \cap \alpha_\zeta$.

Proof. This is proved by induction on the lexicographical order on $\lim(\omega_1)\times\omega^2$. Let (ζ,ξ) be in $\lim(\omega_1)\times\omega^2$ and suppose that the claim is true whenever $(\zeta',\xi') <_{\mathrm{lex}} (\zeta,\xi)$.

Case 1: $\zeta = \zeta_0 + \omega$ for some ζ_0 in $\lim(\omega_1) \cup \{0\}$. By Fact 9.5, α_{ζ_0+k} the least element of $\cap\{E_\xi : \xi < \omega \cdot k\}$ greater than α_ζ. By (*), $\langle q_{\xi',\zeta} : \xi' < \xi\rangle$ decides the element $C^\xi_{\alpha_\zeta}$ of $\vec{C}^\xi$. By Fact 9.3, $\langle q_{\xi',\zeta} : \xi' < \xi\rangle$ forces that $q = E_\xi \cap \alpha_\zeta$ is the unique element of Q_ξ which contains all but finitely many elements of $\{\alpha_{\zeta_0+k} : k < \omega\}$. This finishes case 1.

Case 2: $\zeta \cap \lim(\omega_1)$ is cofinal in ζ. Our induction hypothesis implies that for all ζ' in $\zeta \cap \lim(\omega_1)$, $\alpha_{\zeta'}$ is a limit point of E_ξ. Hence $\{\alpha_{\zeta'} : \zeta' \in \zeta \cap \lim(\omega_1)\}$ is contained in E_ξ. By (*), $\langle q_{\xi',\zeta} : \xi' < \xi\rangle$ decides the element $C^\xi_{\alpha_\zeta}$ of $\vec{C}^\xi$. By Fact 9.3, $\langle q_{\xi',\zeta} : \xi' < \xi\rangle$ forces that $q = E_\xi \cap \alpha_\zeta$ is the unique element of Q_ξ which contains a tail of $\{\alpha_{\zeta'} : \zeta' \in \zeta \cap \lim(\omega_1)\}$. This finishes case 2 and the proof of the claim. □

With the claim in hand, we can apply the recursion theorem in L to find objects $\langle \alpha_\zeta : \zeta < \omega_1\rangle$ and $\langle q_{\xi,\zeta} : \xi < \omega^2; \zeta \in \lim(\omega_1)\rangle$ in L which satisfy the equations of the recursion. By Claim 9, we moreover have that, for each $\xi < \omega^2$, $E_\xi = \cup\{q_{\xi,\zeta} : \zeta \in \lim(\omega_1)\}$ and hence E_ξ is in L for all ξ. Note, however, that if q is in $Q_{\vec{C},E}$, then for all but countably many $\nu \in E$, $q \cup \{\nu\}$ is in $Q_{\vec{C},E}$ and, by genericity, E_1 does not contain any club from L, a contradiction. $\square$

It is known that the length ω^2 in this iteration is the shortest possible.

Theorem 9.10. *A CS iteration of length less than ω^2 of totally proper forcings satisfying the ω_1-iterability condition does not add reals.*

10. Open Problems

We will finish with some open problems. While the example discussed in Section 9 illustrates that we can not drop $(< \omega_1)$-properness from Theorem 4.5 entirely, it does not answer the following problem, which is ultimately what is of greatest interest.

Problem 10.1. *Assume it is consistent that there is a supercompact cardinal. Is the forcing axiom for completely proper forcings consistent with* CH*?*

We have seen that a positive answer to this question implies that measuring is consistent with CH (modulo a large cardinal assumption).

Problem 10.2. *Assume there is a measurable cardinal. Is there a countable support iteration of completely proper forcings which adds a new real?*

Of course if this question has a positive answer under the assumption of any large cardinal hypothesis, this would be of great interest. One might view that the "problem" with the example in Section 9 is that there are an insufficient number of embeddings $M \to N$ to give the definition of complete properness its intended strength. Since Woodin cardinals can be used to generate embeddings similar to $M \to N$ via the "countable tower" (see [7]), the existence of Woodin cardinals may be a natural hypothesis to consider in this context.

While club guessing on ω_1 is easily seen to be preserved by countably closed forcings, strong club guessing is not. This can be used to show that $2^{<\omega_1}$ forces that $Q_{\vec{C},E}$ does not preserve stationary subsets of ω_1 (where $Q_{\vec{C},E}$ is the forcing defined in Section 9.1). Shelah has proved the following iteration theorem in addition to Theorem 4.5.

Theorem 10.3. *A countable support iteration of forcings which are:*

1. *completely proper and*
2. *remain proper in every totally proper forcing extension*

does not add new reals.

In the context of totally proper forcings, the requirement that the forcing remains proper in every totally proper forcing extension is usually met by forcings which have only a countable "working part" and no "side conditions." The example in Section 7.1 has no side conditions (and remains proper in every totally proper extension). The examples in Sections 7.2 and 7.3 do have side conditions. The side condition in the forcing for measure a sequence $\vec{D}$ of closed sets can be removed if either the map $\alpha \mapsto \text{otp}(D_\alpha)$ is regressive or if D_α is a clopen subset of α for each $\alpha < \omega_1$.

It should be remarked that Shelah's two iteration theorems can not (apparently) be combined to a single theorem where we require that each iterand satisfies one of the two hypotheses. While the forcing associated to measuring is cut from the same block as the example in Section 9.1, we do not expect that the forcing to destroy a Souslin tree should "cause problems."

Problem 10.4. *Is it true that countable support iteration of forcings which are:*

(1) *completely proper and*
(2) *remain proper after forcing with* $2^{<\omega_1}$

does not add new reals?

The following is likely a closely related question.

Problem 10.5. *If* P *is a countable support iteration of completely proper forcings which adds a new real, must* $P \times 2^{<\omega_1}$ *collapse* ω_1*?*

The answer to this question, however, is not clear even in the case of the example in Section 9.

Problem 10.6. *If* P *is a countable support iteration of completely proper forcings which adds a new real, must* P *add a club* $E \subseteq \omega_1$ *such that for some* $\delta < \omega_1$, E *contains no ground model subsets of ordertype* δ*?*

If P is example in Section 9, then $\bigcap_{\xi<\omega^2} E_\xi$ contains no ground model infinite set.

References

[1] Uri Abraham and Stevo Todorčević. Partition properties of ω_1 compatible with CH. *Fund. Math.*, 152(2):165–181, 1997.
[2] Keith Devlin and Saharon Shelah. A weak version of $\diamondsuit$ which follows from $2^{\aleph_0} < 2^{\aleph_1}$. *Israel Journal of Math*, 29(2–3):239–247, 1978.
[3] Keith J. Devlin and Håvard Johnsbråten. *The Souslin problem*. Lecture Notes in Mathematics, Vol. 405. Springer-Verlag, Berlin, 1974.
[4] T. Eisworth and J. Roitman. CH with no Ostaszewski spaces. *Trans. Amer. Math. Soc.*, 351(7):2675–2693, 1999.
[5] Todd Eisworth and Peter Nyikos. First countable, countably compact spaces and the continuum hypothesis. *Trans. Amer. Math. Soc.*, 357(11):4269–4299, 2005.
[6] Matthew Foreman, Menachem Magidor, and Saharon Shelah. Martin's Maximum, saturated ideals, and nonregular ultrafilters. I. *Ann. of Math. (2)*, 127(1):1–47, 1988.
[7] Paul B. Larson. *The stationary tower*, volume 32 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2004. Notes on a course by W. Hugh Woodin.
[8] Justin Tatch Moore. ω_1 and ω_1 may be the only minimal uncountable order types. *Michigan Math. Journal*, 55(2):437–457, 2007.
[9] Adam Ostaszewski. On countably compact, perfectly normal spaces. *J. London Math. Soc.*, 14(3):505–516, 1976.
[10] Saharon Shelah. *Proper and improper forcing*. Springer-Verlag, Berlin, second edition, 1998.
[11] Robert Solovay and S. Tennenbaum. Iterated Cohen extensions and Souslin's problem. *Ann. of Math.*, 94:201–245, 1971.
[12] Stevo Todorcevic. A dichotomy for P-ideals of countable sets. *Fund. Math.*, 166(3):251–267, 2000.