Bite 01: A series converges no matter how it is permuted

Colin Tang*

June 12, 2026

1 Preliminaries

Let $(X, +)$ be an abelian group, and let $\|\bullet\|$ be a seminorm, i.e. a map $X \to [0, +\infty)$ satisfying the following three properties:

- $\|0\| = 0$
- For all $x \in X$, we have $\|-x\| = \|x\|$
- The triangle inequality: for all $x, y \in X$, we have $\|x + y\| \leq \|x\| + \|y\|$

Observe that X can be viewed as a pseudometric space, with the pseudometric given by $d(x, y) = \|x - y\|$.

Define $\mathbb{N} \coloneqq \{1, 2, \dots\}$. For a sequence $x_1, x_2, \cdots \in X$, we say that it is a **Cauchy sequence** if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m_1, m_2 > N$, we have $\|x_{m_1} - x_{m_2}\| < \epsilon$.

Fix elements $a_1, a_2, \cdots \in X$. Consider the following two statements:

(I) For all permutations $\sigma \colon \mathbb{N} \to \mathbb{N}$, the sequence $a_{\sigma(1)}, a_{\sigma(1)} + a_{\sigma(2)}, a_{\sigma(1)} + a_{\sigma(2)} + a_{\sigma(3)}, \dots$ is a Cauchy sequence.

(II) For all permutations $\sigma, \tau \colon \mathbb{N} \to \mathbb{N}$, and for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m_1, m_2 > N$, we have

$$\left\|(a_{\sigma(1)} + a_{\sigma(2)} + \cdots + a_{\sigma(m_1)}) - (a_{\tau(1)} + a_{\tau(2)} + \cdots + a_{\tau(m_2)})\right\| < \epsilon.$$

Note that (II) implies (I) (by taking $\sigma = \tau$).

2 Problem

Show that (I) implies (II). (Thus, (I) and (II) are equivalent.)

3 Interpretation

If we assume X is a complete metric space, then the condition (I) is saying that for any σ, the sum $\sum_{i=1}^{\infty} a_{\sigma(i)}$ converges. Condition (II) says that the value of $\sum_{i=1}^{\infty} a_{\sigma(i)}$ does not depend on σ. So the problem statement is essentially: *if a series converges no matter how its terms are permuted, then the series must converge to the same value no matter how its terms are permuted.*

*Carnegie Mellon University, United States of America, cstang@andrew.cmu.edu

4 Solution

We introduce a new statement:[1]

(III) For all permutations $\sigma\colon \mathbb{N} \to \mathbb{N}$, and for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all finite subsets $S \subset \mathbb{N}$ which satisfy $S \cap \{\sigma(1), \sigma(2), \dots, \sigma(N)\} = \emptyset$, we have

$$\left\| \sum_{k \in S} a_k \right\| < \epsilon.$$

The proof proceeds by showing the implications ¬(II) $\implies$ ¬(III) $\implies$ ¬(I).

4.1 Proof of ¬(II) $\implies$ ¬(III)

Assume there exist permutations $\sigma, \tau\colon \mathbb{N} \to \mathbb{N}$ and $\epsilon > 0$ such that for all $N \in \mathbb{N}$, there exist $m_1, m_2 > N$ satisfying

$$\left\| (a_{\sigma(1)} + a_{\sigma(2)} + \cdots + a_{\sigma(m_1)}) - (a_{\tau(1)} + a_{\tau(2)} + \cdots + a_{\tau(m_2)}) \right\| \geq \epsilon.$$

Set $\epsilon' \coloneqq \epsilon/2$ and $\sigma' \coloneqq \sigma$. We will show that for all $N' \in \mathbb{N}$, there exists a finite subset $S' \subset \{\sigma'(N'+1), \sigma'(N'+2), \dots\}$ such that

$$\left\| \sum_{k \in S'} a_k \right\| \geq \epsilon'.$$

Let $N' \in \mathbb{N}$ be arbitrary. Choose $N \geq N'$ sufficiently large, such that

$$\{\sigma(1), \sigma(2), \dots, \sigma(N')\} \subseteq \{\tau(1), \tau(2), \dots, \tau(N)\}.$$

Now, for any $m_1, m_2 > N$, we have that the terms $a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(N')}$ occur in both $(a_{\sigma(1)} + a_{\sigma(2)} + \cdots + a_{\sigma(m_1)})$ and $(a_{\tau(1)} + a_{\tau(2)} + \cdots + a_{\tau(m_2)})$. Thus we may cancel them out, and write

$$\begin{aligned}(a_{\sigma(1)} + a_{\sigma(2)} + \cdots + a_{\sigma(m_1)}) - (a_{\tau(1)} + a_{\tau(2)} + \cdots + a_{\tau(m_2)}) \\ = \sum_{k \in \{\sigma(N'+1), \sigma(N'+2), \dots, \sigma(m_1)\}} a_k - \sum_{k \in \{\tau(1), \tau(2), \dots, \tau(m_2)\} \setminus \{\sigma(1), \sigma(2), \dots, \sigma(N')\}} a_k.\end{aligned}$$

By the triangle inequality, we have

$$\begin{aligned}&\left\| (a_{\sigma(1)} + a_{\sigma(2)} + \cdots + a_{\sigma(m_1)}) - (a_{\tau(1)} + a_{\tau(2)} + \cdots + a_{\tau(m_2)}) \right\| \\ &\leq \left\| \sum_{k \in \{\sigma(N'+1), \sigma(N'+2), \dots, \sigma(m_1)\}} a_k \right\| + \left\| - \sum_{k \in \{\tau(1), \tau(2), \dots, \tau(m_2)\} \setminus \{\sigma(1), \sigma(2), \dots, \sigma(N')\}} a_k \right\| \\ &= \left\| \sum_{k \in \{\sigma(N'+1), \sigma(N'+2), \dots, \sigma(m_1)\}} a_k \right\| + \left\| \sum_{k \in \{\tau(1), \tau(2), \dots, \tau(m_2)\} \setminus \{\sigma(1), \sigma(2), \dots, \sigma(N')\}} a_k \right\|.\end{aligned}$$

Since $\left\| (a_{\sigma(1)} + a_{\sigma(2)} + \cdots + a_{\sigma(m_1)}) - (a_{\tau(1)} + a_{\tau(2)} + \cdots + a_{\tau(m_2)}) \right\| \geq \epsilon$, we see that we have

$$\max\left(\left\| \sum_{k \in \{\sigma(N'+1), \sigma(N'+2), \dots, \sigma(m_1)\}} a_k \right\|, \left\| \sum_{k \in \{\tau(1), \tau(2), \dots, \tau(m_2)\} \setminus \{\sigma(1), \sigma(2), \dots, \sigma(N')\}} a_k \right\| \right) \geq \epsilon/2 = \epsilon'$$

and hence at least one of the choices $S' = \{\sigma(N'+1), \sigma(N'+2), \dots, \sigma(m_1)\}$ or $S' = \{\tau(1), \tau(2), \dots, \tau(m_2)\} \setminus \{\sigma(1), \sigma(2), \dots, \sigma(N')\}$ will work.

[1] Actually, in (III) the outermost quantifier $\forall\sigma$ may be replaced by $\exists\sigma$: if the statement is true for one choice of σ, then it is true for all choices of σ.

4.2 Proof of ¬(III) $\implies$ ¬(I)

Assume there exists a permutation $\sigma\colon \mathbb{N} \to \mathbb{N}$ and $\epsilon > 0$ such that for all $N \in \mathbb{N}$, there exists a finite subset $S_N \subset \{\sigma(N+1), \sigma(N+2), \dots\}$ satisfying

$$\left\| \sum_{k \in S_N} a_k \right\| \geq \epsilon.$$

Set $\epsilon' \coloneqq \epsilon$. We will construct a permutation $\sigma'\colon \mathbb{N} \to \mathbb{N}$ such that for all $N' \in \mathbb{N}$, there exist $m_2' > m_1' > N'$ such that

$$\|a_{\sigma'(m_1'+1)} + a_{\sigma'(m_1'+2)} + \cdots + a_{\sigma'(m_2')}\| \geq \epsilon'.$$

First, let us find a subsequence $N_1 < N_2 < \dots$ such that $S_{N_1}, S_{N_2}, \dots$ are pairwise disjoint. Now, choose σ' such that in the sequence $\sigma'(1), \sigma'(2), \dots$, the following holds:

- For each $r \in \mathbb{N}$, the elements of S_{N_r} appear consecutively in this sequence (i.e. there exist $i_r < j_r$ such that $S_{N_r} = \{\sigma'(i_r+1), \sigma'(i_r+2), \dots, \sigma'(j_r)\}$).

Now, given $N' \in \mathbb{N}$, simply take sufficiently large r such that $j_r > i_r > N'$. Then, setting $m_2' = j_r$ and $m_1' = i_r$, we have

$$\begin{aligned}
&\|a_{\sigma'(m_1'+1)} + a_{\sigma'(m_1'+2)} + \cdots + a_{\sigma'(m_2')}\| \\
&= \|a_{\sigma'(i_r+1)} + a_{\sigma'(i_r+2)} + \cdots + a_{\sigma'(j_r)}\| \\
&= \left\| \sum_{k \in S_{N_r}} a_k \right\| \\
&\geq \epsilon = \epsilon',
\end{aligned}$$

as desired.

5 Remarks

If $X = \mathbb{R}$ and $\|\bullet\|$ is taken to be the Euclidean norm, then by the Riemann rearrangement theorem, we have that (I), (II), (III) are equivalent to the assertion that the series $\sum_{i=1}^{\infty} a_i$ is absolutely convergent. However, this is not true for general X, in the sense that if $X = \mathbb{Q}_p$ (the p-adic numbers) and $\|\bullet\|$ is taken to be the p-adic absolute value $|\bullet|_p$, then (I), (II), (III) are equivalent to the assertion that $|a_i|_p \to 0$ as $i \nearrow +\infty$, which does not imply that $\sum_{i=1}^{\infty} |a_i|_p < +\infty$.

Suppose $(X_1, \|\bullet\|_1)$ and $(X_2, \|\bullet\|_2)$ are complete metric spaces such that for any $x_1 \in X_1$ there exists a continuous map $f\colon X_1 \to X_2$ satisfying

- $f(x_1) \neq 0$
- For all $x, y \in X_1$, we have $f(x+y) = f(x) + f(y)$.

Let $a_1, a_2, \dots \in X_1$, and assume that (I) holds of $a_1, a_2, \dots$ in X_1. If (II) holds of $f(a_1), f(a_2), \dots$ in X_2, then we may conclude that (II) holds of $a_1, a_2, \dots$ in X_1. As an example, for any Banach space V, we may set $X_1 = V$ and $X_2 = \mathbb{R}$, where f can be shown to exist by the Hahn-Banach theorem.