# Ultraproducts of measure preserving actions and graph combinatorics

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## 0 Introduction

In this paper we apply the method of ultraproducts to the study of graph combinatorics associated with measure preserving actions of infinite, countable groups, continuing the work in Conley-Kechris [CK].

We employ the ultraproduct construction as a flexible method to produce measure preserving actions $a$ of a countable group $\Gamma$ on a standard measure space $(X, \mu)$ (i.e., a standard Borel space with its $\sigma$-algebra of Borel sets and a Borel probability measure) starting from a sequence of such actions $a_n$ on $(X_n, \mu_n), n \in \mathbb{N}$. One uses a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ to generate the ultraproduct action $\prod_n a_n/\mathcal{U}$ of $(a_n)$ on a measure space $(X_\mathcal{U}, \mu_\mathcal{U})$, obtained as the ultraproduct of $((X_n, \mu_n))_{n\in\mathbb{N}}$ via the Loeb measure construction. The measure algebra of the space $(X_\mathcal{U}, \mu_\mathcal{U})$ is non-separable, but by taking appropriate countably generated subalgebras of this measure algebra one generates factors $a$ of the action $\prod_n a_n/\mathcal{U}$ which are now actions of $\Gamma$ on a standard measure space $(X, \mu)$ and which have various desirable properties.

In §2, we discuss the construction of the ultrapower $(X_\mathcal{U}, \mu_\mathcal{U})$ of a sequence of standard measure spaces $(X_n, \mu_n), n \in \mathbb{N}$, with respect to a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$, via the Loeb measure construction. We follow largely the exposition in Elek-Szegedy [ES], which dealt with the case of finite spaces $X_n$ with $\mu_n$ the counting measure.

In §3, we define the ultraproduct action $\prod_n a_n/\mathcal{U}$ on $(X_\mathcal{U}, \mu_\mathcal{U})$ associated with $a$ sequence $a_n, n \in \mathbb{N}$, of measure preserving actions of a countable group $\Gamma$ on $(X_n, \mu_n)$ and discuss its freeness properties. When $a_n = a$ for all $n$, we put $a_\mathcal{U} = \prod_n a_n/\mathcal{U}$.

In §4, we characterize the factors of the action $\prod_n a_n/\mathcal{U}$ associated with countably generated $\sigma$-subalgebras of the measure algebra of $(X_\mathcal{U}, \mu_\mathcal{U})$.

For a measure space $(X, \mu)$ and a countable group $\Gamma$, we denote by $A(\Gamma, X, \mu)$ the space of measure preserving actions of $\Gamma$ on $(X, \mu)$ (where, as usual, actions are identified if they agree a.e.). This space carries the *weak topology* generated by the maps $a \in A(\Gamma, X, \mu) \mapsto \gamma^a \cdot A$ $(\gamma \in \Gamma, A \in \mathrm{MALG}_\mu)$, from $A(\Gamma, X, \mu)$ into the measure algebra $\mathrm{MALG}_\mu$ (with the usual metric $d_\mu(A, B) = \mu(A \Delta B)$), and where we put $\gamma^a \cdot x = a(\gamma, x)$. When $(X, \mu)$ is standard, $A(\Gamma, X, \mu)$ is a Polish space.

If $a \in A(\Gamma, X, \mu), a_n \in A(\Gamma, X_n, \mu_n), n \in \mathbb{N}$, and $\mathcal{U}$ is a non-principal ultrafilter on $\mathbb{N}$, we say that $a$ is *weakly $\mathcal{U}$-contained* in $(a_n)$, in symbols

$$a \prec_{\mathcal{U}} (a_n),$$

if for every finite $F \subseteq \Gamma, A_1, \dots, A_N \in \mathrm{MALG}_\mu, \epsilon > 0$, for $\mathcal{U}$-almost all $n$:

$$\exists B_{1,n} \dots \exists B_{N,n} \in \mathrm{MALG}_{\mu_n} \forall \gamma \in F \forall i, j \leq N$$

$$|\mu(\gamma^a \cdot A_i \cap A_j) - \mu_n(\gamma^{a_n} \cdot B_{i,n} \cap B_{j,n})| < \epsilon,$$

(where a property $P(n)$ is said to hold for *$\mathcal{U}$-almost all $n$* if $\{n \colon P(n)\} \in \mathcal{U}$). In case $a_n = b$ for all $n$, then $a \prec_{\mathcal{U}} (a_n) \Leftrightarrow a \prec b$ (in the sense of weak containment of actions, see Kechris [Ke2]).

If $a, b_n \in A(\Gamma, X, \mu), n \in \mathbb{N}$, we write

$$\lim_{n \to \mathcal{U}} b_n = a$$

if for each open nbhd $V$ of $a$ in $A(\Gamma, X, \mu), b_n \in V$, for $\mathcal{U}$-almost all $n$. Finally $a \cong b$ denotes isomorphism (conjugacy) of actions.

We show the following (in 4.3):

**Theorem 1** *Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. Let $(X, \mu)$, $(X_n, \mu_n)$, $n \in \mathbb{N}$ be non-atomic, standard measure spaces and let $a \in A(\Gamma, X, \mu)$, $a_n \in A(\Gamma, X_n, \mu_n)$. Then the following are equivalent:*

*(1)* $a \prec_{\mathcal{U}} (a_n)$,

*(2) $a$ is a factor of* $\prod_n a_n / \mathcal{U}$,

*(3)* $a = \lim_{n \to \mathcal{U}} b_n$, *for some sequence* $(b_n)$, *with*

$$b_n \in A(\Gamma, X, \mu), b_n \cong a_n, \forall n \in \mathbb{N},$$

In particular, for $a \in A(\Gamma, X, \mu), b \in A(\Gamma, Y, \nu), a \prec b$ is equivalent to "$a$ is a factor of $b_{\mathcal{U}}$". Moreover one has the following curious compactness property of $A(\Gamma, X, \mu)$ as a consequence of Theorem 1: If $a_n \in A(\Gamma, X, \mu), n \in \mathbb{N}$, then there is $n_0 < n_1 < n_2 < \dots$ and $b_{n_i} \in A(\Gamma, X, \mu), b_{n_i} \cong a_{n_i}$, such that $(b_{n_i})$ converges in $A(\Gamma, X, \mu)$.

In §5, we apply the ultraproduct construction to the study of combinatorial parameters associated to group actions. Given an infinite group $\Gamma$ with a finite set of generators $S$, not containing 1, and given a free action $a$ of $\Gamma$ on a standard space $(X, \mu)$, the (simple, undirected) graph $G(S, a)$ has vertex set $X$ and edge set $E(S, a)$, where

$$(x, y) \in E(S, a) \Leftrightarrow x \neq y \ \& \ \exists s \in S(s^a \cdot x = y \text{ or } s^a \cdot y = x).$$

As in Conley-Kechris [CK], we define the associated parameters $\chi_\mu(S, a)$ (the *measurable chromatic number*), $\chi_\mu^{ap}(S, a)$ (the *approximate chromatic number*) and $i_\mu(S, a)$ (the *independence number*), as follows:

- $\chi_\mu(S, a)$ is the smallest cardinality of a standard Borel space $Y$ for which there is a ($\mu-$)measurable coloring $c\colon X \to Y$ of $G(S, a)$ (i.e., $xE(S, a)y \Rightarrow c(x) \neq c(y)$).
- $\chi_\mu^{ap}(S, a)$ is the smallest cardinality of a standard Borel space $Y$ such that for each $\epsilon > 0$, there is a Borel set $A \subseteq X$ with $\mu(X \setminus A) < \epsilon$ and a measurable coloring $c\colon A \to Y$ of the induced subgraph $G(S, a)|A = (A, E(S, A) \cap A^2)$.
- $i_\mu(S, a)$ is the supremum of the measures of Borel independent sets, where $A \subseteq X$ is *independent* if no two elements of $A$ are adjacent.

Given a (simple, undirected) graph $G = (X, E)$, where $X$ is the set of vertices and $E$ the set of edges, a *matching* in $G$ is a subset $M \subseteq E$ such that no two edges in $M$ have a common vertex. We denote by $X_M$ the set of matched vertices, i.e., the set of vertices belonging to an edge in $M$. If $X_M = X$ we say that $M$ is a *perfect matching.*

For a free action $a$ of $\Gamma$ as before, we also define the parameter

$$m(S, a) = \text{ the } \textit{matching number,}$$

where $m(S, a)$ is 1/2 of the supremum of $\mu(X_M)$, with $M$ a Borel (as a subset of $X^2$) matching in $G(S, a)$. If $m(S, a) = 1/2$ and the supremum is attained, we say that $G(S, a)$ admits an *a.e. perfect matching.*

The parameters $i_\mu(S, a), m(S, a)$ are monotone increasing with respect to weak containment, while $\chi_\mu^{ap}(S, a)$ is decreasing. Below we let $a \sim_w b$ denote *weak equivalence* of actions, where $a \sim_w b \Leftrightarrow a \prec b$ & $b \prec a$, and we let $a \sqsubseteq b$ denote that $a$ is a factor of $b$. We now have (see 5.2)

**Theorem 2** *Let $\Gamma$ be an infinite, countable group and $S$ a finite set of generators. Then for any free action $a$ of $\Gamma$ on a non-atomic, standard measure space $(X, \mu)$, there is a free action $b$ of $\Gamma$ on $(X, \mu)$ such that*

*(i)* $a \sim_w b$ *and* $a \sqsubseteq b$,

*(ii)* $\chi_\mu^{ap}(S, a) = \chi_\mu^{ap}(S, b) = \chi_\mu(S, b)$,

*(iii)* $i_\mu(S, a) = i_\mu(S, b)$ *and* $i_\mu(S, b)$ *is attained,*

*(iv)* $m(S, a) = m(S, b)$ *and* $m(S, b)$ *is attained.*

In §6, we study analogs of the classical Brooks' Theorem for finite graphs, which asserts that the chromatic number of a finite graph $G$ with degree bounded by $d$ is $\leq d$ unless $d = 2$ and $G$ contains an odd cycle or $d \geq 3$ and $G$ contains the complete subgraph with $d + 1$ vertices.

Let $\Gamma, S$ be as in the preceding discussion, so that the graph $G(S, a)$ associated with a free action $a$ of $\Gamma$ on a standard space $(X, \mu)$ has degree $d = |S^{\pm 1}|$, where $S^{\pm 1} = S \cup S^{-1}$. It was shown in Conley-Kechris [CK] that $\chi_\mu^{ap}(S, a) \leq d$, so one has an "approximate" version of Brooks' Theorem. Using this and the results of §5, we now have (see 6.11):

**Theorem 3** *Let $\Gamma$ be an infinite group and $S$ a finite set of generators. Then for any free action $a$ of $\Gamma$ on a non-atomic, standard space $(X, \mu)$, there is a free action $b$ on $(X, \mu)$ such that $a \sim_w b$ and $\chi_\mu(S, b) \leq d$ $(= |S^{\pm 1}|)$.*

It is not the case that for *every* free action $a$ of $\Gamma$ we have $\chi_\mu(S, a) \leq d$, but the only counterexamples known are $\Gamma = \mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ (with the usual sets of generators) and Conley-Kechris [CK] show that these are the only counterexamples if $\Gamma$ has finitely many ends.

The previous result can be used to answer a question in probability theory (see Aldous-Lyons [AL]), namely whether for any $\Gamma, S$, there is an invariant, random $d$-coloring of the Cayley graph $\mathrm{Cay}(\Gamma, S)$ (an earlier result of Schramm (unpublished, 1997) shows that this is indeed the case with $d$ replaced by $d+1$). A *random $d$-coloring* is a probability measure on the Borel sets of the space of $d$-colorings of the Cayley graph $\mathrm{Cay}(\Gamma, S)$ and invariance refers to the canonical shift action of $\Gamma$ on this space.

We now have (see 6.4):

**Theorem 4** *Let $\Gamma$ be an infinite group and $S$ a finite set of generators with $d = |S^{\pm 1}|$. Then there is an invariant, random $d$-coloring. Moreover for any free action $a$ of $\Gamma$ on a non-atomic, standard space $(X, \mu)$, there is such a coloring weakly contained in $a$.*

Let $\mathrm{Aut}_{\Gamma,S}$ be the automorphism group of the Cayley graph $\mathrm{Cay}(\Gamma, S)$ with the pointwise convergence topology. This is a Polish locally compact group containing $\Gamma$ as a closed subgroup. One can consider invariant, random colorings under the canonical action of $\mathrm{Aut}_{\Gamma,S}$ on the space of colorings, which we call $\mathrm{Aut}_{\Gamma,S}$*-invariant, random colorings.* This appears to be a stronger notion but we note in 6.6 that the existence of a $\mathrm{Aut}_{\Gamma,S}$-invariant, random $d$-coloring is equivalent to the existence of an invariant, random $d$-coloring, so Theorem 4 works as well for $\mathrm{Aut}_{\Gamma,S}$-invariant, random colorings.

One can also ask whether the last statement in Theorem 4 can be improved to "is a factor of" instead of "weakly contained in". This again fails for $\Gamma = \mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ and $a$ the shift action of $\Gamma$ on $[0,1]^\Gamma$, a case of primary interest, but holds for all other $\Gamma$ that have finitely many ends. Moreover in the case of the shift action one has also $\mathrm{Aut}_{\Gamma,S}$-invariance (see 6.7).

**Theorem 5** *Let $\Gamma$ be an infinite group and $S$ a finite set of generators with $d = |S^{\pm 1}|$. If $\Gamma$ has finitely many ends but is not isomorphic to $\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$, then there is a $\mathrm{Aut}_{\Gamma,S}$-invariant, random $d$-coloring which is a factor of the shift action of $\mathrm{Aut}_{\Gamma,S}$ on $[0,1]^\Gamma$.*

In §7, we discuss various results about a.e. perfect matchings and invariant, random matchings. Lyons-Nazarov [LN] showed that if $\Gamma$ is a non-amenable group with a finite set of generators $S$ and $\mathrm{Cay}(\Gamma, S)$ is bipartite (i.e., has no odd cycles), then there is a $\mathrm{Aut}_{\Gamma,S}$-invariant, random perfect matching of its Cayley graph, which is a factor of the shift action of $\mathrm{Aut}_{\Gamma,S}$ on $[0,1]^\Gamma$. This also implies that $m(S, s_\Gamma) = \frac{1}{2}$, where $s_\Gamma$ is the shift action of $\Gamma$ on $[0,1]^\Gamma$, and in fact the graph associated with this action has an a.e. perfect matching. We do not know if $m(S, a) = \frac{1}{2}$ actually holds for *every* $\Gamma, S$ and *every* free action $a$. We note in 7.5 that the only possible counterexamples are those $\Gamma, S$ for which $\Gamma$ is not amenable and $S$ consists of elements of odd order. However we show in 7.6 the following:

**Theorem 6** *Let $\Gamma = (\mathbb{Z}/3\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z})$ with the usual set of generators $S = \{s, t\}$, where $s^3 = t^3 = 1$. Then for any free action $a$ of $\Gamma$ on a non-atomic, standard measure space $(X, \mu)$, $G(S, a)$ admits an a.e. perfect matching.*

In §8, we study independence numbers. In Conley-Kechris [CK], the following was shown: Let $\Gamma, S$ be as before. Then the set of independence numbers $i_\mu(S,a)$, as $a$ varies over all free actions of $\Gamma$, is a closed interval. The question was raised about the structure of the set of all $i_\mu(S,a)$, where $a$ varies over all free, *ergodic* actions of $\Gamma$. We show the following (in 8.1).

**Theorem 7** *Let $\Gamma$ be an infinite group with $S$ a finite set of generators. If $\Gamma$ has property* (T)*, the set of $i_\mu(S,a)$ as $a$ varies over all the free, ergodic actions of $\Gamma$ is closed.*

We do not know what happens in general if $\Gamma$ does not have property (T) but we show in 8.2 that for certain groups of the form $\mathbb{Z} * \Gamma$ and generators $S$, the set of $i_\mu(S,a)$, for free, ergodic $a$, is infinite.

In §9, we discuss the notion of sofic equivalence relations and sofic actions, recently introduced in Elek-Lippner [EL1]. We use ultraproducts and a result of Abért-Weiss [AW] to give (in 9.6) an alternative proof of the theorem of Elek-Lippner [EL1] that the shift action of an infinite countable sofic group in sofic and discuss some classes of groups $\Gamma$ for which *every* free action is sofic.

Elek-Lippner [EL1] raised the question of whether *every* free action of a sofic group is sofic.

**Addendum.** After receiving a preliminary version of this paper, Miklós Abért informed us that he and Gábor Elek have independently developed similar ideas concerning the use of ultraproducts in studying group actions and their connections with weak containment and combinatorics. Their results are included in [AE]. In particular, [AE] contains versions of 4.7 and Theorem 5.2 (i), (ii) below.

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# 1 Preliminaries

We review here some standard terminology and notation that will be used throughout the paper.

**(A)** A *standard measure space* is a measure space $(X, \mu)$, where $X$ is standard Borel space (i.e., a Polish space with its $\sigma$-algebra of Borel sets) and $\mu$ a probability measure on the $\sigma$-algebra $\boldsymbol{B}(X)$ of Borel sets. We do not assume in this paper that $(X, \mu)$ is non-atomic, since we do want to include in this definition also finite measure spaces. If $(X, \mu)$ is supposed to be non-atomic in a given context, this will be stated explicitly.

The *measure algebra* $\mathrm{MALG}_\mu$ of a measure space $(X, \mu)$ is the Boolean $\sigma$-algebra of measurable sets modulo null sets equipped with the measure $\mu$.

As a general convention in dealing with measure spaces, we will often neglect null sets, if there is no danger of confusion.

**(B)** If $(X, \mu)$ is a standard measure space and $E \subseteq X^2$ a countable Borel equivalence relation on $X$ (i.e., one whose equivalence classes are countable), we say that $E$ is *measure preserving* if for all Borel bijections $\varphi\colon A \to B$, where $A, B$ are Borel subsets of $X$, such that $\varphi(x) E x$, $\mu$-a.e.$(x \in A)$, we have that $\varphi$ preserves the measure $\mu$.

Such an equivalence relation is called *treeable* if there is a Borel acyclic graph on $X$ whose connected components are the equivalence classes.

**(C)** If $\Gamma$ is an infinite, countable group and $S$ a finite set of generators, not containing 1, the *Cayley graph* $\mathrm{Cay}(\Gamma, S)$, is the (simple, undirected) graph with set of vertices $\Gamma$ and in which $\gamma, \delta \in \Gamma$ are connected by an edge iff $\exists s \in S(\gamma s = \delta \text{ or } \delta s = \gamma)$.

Finally for such $\Gamma, S$ the number of *ends* of $\mathrm{Cay}(\Gamma, S)$ is the supremum of the number of infinite components, when any finite set of vertices is removed. This number is independent of $S$ and it is equal to $1, 2$ or $\infty$.

## 2 Ultraproducts of standard measure spaces

**(A)** Let $(X_n, \mu_n), n \in \mathbb{N}$, be a sequence of standard measure spaces and denote by $\boldsymbol{B}(X_n)$ the $\sigma$-algebra of Borel sets of $X_n$. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. For $P \subseteq \mathbb{N} \times X$ ($X$ some set) we write

$$\mathcal{U}n P(n, x) \Leftrightarrow \{n \colon P(n, x)\} \in \mathcal{U}.$$

If $\mathcal{U}n P(n, x)$ we also say that for $\mathcal{U}$-*almost all* $n, P(n, x)$ holds. On $\prod_n X_n$ define the equivalence relation

$$(x_n) \sim_{\mathcal{U}} (y_n) \Leftrightarrow \mathcal{U}n(x_n = y_n),$$

let $[(x_n)]_{\mathcal{U}}$ be the $(\sim_{\mathcal{U}})$-equivalence class of $(x_n)$ and put

$$X_{\mathcal{U}} = (\prod_n X_n)/\mathcal{U} = \{[(x_n)]_{\mathcal{U}} \colon (x_n) \in \prod_n X_n\}.$$

Given now $(A_n) \in \prod_n \boldsymbol{B}(X_n)$, we define $[(A_n)]_{\mathcal{U}} \subseteq X_{\mathcal{U}}$ by

$$[(x_n)]_{\mathcal{U}} \in [(A_n)]_{\mathcal{U}} \Leftrightarrow \mathcal{U}n(x_n \in A_n).$$

Note that

$$\begin{aligned} [(\sim A_n)]_{\mathcal{U}} &= \sim [(A_n)]_{\mathcal{U}} \\ [(A_n \cup B_n)]_{\mathcal{U}} &= [(A_n)]_{\mathcal{U}} \cup [(B_n)]_{\mathcal{U}} \\ [(A_n \cap B_n)]_{\mathcal{U}} &= [(A_n)]_{\mathcal{U}} \cap [(B_n)]_{\mathcal{U}}, \end{aligned}$$

where $\sim$ denotes complementation. Put

$$\boldsymbol{B}^0_{\mathcal{U}} = \{[(A_n)]_{\mathcal{U}} \colon (A_n) \in \prod_n \boldsymbol{B}(X_n)\},$$

so that $\boldsymbol{B}^0_{\mathcal{U}}$ is a Boolean algebra of subsets of $X_{\mathcal{U}}$.

For $[(A_n)]_{\mathcal{U}} \in \boldsymbol{B}^0_{\mathcal{U}}$, put

$$\mu_{\mathcal{U}}([(A_n)]_{\mathcal{U}}) = \lim_{n \to \mathcal{U}} \mu_n(A_n),$$

where $\lim_{n \to \mathcal{U}} r_n$ denotes the ultrafilter limit of the sequence $(r_n)$. It is easy to see that $\mu_{\mathcal{U}}$ is a finitely additive probability Borel measure on $\boldsymbol{B}^0_{\mathcal{U}}$. We will extend it to a (countably additive) probability measure on a $\sigma$-algebra containing $\boldsymbol{B}^0_{\mathcal{U}}$.

**Definition 2.1** *A set $N \subseteq X_{\mathcal{U}}$ is null if $\forall \epsilon > 0 \exists A \in \boldsymbol{B}^0_{\mathcal{U}}$ ($N \subseteq A$ and $\mu_{\mathcal{U}}(A) < \epsilon$). Denote by $\boldsymbol{N}$ the collection of nullsets.*

**Proposition 2.2** *The collection $\boldsymbol{N}$ is a $\sigma$-ideal of subsets of $X_{\mathcal{U}}$.*

**Proof**. It is clear that $\boldsymbol{N}$ is closed under subsets. We will now show that it is closed under countable unions.

**Lemma 2.3** *Let $A^i \in \boldsymbol{B}^0_{\mathcal{U}}, i \in \mathbb{N}$, and assume that $\lim_{m \to \infty} \mu_{\mathcal{U}}(\bigcup_{i=0}^m A^i) = t$. Then there is $A \in \boldsymbol{B}^0_{\mathcal{U}}$ with $\mu_{\mathcal{U}}(A) = t$ and $\bigcup_i A^i \subseteq A$.*

Granting this let $N^i \in \boldsymbol{N}, i \in \mathbb{N}, \epsilon > 0$ be given. Let $N^i \subseteq A^i \in \boldsymbol{B}^0_{\mathcal{U}}$ with $\mu_{\mathcal{U}}(A^i) \leq \epsilon/2^i$. Then $\mu_{\mathcal{U}}(\bigcup_{i=0}^m A^i) \leq \epsilon$ and $\mu_{\mathcal{U}}(\bigcup_{i=0}^m A^i)$ increases with $m$. So

$$\lim_{m \to \mathcal{U}} \mu_{\mathcal{U}}(\bigcup_{i=0}^{m} A^i) = t \leq \epsilon$$

and by the lemma there is $A \in \boldsymbol{B}^0_{\mathcal{U}}$ with $\mu_{\mathcal{U}}(A) \leq \epsilon$ and $\bigcup_i N^i \subseteq \bigcup_i A^i \subseteq A$. So $\bigcup_i N^i$ is null.

**Proof of 2.3**. Put $B^m = \bigcup_{i=0}^m A^i$, so that $\mu_{\mathcal{U}}(B^m) = t_m \to t$. Let $A^i = [(A^i_n)]_{\mathcal{U}}$, so that $B^m = [(B^m_n)]_{\mathcal{U}}$, with $B^m_n = \bigcup_{i=0}^m A^i_n$. Let

$$T_m = \left\{ n \geq m \colon |\mu_n(B^m_n) - t_m| \leq \frac{1}{2^m} \right\},$$

so that $\bigcap_m T_m = \emptyset$ and $T_m \in \mathcal{U}$, as $t_m = \mu_{\mathcal{U}}(B^m) = \lim_{n \to \mathcal{U}} \mu_n(B^m_n)$.

Let $m(n) =$ largest $m$ such that $n \in \bigcap_{\ell \leq m} T_\ell$. Then $m(n) \to \infty$ as $n \to \mathcal{U}$, since for each $M$, $\{n \colon m(n) \geq M\} \supseteq \bigcap_{m=0}^M T_m \in \mathcal{U}$. Also $n \in T_{m(n)}$. So

$$|\mu_{m(n)}(B^{m(n)}_n) - t_{m(n)}| \leq \frac{1}{2^{m(n)}},$$

thus

$$\lim_{n \to \mathcal{U}} \mu_n(B^{m(n)}_n) = t.$$

Let $A = [(B^{m(n)}_n)]_{\mathcal{U}}$. Then $\mu_{\mathcal{U}}(A) = t$. Also for each $i$,

$$\{n \colon A^i_n \subseteq B^{m(n)}_n\} \supseteq \{n \colon m(n) \geq i\} \in \mathcal{U},$$

so $A^i = [(A^i_n)]_{\mathcal{U}} \subseteq [(B^{m(n)}_n)]_{\mathcal{U}} = A$, thus $\bigcup_i A^i \subseteq A$. $\dashv$

Put

$$\boldsymbol{B}_{\mathcal{U}} = \{A \subseteq X_{\mathcal{U}} \colon \exists A' \in \boldsymbol{B}^0_{\mathcal{U}} (A \Delta A' \in \boldsymbol{N})\},$$

and for $A \in \boldsymbol{B}_{\mathcal{U}}$ put

$$\mu_{\mathcal{U}}(A) = \mu_{\mathcal{U}}(A')$$

where $A' \in \boldsymbol{B}^0_{\mathcal{U}}$, $A \Delta A' \in \boldsymbol{N}$. This is clearly well defined and agrees with $\mu_{\mathcal{U}}$ on $\boldsymbol{B}^0_{\mathcal{U}}$.

**Proposition 2.4** *The class $\boldsymbol{B}_{\mathcal{U}}$ is a $\sigma$-algrebra of subsets of $X_{\mathcal{U}}$ containing $\boldsymbol{B}^0_{\mathcal{U}}$ and $\mu_{\mathcal{U}}$ is a probability measure on $\boldsymbol{B}_{\mathcal{U}}$.*

**Proof**. It is easy to see that $\boldsymbol{B}_{\mathcal{U}}$ is a Boolean algebra containing $\boldsymbol{B}^0_{\mathcal{U}}$ and $\mu_{\mathcal{U}}$ is a finitely additive probability measure on $\boldsymbol{B}_{\mathcal{U}}$. It only remains to show that if $A_n \in \boldsymbol{B}_{\mathcal{U}}, n \in \mathbb{N}$, are pairwise disjoint, then $\bigcup_n A_n \in \boldsymbol{B}_{\mathcal{U}}$ and $\mu_{\mathcal{U}}(\bigcup_n A_n) = \sum_n \mu_{\mathcal{U}}(A_n)$.

For $A, A' \in \boldsymbol{B}_{\mathcal{U}}$, let

$$A \equiv A' \Leftrightarrow A \Delta A' \in \boldsymbol{N}.$$

Let now $A'_n \in \boldsymbol{B}^0_{\mathcal{U}}$ be such that $A_n \equiv A'_n$. By disjointifying, we can assume that the $A'_n$ are disjoint. Note also that $\bigcup_n A_n \equiv \bigcup_n A'_n$. It is thus enough to find $A' \in \boldsymbol{B}^0_{\mathcal{U}}$ with $A' \equiv \bigcup_n A'_n$ and $\mu_{\mathcal{U}}(A') = \sum_n \mu_{\mathcal{U}}(A'_n)$ $(= \sum_n \mu_{\mathcal{U}}(A_n))$.

By Lemma 2.3, there is $A' \in \boldsymbol{B}^0_{\mathcal{U}}$ with $\bigcup_n A'_n \subseteq A'$ and $\mu_{\mathcal{U}}(A') = \sum_n \mu_{\mathcal{U}}(A'_n)$. Then for each $N$,

$$A' \setminus \bigcup_n A'_n \subseteq A' \setminus \bigcup_{n=0}^{N} A'_n \in \boldsymbol{B}^0_{\mathcal{U}}$$

and

$$\mu_{\mathcal{U}}(A' \setminus \bigcup_{n=0}^{N} A'_n) = \mu_{\mathcal{U}}(A') - \sum_{n=0}^{N} \mu_{\mathcal{U}}(A'_n) \to 0$$

as $N \to \infty$. So

$$A' \Delta \bigcup_n A'_n = A' \setminus \bigcup_n A'_n \in \boldsymbol{N}$$

i.e., $A' \equiv \bigcup_n A'_n$. $\dashv$

Finally, note that for $A \in \boldsymbol{B}_{\mathcal{U}}, \mu_{\mathcal{U}}(A) = 0 \Leftrightarrow A \in \boldsymbol{N}$.

**(B)** The following is straightforward.

**Proposition 2.5** *The measure $\mu_{\mathcal{U}}$ is non-atomic if and only if* $\forall \epsilon > 0$ $\forall (A_n) \in \prod_n \boldsymbol{B}(X_n)$ $\big((\mathcal{U}n(\mu_n(A_n) \geq \epsilon) \Rightarrow \exists \delta > 0\ \exists (B_n) \in \prod_n \boldsymbol{B}(X_n)$ $[\mathcal{U}n(B_n \subseteq A_n \;\&\; \delta \leq \mu_n(B_n), \mu_n(A_n \setminus B_n))]\big)$.

For example, this condition is satisfied if each $(X_n, \mu_n)$ is non-atomic or if each $X_n$ is finite, $\mu_n$ is normalized counting measure and $\lim_{n \to \mathcal{U}} \operatorname{card}(X_n) = \infty$.

Let $\mathrm{MALG}_{\mu_{\mathcal{U}}}$ be the measure algebra of $(X, \boldsymbol{B}_{\mathcal{U}}, \mu_{\mathcal{U}})$. If $\mu_{\mathcal{U}}$ is non-atomic, fix also a function $S_{\mathcal{U}} \colon \mathrm{MALG}_{\mu_{\mathcal{U}}} \to \mathrm{MALG}_{\mu_{\mathcal{U}}}$ such that $S_{\mathcal{U}}(A) \subseteq A$ and

$$\mu_{\mathcal{U}}(S_{\mathcal{U}}(A)) = \frac{1}{2}\mu_{\mathcal{U}}(A).$$

Let now $\boldsymbol{B}_0 \subseteq \mathrm{MALG}_{\mu_{\mathcal{U}}}$ be a countable subalgebra closed under $S_{\mathcal{U}}$. Let $\boldsymbol{B} = \sigma(\boldsymbol{B}_0) \subseteq \mathrm{MALG}_{\mu_{\mathcal{U}}}$ be the $\sigma$-subalgebra of $\mathrm{MALG}_{\mu_{\mathcal{U}}}$ generated by $\boldsymbol{B}_0$. Since every element of $\boldsymbol{B}$ can be approximated (in the sense of the metric $d(A,B) = \mu_{\mathcal{U}}(A \Delta B)$) by elements of $\boldsymbol{B}_0$, it follows that $\boldsymbol{B}$ is countably generated and non-atomic. It follows (see, e.g., Kechris [Ke1, 17.44]) that the measure algebra $(\boldsymbol{B}, \mu_{\mathcal{U}} | \boldsymbol{B})$ is isomorphic to the measure algebra of (any) non-atomic, standard measure space, in particular $\mathrm{MALG}_\rho$, where $\rho$ is the usual product measure on the Borel sets of $2^{\mathbb{N}}$. Then we can find a Cantor scheme $(B_s)_{s \in 2^{<\mathbb{N}}}$, with $B_s \in \boldsymbol{B}_{\mathcal{U}}$, $B_\emptyset = X$, $B_{s\hat{}0} \cap B_{s\hat{}1} = \emptyset$, $B_s = B_{s\hat{}0} \cup B_{s\hat{}1}$, $\mu_{\mathcal{U}}(B_s) = 2^{-n}$, and $(B_s)$ viewed now as members of $\mathrm{MALG}_{\mu_{\mathcal{U}}}$, belong to $\boldsymbol{B}$ and generate $\boldsymbol{B}$. Then define

$$\varphi\colon X_{\mathcal{U}} \to 2^{\mathbb{N}}$$

by

$$\varphi(x) = \alpha \Leftrightarrow x \in \bigcap_n B_{\alpha|n}.$$

Then $\varphi^{-1}(N_s) = B_s$, where $N_s = \{\alpha \in 2^{\mathbb{N}} \colon s \subseteq \alpha\}$ for $s \in 2^{<\mathbb{N}}$. Thus $\varphi$ is $\boldsymbol{B}_{\mathcal{U}}$-measurable (i.e., the inverse image of a Borel set in $2^{\mathbb{N}}$ is in $\boldsymbol{B}_{\mathcal{U}}$) and $\varphi_* \mu_{\mathcal{U}} = \rho$, so that $(2^{\mathbb{N}}, \rho)$ is a factor of $(X_{\mathcal{U}}, \mu_{\mathcal{U}})$ and $A \mapsto \varphi^{-1}(A)$ is an isomorphism of the measure algebra $\mathrm{MALG}_\rho$ with $(\boldsymbol{B}, \mu_{\mathcal{U}} | \boldsymbol{B})$.

# 3 Ultraproducts of measure preserving actions

**(A)** Let $(X_n, \mu_n), \mathcal{U}$ be as in §2. Let $\Gamma$ be a countable group and let $\{\alpha_n\}$ be a sequence of Borel actions $\alpha_n \colon \Gamma \times X_n \to X_n$, such that $\alpha_n$ preserves $\mu_n, \forall n \in \mathbb{N}$. We can define then the action $\alpha_{\mathcal{U}} \colon \Gamma \times X_{\mathcal{U}} \to X_{\mathcal{U}}$ by

$$\gamma^{\alpha_{\mathcal{U}}} \cdot [(x_n)]_{\mathcal{U}} = [(\gamma^{\alpha_n} \cdot x_n)]_{\mathcal{U}},$$

where we let $\gamma^{\alpha_{\mathcal{U}}} \cdot x = \alpha_{\mathcal{U}}(\gamma, x)$ and similarly for each $\alpha_n$.

**Proposition 3.1** *The action $\alpha_{\mathcal{U}}$ preserves $\boldsymbol{B}^0_{\mathcal{U}}$, $\boldsymbol{B}_{\mathcal{U}}$ and the measure $\mu_{\mathcal{U}}$.*

**Proof**. First let $A = [(A_n)]_{\mathcal{U}} \in \boldsymbol{B}^0_{\mathcal{U}}$. We verify that $\gamma^{\alpha_{\mathcal{U}}} \cdot A = [(\gamma^{\alpha_n} \cdot A_n)]_{\mathcal{U}}$, from which it follows that the action preserves $\boldsymbol{B}^0_{\mathcal{U}}$. Indeed

$$\begin{aligned}[(x_n)]_{\mathcal{U}} \in \gamma^{\alpha_{\mathcal{U}}} \cdot [(A_n)]_{\mathcal{U}} &\Leftrightarrow (\gamma^{-1})^{\alpha_{\mathcal{U}}} \cdot [(x_n)]_{\mathcal{U}} \in [(A_n)] \\ &\Leftrightarrow \mathcal{U}n((\gamma^{-1})^{\alpha_n} \cdot x_n \in A_n) \\ &\Leftrightarrow \mathcal{U}n(x_n \in \gamma^{\alpha_n} \cdot A_n) \\ &\Leftrightarrow [(x_n)]_{\mathcal{U}} \in [(\gamma^{\alpha_n} \cdot A_n)]_{\mathcal{U}}.\end{aligned}$$

Also

$$\begin{aligned}\mu_{\mathcal{U}}(\gamma^{\alpha_{\mathcal{U}}} \cdot A) &= \lim_{n \to \mathcal{U}} \mu_n(\gamma^{\alpha_n} \cdot A_n) \\ &= \lim_{n \to \mathcal{U}} \mu_n(A_n) = \mu_{\mathcal{U}}(A),\end{aligned}$$

so the action preserves $\mu_{\mathcal{U}} | \boldsymbol{B}^0_{\mathcal{U}}$.

Next let $A \in \boldsymbol{N}$ and for each $\epsilon > 0$ let $A \subseteq A_\epsilon \in \boldsymbol{B}^0_{\mathcal{U}}$ with $\mu_{\mathcal{U}}(A_\epsilon) < \epsilon$. Then $\gamma^{\alpha_{\mathcal{U}}} \cdot A \subseteq \gamma^{\alpha_{\mathcal{U}}} \cdot A_\epsilon$ and $\mu_{\mathcal{U}}(\gamma^{\alpha_{\mathcal{U}}} \cdot A_\epsilon) < \epsilon$, so $\gamma^{\alpha_{\mathcal{U}}} \cdot A \in \boldsymbol{N}$, i.e., $\boldsymbol{N}$ is invariant under the action.

Finally, let $A \in \boldsymbol{B}_{\mathcal{U}}$ and let $A' \in \boldsymbol{B}^0_{\mathcal{U}}$ be such that $A \Delta A' \in \boldsymbol{N}$, so that $\gamma^{\alpha_{\mathcal{U}}}(A) \Delta \gamma^{\alpha_{\mathcal{U}}}(A') \in \boldsymbol{N}$, thus $\gamma^{\alpha_{\mathcal{U}}}(A) \in \boldsymbol{B}_{\mathcal{U}}$ and $\mu_{\mathcal{U}}(\gamma^{\alpha_{\mathcal{U}}} \cdot A) = \mu_{\mathcal{U}}(\gamma^{\alpha_{\mathcal{U}}} \cdot A') = \mu_{\mathcal{U}}(A') = \mu_{\mathcal{U}}(A)$. $\dashv$

If $(X, \mu)$ is a probability space and $\alpha, \beta \colon \Gamma \times X \to X$ are measure preserving actions of $\Gamma$, we say the $\alpha, \beta$ are equivalent if $\forall \gamma \in \Gamma (\gamma^\alpha = \gamma^\beta, \mu\text{-a.e.})$. We let $A(\Gamma, X, \mu)$ be the space of equivalence classes and we call the elements of $A(\Gamma, X, \mu)$ also measure preserving actions. Note that if for each $n$, $\alpha_n, \alpha'_n$ as above are equivalent, then it is easy to check that $\alpha_{\mathcal{U}}, \alpha'_{\mathcal{U}}$ are also equivalent, thus if $a_n \in A(\Gamma, X_n, \mu_n), n \in \mathbb{N}$, is a sequence of measure preserving actions and we pick $\alpha_n$ a representative of $a_n$, then we can define unambiguously the ultraproduct action

$$\prod_n a_n / \mathcal{U}$$

with representative $\alpha_{\mathcal{U}}$. This is a measure preserving action of $\Gamma$ on $(X_{\mathcal{U}}, \mu_{\mathcal{U}})$, i.e., $\prod_n a_n / \mathcal{U} \in A(\Gamma, X_{\mathcal{U}}, \mu_{\mathcal{U}})$. When $a_n = a$ for all $n$, we put

$$a_{\mathcal{U}} = \prod_n a / \mathcal{U}.$$

**(B)** Recall that if $a \in A(\Gamma, X, \mu), b \in A(\Gamma, Y, \nu)$, we say that $b$ is a *factor* of $a$, in symbols

$$b \sqsubseteq a,$$

if there is a measurable map $\varphi\colon X \to Y$ such that $\varphi_*\mu = \nu$ and $\varphi(\gamma^a \cdot x) = \gamma^b \cdot \varphi(x), \mu$-a.e.$(x)$. We denote by $\mathrm{MALG}_\mu$ the measure algebra of $(X, \mu)$. Clearly $\Gamma$ acts on $\mathrm{MALG}_\mu$ by automorphisms of the measure algebra. If $(Y, \nu)$ is a non-atomic, standard measure space, the map $A \in \mathrm{MALG}_\nu \mapsto \varphi^{-1}(A) \in \mathrm{MALG}_\mu$ is an isomorphism of $\mathrm{MALG}_\nu$ with a countably generated, non-atomic, $\sigma$-subalgebra $\boldsymbol{B}$ of $\mathrm{MALG}_\mu$, which is $\Gamma$-invariant, and this isomorphism preserves the $\Gamma$-actions. Conversely, we can see as in §1,**(B)** that every countably generated, non-atomic, $\sigma$-subalgebra $\boldsymbol{B}$ of $\mathrm{MALG}_\mu$, which is $\Gamma$-invariant, gives rise to a factor of $a$ as follows: First fix an isomorphism $\pi$ between the measure algebra $(\boldsymbol{B}, \mu|\boldsymbol{B})$ and the measure algebra of $(Y, \nu)$, where $Y = 2^{\mathbb{N}}$ and $\nu = \rho$ is the usual product measure. Use this to define the Cantor scheme $(B_s)_{s \in 2^{<\mathbb{N}}}$ for $\boldsymbol{B}$ as in §2, **(B)** and define $\varphi\colon X \to Y$ as before. Now the isomorphism $\pi$ gives an action of $\Gamma$ on the measure algebra of $(Y, \nu)$, which by definition preserves the $\Gamma$-actions on $(\boldsymbol{B}, \mu|\boldsymbol{B})$ and $\mathrm{MALG}_\nu$. The $\Gamma$-action on $\mathrm{MALG}_\nu$ is induced by a (unique) action $b \in A(\Gamma, Y, \nu)$ (see, e.g., Kechris [Ke1, 17.46]) and then it is easy to check that $\varphi$ witnesses that $b \sqsubseteq a$ (notice that for each $s \in 2^{<\mathbb{N}}, \gamma \in \Gamma, \varphi(\gamma^a \cdot x) \in N_s \Leftrightarrow \gamma^b \cdot \varphi(x) \in N_s, \mu$-a.e.$(x)$).

In particular, the factors $b \in A(\Gamma, Y, \nu)$ of $a = \prod_n a_n/\mathcal{U}$ where $(Y, \nu)$ is a non-atomic, standard measure space, correspond exactly to the countably generated, non-atomic, $\Gamma$-invariant (for $a$) $\sigma$-subalgebras of $\mathrm{MALG}_{\mu_\mathcal{U}}$. For non-atomic $\mu_\mathcal{U}$, we can construct such subalgebras as follows: Start with a countable Boolean subalgebra $\boldsymbol{B}_0 \in \mathrm{MALG}_{\mu_\mathcal{U}}$, which is closed under the $\Gamma$-action and the function $S_\mathcal{U}$ of §2, **(B)**. Then let $\boldsymbol{B} = \sigma(\boldsymbol{B}_0)$ be the $\sigma$-subalgebra of $\mathrm{MALG}_{\mu_\mathcal{U}}$ generated by $\boldsymbol{B}_0$. This has all the required properties.

**(C)** We will next see how to ensure, in the notation of the preceding paragraph, that the factor corresponding to $\boldsymbol{B}$ is a free action. Recall that $a \in A(\Gamma, X, \mu)$ is *free* if $\forall \gamma \in \Gamma \setminus \{1\}(\gamma^a \cdot x \neq x$, $\mu$-a.e. $(x))$.

**Proposition 3.2** *The action $a = \prod_n a_n/\mathcal{U}$ is free iff for each $\gamma \in \Gamma \setminus \{1\}$,*

$$\lim_{n \to \mathcal{U}} \mu_n(\{x \colon \gamma^{a_n} \cdot x \neq x\}) = 1.$$

**Proof**. Note that, modulo null sets,

$$\{x \in X_\mathcal{U} \colon \gamma^a \cdot x \neq x\} = [(A_n)]_\mathcal{U},$$

where $A_n = \{x \in X_n \colon \gamma^{a_n} \cdot x \neq x\}$. ⊣

In particular, if all $a_n$ are free, so is $\prod_n a_n/\mathcal{U}$.

**Proposition 3.3** *Suppose the action $a = \prod_n a_n/\mathcal{U}$ is free. Then for each $A \in \mathrm{MALG}_{\mu_\mathcal{U}}, A \neq \emptyset$ and $\gamma \in \Gamma \setminus \{1\}$, there is $B \in \mathrm{MALG}_{\mu_\mathcal{U}}$ with $B \subseteq A$, $\mu_\mathcal{U}(B) \geq \frac{1}{16}\mu_\mathcal{U}(A)$ and $\gamma^a \cdot B \cap B = \emptyset$.*

**Proof**. It is clearly enough to show that if $\gamma \neq 1$, $A \in \boldsymbol{B}^0_\mathcal{U}, \mu_\mathcal{U}(A) > 0$, then there is $B \in \boldsymbol{B}^0_\mathcal{U}, B \subseteq A$, with $\mu_\mathcal{U}(B) \geq \frac{1}{16}\mu_\mathcal{U}(A)$ and $\gamma^a \cdot B \cap B = \emptyset$.

Let $A = [(A_n)]_\mathcal{U}$ and $\mu_\mathcal{U}(A) = \epsilon > 0$. Then there is $U \subseteq \mathbb{N}, U \in \mathcal{U}$ with $n \in U \Rightarrow (\mu_n(A_n) > \frac{\epsilon}{2}$ and $\mu(\{x \in X_n \colon \gamma^{a_n} \cdot x \neq x\}) > 1 - \frac{\epsilon}{4})$. We can assume that each $X_n$ is Polish and $\gamma^{a_n}$ is represented (a.e.) by a homeomorphism $\gamma^{\alpha_n}$ of $X_n$. Let

$$C_n = \{x \in A_n \colon \gamma^{\alpha_n} \cdot x \neq x\},$$

so that $\mu_n(C_n) > \frac{\epsilon}{4}$. Fix also a countable basis $(V^n_i)_{i \in \mathbb{N}}$ for $X_n$.

If $x \in C_n$, let $V^x_n$ be a basic open set such that $\gamma^{\alpha_n} \cdot V^x_n \cap V^x_n = \emptyset$ (this exists by the continuity of $\gamma^{\alpha_n}$ and the fact that $\gamma^{\alpha_n} \cdot x \neq x$). It follows that there is $x_0 \in C_n$ with $\mu_n(C_n \cap V^{x_0}_n) > 0$ and $\gamma^{\alpha_n} \cdot (C_n \cap V^{x_0}_n) \cap (C_n \cap V^{x_0}_n) = \emptyset$. Thus there is $C \subseteq C_n$ with $\mu_n(C) > 0$ and $\gamma^{a_n} \cdot C \cap C = \emptyset$. By Zorn's Lemma or transfinite induction there is an element $B_n$ of $\mathrm{MALG}_{\mu_n}$ which is maximal, under inclusion, among all $D \in \mathrm{MALG}_{\mu_\mathcal{U}}$ satisfying: $D \subseteq C_n$ (viewing $C_n$ as an element of the measure algebra), $\mu_n(D) > 0$, $\gamma^{a_n} \cdot D \cap D = \emptyset$. We claim that $\mu_n(B_n) \geq \frac{\epsilon}{16}$. Indeed let

$$E_n = C_n \setminus (B_n \cup \gamma^{a_n} \cdot B_n \cup (\gamma^{-1})^{a_n} \cdot B_n).$$

If $\mu_n(B_n) < \frac{\epsilon}{16}$, then $E_n \neq \emptyset$, so as before we can find $F_n \subseteq E_n$ with $\mu_n(F_n) > 0$ and $\gamma^{a_n} \cdot F_n \cap F_n = \emptyset$. Then $\gamma^{a_n} \cdot (B_n \cup F_n) \cap (B_n \cup F_n) = \emptyset$, contradicting to maximality of $B_n$. So $\mu_n(B_n) \geq \frac{\epsilon}{16}$. Let now $B = [(B_n)]_\mathcal{U}$.$\dashv$

**Remark 3.4** The above argument can be simplified by using [KST, 4.6]. Consider the graph $G_n$ on $X_n$ whose edges consist of all distinct $x, y$ such that $y = (\gamma^{\pm 1})^{\alpha_n} \cdot x$. It has maximum degree 2, so admits a Borel 3-coloring. Thus there is an independent (for $G_n$) set $B_n \subseteq C_n$ with $\mu_n(B_n) \geq \mu_n(C_n)/3$. Then $\gamma^{a_n} \cdot B_n \cap B_n = \emptyset$ and actually $\mu_n(B_n) \geq \frac{\epsilon}{12}$.

So if the action $a = \prod_n a_n/\mathcal{U}$ is free, let

$$T_\mathcal{U} \colon \Gamma \times \mathrm{MALG}_{\mu_\mathcal{U}} \to \mathrm{MALG}_{\mu_\mathcal{U}}$$

be a function such that for each $\gamma \neq 1, A \in \mathrm{MALG}_{\mu_\mathcal{U}} \setminus \{\emptyset\}, T_\mathcal{U}(\gamma, A) \subseteq A, \mu(T_\mathcal{U}(\gamma, A)) \geq \frac{1}{16}\mu(A)$ and $\gamma^a \cdot T_\mathcal{U}(\gamma, A) \cap T_\mathcal{U}(\gamma, A) = \emptyset$. Now, if in the earlier construction of countably generated, non-atomic, $\Gamma$-invariant $\sigma$-subalgebras of $\mathrm{MALG}_{\mu_{\mathcal{U}}}$, we start with a countable Boolean subalgebra $\boldsymbol{B}_0$ closed under the $\Gamma$-action, the function $S_{\mathcal{U}}$ of §2, **(B)** and $T_{\mathcal{U}}$ (i.e., $\forall\gamma(A \in \boldsymbol{B}_0 \Rightarrow T_{\mathcal{U}}(\gamma, A) \in B_0))$, then the factor $b$ corresponding to $\boldsymbol{B} = \sigma(\boldsymbol{B}_0)$ is a free action.

# 4 Characterizing factors of ultraproducts

*In sections §4–8 all measure spaces will be non-atomic and standard. Also* $\Gamma$ *is an arbitrary countable infinite group.*

**(A)** For such a measure space $(X, \mu)$, $\mathrm{Aut}(X, \mu)$ is the Polish group of measure preserving automorphisms of $(X, \mu)$ equipped with the *weak topology* generated by the maps $T \mapsto T(A), A \in \mathrm{MALG}_\mu$, from $\mathrm{Aut}(X, \mu)$ into $\mathrm{MALG}_\mu$ (equipped with the usual metric $d_\mu(A, B) = \mu(A\Delta B)$). We can identify $A(\Gamma, X, \mu)$ with the space of homomorphisms of $\Gamma$ into $\mathrm{Aut}(X, \mu)$, so that it becomes a closed subspace of $\mathrm{Aut}(X, \mu)^\Gamma$ with the product topology, thus also a Polish space.

**Definition 4.1** *Let* $a \in A(\Gamma, X, \mu), a_n \in A(\Gamma, X_n, \mu_n), n \in \mathbb{N}$. *Let* $\mathcal{U}$ *be a non-principal ultrafilter on* $\mathbb{N}$. *We say that* $a$ *is weakly* $\mathcal{U}$*-contained in* $(a_n)$*, in symbols*

$$a \prec_{\mathcal{U}} (a_n),$$

*if for every finite* $F \subseteq \Gamma, A_1, \dots, A_N \in \mathrm{MALG}_\mu, \epsilon > 0$*, for* $\mathcal{U}$*-almost all* $n$*:*

$$\exists B_{1,n} \dots B_{N,n} \in \mathrm{MALG}_{\mu_n} \forall\gamma \in F \forall i, j \leq N$$

$$|\mu(\gamma^a \cdot A_i \cap A_j) - \mu_n(\gamma^{a_n} \cdot B_{i,n} \cap B_{j,n})| < \epsilon.$$

Note that if $a_n = b$ for all $n$, then $a \prec_{\mathcal{U}} (a_n) \Leftrightarrow a \prec b$ in the sense of weak containment of actions, see Kechris [Ke2].

One can also trivially see that $a \prec_{\mathcal{U}} (a_n)$ is equivalent to the statement:

For every finite $F \subseteq \Gamma, A_1, \dots A_N \in \mathrm{MALG}_\mu, \epsilon > 0$, there are $[(B_{1,n})]_{\mathcal{U}}, \dots, [(B_{N,n})]_{\mathcal{U}} \in \boldsymbol{B}^0_{\mathcal{U}}(X_{\mathcal{U}})$ such that for $\mathcal{U}$-almost all $n$:

$$\forall\gamma \in F \forall i, j \leq N |\mu(\gamma^a \cdot A_i \cap A_j) - \mu_n(\gamma^{a_n} \cdot B_{i,n} \cap B_{j,n})| < \epsilon).$$

**Definition 4.2** *For $a, b_n \in A(\Gamma, X, \mu)$, we write*

$$\lim_{n \to \mathcal{U}} b_n = a$$

*if for each open nbhd $V$ of $a$ in $A(\Gamma, X, \mu)$, $\mathcal{U}n(b_n \in V)$.*

Since the sets of the form

$$V = \{b \colon \forall \gamma \in F \forall i, j \leq N |\mu(\gamma^a \cdot A_i \cap A_j) - \mu(\gamma^b \cdot A_i \cap A_j)| < \epsilon\},$$

for $A_1, \dots, A_N$ a Borel partition of $X, \epsilon > 0, F \subseteq \Gamma$ finite containing 1, form a nbhd basis of $a, \lim_{n \to \mathcal{U}} b_n = a$ iff $\mathcal{U}n(b_n \in V)$, for any $V$ of the above form.

Below $\cong$ denotes isomorphism of actions.

**Theorem 4.3** *Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. Let $a \in A(\Gamma, X, \mu)$, and let $a_n \in A(\Gamma, X_n, \mu_n), n \in \mathbb{N}$. Then the following are equivalent*

*(1) $a \prec_{\mathcal{U}} (a_n)$,*

*(2) $a \sqsubseteq \prod_n a_n / \mathcal{U}$,*

*(3) $a = \lim_{n \to \mathcal{U}} b_n$, for some sequence $(b_n)$, $b_n \in A(\Gamma, X, \mu)$ with $b_n \cong a_n, n \in \mathbb{N}$.*

**Proof**. Below put $b = \prod_n a_n / \mathcal{U}$.

(1) $\Rightarrow$ (2): Let $1 \in F_0 \subseteq F_1 \subseteq \dots$ be a sequence of finite subsets of $\Gamma$ with $\Gamma = \bigcup_m F_m$. We can assume that $X = 2^{\mathbb{N}}, \mu = \rho$ (the usual product measure on $2^{\mathbb{N}}$). Let $N_s = \{\alpha \in 2^{\mathbb{N}} \colon s \subseteq \alpha\}$, for $s \in 2^{<\mathbb{N}}$.

By (1), we can find for each $m \in \mathbb{N}$ and for each $s \in 2^{\leq m}$, $[(B_n^{s,m})]_{\mathcal{U}} \in \boldsymbol{B}_{\mathcal{U}}^0$ such that $U_m \in \mathcal{U}$, where

$$U_m = \{n \geq m \colon \forall \gamma \in F_m \forall s, t \in 2^{\leq m}$$

$$|\mu(\gamma^a \cdot N_s \cap N_t) - \mu_n(\gamma^{a_n} \cdot B_n^{s,m} \cap B_n^{t,m})| < \epsilon_m\},$$

where $\epsilon_m \to 0$. Since $\bigcap_m U_m = \emptyset$, let $m(n) =$ largest $m$ such that $n \in \bigcap_{i \leq m} U_i$. Then $n \in U_{n(m)}$ and $\lim_{n \to \mathcal{U}} m(n) = \infty$. Put

$$B_s = [(B_n^{s,m(n)})]_{\mathcal{U}} \in \boldsymbol{B}_{\mathcal{U}}^0.$$

Since $n \in U_{m(n)}$ for all $n$, it follows (taking $\gamma = 1, s = t$ in the definition of $U_m$) that for all $n$ with $m(n) > \text{length}(s)$,

$$|\mu(N_s) - \mu_n(B_n^{s,m(n)})| < \epsilon_{m(n)}. \tag{$*$}$$

So for any $\epsilon > 0$, if $M > \text{length}(s)$ and $\epsilon_M < \epsilon$, then $\mathcal{U}n(m(n) > M)$, so $(*)$ holds with $\epsilon$ replacing $\epsilon_{m(n)}$ for $\mathcal{U}$-almost all $n$, thus

$$\mu_{\mathcal{U}}(B_s) = \lim_{n \to \mathcal{U}} \mu_n(B_n^{s,m(n)}) = \mu(N_s).$$

In general, we have that for all $\gamma \in F_{m(n)}$ and all $s, t \in 2^{\leq m(n)}$,

$$|\mu(\gamma^a \cdot N_s \cap N_t) - \mu_n(\gamma^{a_n} \cdot B_n^{s,m(n)} \cap B_n^{t,m(n)})| < \epsilon_{m(n)}.$$

So if $\gamma \in F, s, t \in 2^{<\mathbb{N}}, \epsilon > 0$, and if $M$ is large enough so that $M > \max\{\text{length}(s), \text{length}(t)\}, \gamma \in F_M, \epsilon_M < \epsilon$, then on $\{n \colon m(n) \geq M\} \in \mathcal{U}$ we have

$$|\mu(\gamma^a \cdot N_s \cap N_t) - \mu_n(\gamma^{a_n} \cdot B_n^{s,m(n)} \cap B_n^{t,m(n)})| < \epsilon,$$

so

$$\mu_{\mathcal{U}}(\gamma^{\prod_n a_n/\mathcal{U}} \cdot B_s \cap B_t) = \mu(\gamma^a \cdot N_s \cap N_t). \tag{$**$}$$

Viewing each $B_s$ as an element of $\text{MALG}_{\mu_{\mathcal{U}}}$, we have $B_\emptyset = X_{\mathcal{U}}$, $B_{s\hat{}0} \cap B_{s\hat{}1} = \emptyset$, $B_s = B_{s\hat{}0} \cup B_{s\hat{}1}$ (for the last take $\gamma = 1, t = s\hat{}i$ in $(**)$) and $\mu_{\mathcal{U}}(B_s) = 2^{-n}$, if $s \in 2^n$. Then the map $\pi(N_s) = B_s$ gives a measure preserving isomorphism of the Boolean subalgebra $\boldsymbol{A}_0$ of $\text{MALG}_\mu$ generated by $(N_s)$ and the Boolean algebra $\boldsymbol{B}_0$ in $\text{MALG}_{\mu_{\mathcal{U}}}$ generated by $(B_s)$. Let $\boldsymbol{B}$ be the $\sigma$-subalgebra of $\text{MALG}_{\mu_{\mathcal{U}}}$ generated by $(B_s)$. Since $\pi$ is an isometry of $\boldsymbol{A}_0$ with $\boldsymbol{B}_0$ (with the metrics they inherit from the measure algebra), and $\boldsymbol{A}_0$ is dense in $\text{MALG}_\mu$, $\boldsymbol{B}_0$ is dense in $\boldsymbol{B}$, it follows that $\pi$ extends uniquely to an isometry, also denoted by $\pi$, from $\text{MALG}_\mu$ onto $\boldsymbol{B}$. Since $\pi(\emptyset) = \emptyset$, $\pi$ is actually an isomorphism of the measure algebra $\text{MALG}_\mu$ with the measure algebra $\boldsymbol{B}$ (see Kechris [Ke2, pp. 1-2]). It is thus enough to show that $\boldsymbol{B}$ is $\Gamma$-invariant (for $b$) and that $\pi$ preserves the $\Gamma$-action (i.e., it is $\Gamma$-*equivariant*).

It is enough to show that $\pi(\gamma^a \cdot N_s) = \gamma^b \cdot B_s$ (since $(B_s)$ generates $\boldsymbol{B}$).

Fix $\gamma \in \Gamma, \epsilon > 0, s \in 2^{<\mathbb{N}}$. There is $A \in \boldsymbol{A}_0$ with $\mu(\gamma^a \cdot N_s \Delta A) < \epsilon/2$. Now $A = \bigsqcup_{i=1}^{m_1} N_{t_i}, \sim A = \bigsqcup_{j=1}^{m_2} N_{t'_j}$ and $\sim N_s = \bigsqcup_{k=1}^{m_3} N_{s_k}$ (disjoint unions), so

$$\begin{aligned}\gamma^a \cdot N_s \Delta A &= (\gamma^a \cdot N_s \cap (\sim A)) \sqcup (\gamma^a \cdot (\sim N_s) \cap A) \\ &= (\bigsqcup_{j=1}^{m_2} \gamma^a \cdot N_s \cap N_{t'_j}) \sqcup (\bigsqcup_{k=1}^{m_3} \bigsqcup_{i=1}^{m_1} (\gamma^a \cdot N_{s_k} \cap N_{t_i})).\end{aligned}$$

If $B = \pi(A) \in \boldsymbol{B}_0$, then we also have

$$\begin{aligned}\gamma^b \cdot B_s \Delta B =& (\bigsqcup_{j=1}^{m_2} \gamma^b \cdot B_s \cap B_{t'_j}) \sqcup \\ & (\bigsqcup_{k=1}^{m_3} \bigsqcup_{i=1}^{m_1} (\gamma^b \cdot B_{s_k} \cap B_{t_i})),\end{aligned}$$

so by $(**)$

$$\mu_{\mathcal{U}}(\gamma^b \cdot B_s \Delta B) = \mu(\gamma^a \cdot N_s \Delta A) < \epsilon/2.$$

Since $\pi$ preserves measure, we also have $\mu_{\mathcal{U}}(\pi(\gamma^a \cdot N_s)\Delta B) < \epsilon/2$, thus

$$\mu_{\mathcal{U}}(\gamma^b \cdot B_s \Delta \pi(\gamma^a \cdot N_s)) < \epsilon.$$

Therefore $\gamma^b \cdot B_s = \pi(\gamma^a \cdot N_s)$.

(2) $\Rightarrow$ (1): Suppose that $a \sqsubseteq b$. Let $\pi\colon \mathrm{MALG}_\mu \to \mathrm{MALG}_{\mu_{\mathcal{U}}}$ be a measure preserving embedding preserving the $\Gamma$-actions (so that the image $\pi(\mathrm{MALG}_\mu)$ is a $\Gamma$-invariant $\sigma$-subalgebra of $\mathrm{MALG}_{\mu_{\mathcal{U}}}$). Fix $F \subseteq \Gamma$ finite, $A_1, \dots, A_n \in \mathrm{MALG}_\mu$ and $\epsilon > 0$. Let $B^1, \dots, B^N \in \boldsymbol{B}^0_{\mathcal{U}}$ represent $\pi(A_1), \dots, \pi(A_N)$. Let $B^i = [(B^i_n)]_{\mathcal{U}}$. Then for $\gamma \in F, j, k \leq N$,

$$\begin{aligned}\mu(\gamma^a \cdot A_j \cap A_k) &= \mu_{\mathcal{U}}(\gamma^b \cdot B^j \cap B^k) \\ &= \lim_{n \to \mathcal{U}} \mu_n(\gamma^{a_n} \cdot B^j_n \cap B^k_n),\end{aligned}$$

so for $\mathcal{U}$-almost all $n$,

$$|\mu(\gamma^a \cdot A_j \cap A_k) - \mu_n(\gamma^{a_n} \cdot B^j_n \cap B^k_n)| < \epsilon,$$

and thus for $\mathcal{U}$-almost all $n$, this holds for all $\gamma \in F, j, k \leq N$. Thus $a \prec_{\mathcal{U}} (a_n)$.

(3) $\Rightarrow$ (1): Fix such $b_n$, and let $A_1, \dots, A_N \in \mathrm{MALG}_\mu, F \subseteq \Gamma$ finite, $\epsilon > 0$. Then there is $U \in \mathcal{U}$ such that for $n \in U$ we have

$$\forall \gamma \in F \forall i, j \leq N(|\mu(\gamma^a \cdot A_i \cap A_j) - \mu(\gamma^{b_n} \cdot A_i \cap A_j)| < \epsilon).$$

Let $\varphi_n\colon (X, \mu) \to (X_n, \mu_n)$ be an isomorphism that sends $b_n$ to $a_n$ and put $\varphi_n(A_i) = B^i_n$. Then $\varphi_n(\gamma^{b_n} \cdot A_i \cap A_j) = \gamma^{a_n} \cdot B^i_n \cap B^j_n$, so for $n \in U$:

$$\forall \gamma \in F \forall i, j \leq N(|\mu(\gamma^a \cdot A_i \cap A_j) - \mu_n(\gamma^{a_n} \cdot B^i_n \cap B^j_n)| < \epsilon),$$

thus $a \prec_{\mathcal{U}} (a_n)$.

(1) $\Rightarrow$ (3): Suppose $a \prec_{\mathcal{U}} (a_n)$. Let

$$V = \{c \in A(\Gamma, X, \mu) \colon \forall \gamma \in F \forall i, j \leq N(|\mu(\gamma^a \cdot A_i \cap A_j) - \mu(\gamma^c \cdot A_i \cap A_j)| < \epsilon)\},$$

where $A_1, \dots, A_n \in \mathrm{MALG}_\mu$ is a Borel partition of $X$, $\epsilon > 0$ and $F \subseteq \Gamma$ is finite with $1 \in F$, be a basic nbhd of $a$.

**Claim**. For any such $V$, we can find $U \in \mathcal{U}$ such that for $n \in U$ there is $b_n \in V$ with $b_n \cong a_n$.

Assume this for the moment and complete the proof of (1) $\Rightarrow$ (3).

Let $V_0 \supseteq V_1 \supseteq V_2 \supseteq \dots$ be a nbhd basis for $a$ consisting of sets of the above form, and for each $m$ let $U_m \in \mathcal{U}$ be such that for $n \in U_m$, there is $b_{n,m} \in V_m$ with $b_{n,m} \cong a_n$. We can also assume that $\bigcap_m U_m = \emptyset$. Let $m(n) =$ largest $m$ such that $n \in \bigcap_{i \leq m} U_i$. We have $a_n \cong b_{n,m(n)} \in V_{m(n)}$, and for any nbhd $V$ of $a$ as above, if $M$ is so large that $V_M \subseteq V$, then $b_{n,m(n)} \in V_{m(n)} \subseteq V_M \subseteq V$, for $n \in \{n \colon m(n) \geq M\} \in \mathcal{U}$. So $a = \lim_{n \to \mathcal{U}} b_{n,m(n)}$.

**Proof of the claim.** Since $a \prec_{\mathcal{U}} (a_n)$, for any $\delta > 0$, we can find $[(B_{1,n})]_{\mathcal{U}}, \dots, [(B_{N,n})]_{\mathcal{U}} \in \boldsymbol{B}^0_{\mathcal{U}}$ and $U_\delta \in \mathcal{U}$ such that for $n \in U_\delta$ we have

$$\forall \gamma \in F \forall i, j \leq N(|\mu(\gamma^a \cdot A_i \cap A_j) - \mu_n(\gamma^{a_n} \cdot B_{i,n} \cap B_{j,n})| < \delta).$$

Taking $\delta < \epsilon/20N^3$ and $U = U_\delta$, the proof of Proposition 10.1 in Kechris [Ke2] shows that for $n \in U$ there is $b_n \cong a_n$ with $b_n \in V$. $\dashv$

**Corollary 4.4** *Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$ and consider the actions $a \in A(\Gamma, X, \mu), b \in A(\Gamma, Y, \nu)$. Then the following are equivalent:*

*(1) $a \prec b$,*

*(2) $a \sqsubseteq b_{\mathcal{U}}$.*

Theorem 4.3 also has the following curious consequence, a compactness property of the space $A(\Gamma, X, \mu)$.

**Corollary 4.5** *Let $a_n \in A(\Gamma, X, \mu), n \in \mathbb{N}$, be a sequence of actions. Then there is a subsequence $n_0 < n_1 < n_2 < \dots$ and $b_{n_i} \in A(\Gamma, X, \mu), b_{n_i} \cong a_{n_i}$, such that $(b_{n_i})$ converges in $A(\Gamma, X, \mu)$.*

**Proof**. Let $a \in A(\Gamma, X, \mu)$ be such that $a \sqsubseteq \prod_n a_n/\mathcal{U}$ (such exists by §3, **(B)**). Then by 4.3, we can find $b_n \cong a_n$, with $\lim_{n \to \mathcal{U}} b_n = a$. This of course implies that there is $n_0 < n_1 < \dots$ with $\lim_{i \to \infty} b_{n_i} = a$. $\dashv$

Benjy Weiss pointed out that for *free* actions a stronger version of 4.5 follows from his work with Abért, see Abért-Weiss [AW]. In this paper it is shown that if $s_\Gamma$ is the shift action of an infinite group $\Gamma$ on $[0,1]^\Gamma$, then $s_\Gamma \prec a$ for *any* free action $a$ of $\Gamma$. From this it follows that given free $a_n \in A(\Gamma, X, \mu), n \in \mathbb{N}$, there is $b_n \cong a_n$ with $\lim_{n \to \infty} b_n = s_\Gamma$.

Another form of compactness for $A(\Gamma, X, \mu)$ that is an immediate consequence of 4.5 is the following:

Any cover of $A(\Gamma, X, \mu)$ by open, *invariant under* $\cong$ sets, has a finite subcover. Equivalently, the quotient space $A(\Gamma, X, \mu)/\cong$ is compact.

**(B)** Consider now $a \in A(\Gamma, X, \mu)$ and the action $a_\mathcal{U}$ on $(X_\mathcal{U}, \mu_\mathcal{U})$. Clearly $\mu_\mathcal{U}$ is non-atomic as $\mu$ is non-atomic. Fix also a countable Boolean subalgebra $\boldsymbol{A}_0$ of $\mathrm{MALG}_\mu$ which generates $\mathrm{MALG}_\mu$ and is closed under the action $a$. The map

$$\pi(A) = [(A)]_\mathcal{U}$$

(where $(A)$ is the constant sequence $(A_n), A_n = A, \forall n \in \mathbb{N}$) embeds $\boldsymbol{A}_0$ into a Boolean subalgebra $\boldsymbol{C}_0$ of $\mathrm{MALG}_{\mu_\mathcal{U}}$, invariant under $a_\mathcal{U}$, preserving the measure and the $\Gamma$-actions ($a$ on $\boldsymbol{A}_0$ and $a_\mathcal{U}$ on $\boldsymbol{C}_0$).

Let $\boldsymbol{B}_0 \supseteq \boldsymbol{C}_0$ be any countable Boolean subalgebra of $\mathrm{MALG}_{\mu_\mathcal{U}}$ closed under the action $a_\mathcal{U}$ and the function $S_\mathcal{U}$ of §2, **(B)** and let $\boldsymbol{B} = \sigma(\boldsymbol{B}_0)$ be the $\sigma$-algebra generated by $\boldsymbol{B}_0$. Let $b$ be the factor of $a_\mathcal{U}$ corresponding to $\boldsymbol{B}$, so that $b \sqsubseteq a_\mathcal{U}$ and thus $b \prec a$ by 4.4. We also claim that $a \sqsubseteq b$ and thus $a \sim_w b$, where

$$a \sim_w b \Leftrightarrow a \prec b \ \&\ b \prec a.$$

Indeed, let $\boldsymbol{D}_0 = \sigma(\boldsymbol{C}_0)$ be the $\sigma$-subalgebra of $\boldsymbol{B}$ generated by $\boldsymbol{C}_0$. Then $\boldsymbol{D}_0$ is also closed under the action $a_\mathcal{U}$. The map $\pi$ is an isometry of $\boldsymbol{A}_0$ with $\boldsymbol{C}_0$, which are dense in $\mathrm{MALG}_\mu$, $\boldsymbol{D}_0$, resp., so extends uniquely to an isometry, also denoted by $\pi$, of $\mathrm{MALG}_\mu$ with $\boldsymbol{D}_0$. Since $\pi(\emptyset) = \emptyset$, it follows that $\pi$ is an isomorphism of the measure algebra $\mathrm{MALG}_\mu$ with the measure algebra $\boldsymbol{D}_0$ (see Kechris [Ke2, pp. 1-2]). Fix now $\gamma \in \Gamma$. Then $\gamma^a$ on $\mathrm{MALG}_\mu$ is mapped by $\pi$ to an automorphism $\pi(\gamma^a)$ of the measure algebra $\boldsymbol{D}_0$. Since $\pi(\gamma^a \cdot A) = \gamma^{a_\mathcal{U}} \cdot \pi(A)$, for $A \in \boldsymbol{A}_0$, it follows that $\pi(\gamma^a)|\boldsymbol{C}_0 = \gamma^{a_\mathcal{U}}|\boldsymbol{C}_0$, so since $\boldsymbol{C}_0$ generates $\boldsymbol{D}_0$, we have $\pi(\gamma^a) = \gamma^{a_\mathcal{U}}|\boldsymbol{D}_0$, i.e., $\pi$ preserves the $\Gamma$-actions ($a$ on $\mathrm{MALG}_\mu$ and $a_\mathcal{U}$ on $\boldsymbol{D}_0$), thus $a \sqsubseteq b$.

Recall now that $a \in A(\Gamma, X, \mu)$ admits *non-trivial almost invariant sets* if there is a sequence $(A_n)$ of Borel sets such that $\mu(A_n)(1 - \mu(A_n)) \not\to 0$ but $\forall \gamma (\lim_{n\to\infty} \mu(\gamma^a \cdot A_n \Delta A_n) = 0)$. We call an action *a strongly ergodic* (or *$E_0$-ergodic*) if it does not admit non-trivial almost invariant sets. We now have:

**Proposition 4.6** *Let $a \in A(\Gamma, X, \mu)$. Then $a$ is strongly ergodic iff $\forall b \sim_w a$ ($b$ is ergodic) iff $\forall b \prec a$ ($b$ is ergodic).*

**Proof**. Assume first that $a$ is not strongly ergodic and let $(A_n)$ be a sequence of Borel sets such that for some $\delta > 0$, $\delta \leq \mu(A_n) \leq 1 - \delta$ and $\forall \gamma (\lim_{n\to\infty} \mu(\gamma^a \cdot A_n \Delta A_n) = 0)$. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$ and let $A = [(A_n)]_{\mathcal{U}} \in \boldsymbol{B}^0_{\mathcal{U}}$. Then viewing $A$ as an element of $\mathrm{MALG}_{\mu_{\mathcal{U}}}$ we have $\gamma^{a_{\mathcal{U}}} \cdot A = A, \forall \gamma \in \Gamma$, and $0 < \mu_{\mathcal{U}}(A) < 1$. Let $\boldsymbol{B}_0$ be a countable Boolean subalgebra of $\mathrm{MALG}_{\mu_{\mathcal{U}}}$ containing $A$ and closed under $a_{\mathcal{U}}$, the function $S_{\mathcal{U}}$ and containing $\boldsymbol{C}_0$ as before. Let $b$ be the factor of $a_{\mathcal{U}}$ associated with $\boldsymbol{B} = \sigma(\boldsymbol{B}_0)$, so that $a \sim_w b$. Since $A \in \boldsymbol{B}$, clearly $b$ is not ergodic.

Conversely assume $b \prec a$ and $b$ is not ergodic. It follows easily then from the definition of weak containment that $a$ is not strongly ergodic. $\dashv$

Finally we note the following fact that connects weak containment to factors.

**Proposition 4.7** *Let $a, b \in A(\Gamma, X, \mu)$. Then the following are equivalent:*

*(i)* $a \prec b$,

*(ii)* $\exists c \in A(\Gamma, X, \mu)(c \sim_w b \;\&\; a \sqsubseteq c)$.

**Proof**. (ii) clearly implies (i), since $a \sqsubseteq c \Rightarrow a \prec c$ and $\prec$ is transitive.

(i) $\Rightarrow$ (ii) Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. By 4.4, if $a \prec b$ then $a \sqsubseteq b_{\mathcal{U}}$. Then as in the first two paragraphs of §4, **(B)**, we can find an appropriate $\sigma$-subalgebra of $\mathrm{MALG}_{\mu_{\mathcal{U}}}$ invariant under $b_{\mathcal{U}}$, so that if $c$ is the corresponding factor, then $c \sim_w b$ (and in fact moreover $b \sqsubseteq c$) and $a \sqsubseteq c$. $\dashv$

# 5 Graph combinatorics of group actions

Let $\Gamma$ be an infinite group with a finite set of generators $S \subseteq \Gamma$ for which *we assume throughout that* $1 \notin S$. We denote by $\mathrm{FR}(\Gamma, X, \mu)$ the set of free actions in $A(\Gamma, X, \mu)$. If $a \in \mathrm{FR}(\Gamma, X, \mu)$ we associate with $a$ the (simple, undirected) graph $G(S,a) = (X, E(S,a))$, where $X$ is the set of vertices and $E(S,a)$, the set of edges, is given by

$$(x,y) \in E(S,a) \Leftrightarrow x \neq y \ \& \ \exists s \in S^{\pm 1}(s^a \cdot x = y),$$

where $S^{\pm 1} = \{s, s^{-1} \colon s \in S\}$. We also write $xE(S,a)y$ if $(x,y) \in E(S,a)$. As in Conley-Kechris [CK], we associate with this graph the following parameters:

$$\begin{aligned} \chi_\mu(S,a) &= \text{ the } \textit{measurable chromatic number,} \\ \chi_\mu^{ap}(S,a) &= \text{ the } \textit{approximate chromatic number,} \\ i_\mu(S,a) &= \text{ the } \textit{independence number,} \end{aligned}$$

defined as follows:

• $\chi_\mu(S,a)$ is the smallest cardinality of a standard Borel space $Y$ for which there is a ($\mu-$)measurable coloring $c \colon X \to Y$ of $G(S,a)$ (i.e., $xE(S,a)y \Rightarrow c(x) \neq c(y)$).

• $\chi_\mu^{ap}(S,a)$ is the smallest cardinality of a standard Borel space $Y$ such that for each $\epsilon > 0$, there is a Borel set $A \subseteq X$ with $\mu(X \setminus A) < \epsilon$ and a measurable coloring $c \colon A \to Y$ of the induced subgraph $G(S,a)|A = (A, E(S,A) \cap A^2)$.

• $i_\mu(S,a)$ is the supremum of the measures of Borel independent sets, where $A \subseteq X$ is *independent* if no two elements of $A$ are adjacent.

Given a (simple, undirected) graph $G = (X, E)$, where $X$ is the set of vertices and $E$ the set of edges, a *matching* in $G$ is a subset $M \subseteq E$ such that no two edges in $M$ have a common point. We denote by $X_M$ the set of matched vertices, i.e., the set of points belonging to an edge in $M$. If $X_M = X$ we say that $M$ is a *perfect matching*.

For $a \in \mathrm{FR}(\Gamma, X, \mu)$ as before, we also define the parameter

$$m(S,a) = \text{ the } \textit{matching number,}$$

where $m(S,a)$ is $1/2$ of the supremum of $\mu(X_M)$, with $M$ a Borel (as a subset of $X^2$) matching in $G(S,a)$. If $m(S,a) = 1/2$ and the supremum is attained, we say that $G(S,a)$ admits an *a.e. perfect matching*.

Note that we can view a matching $M$ in $G(S,a)$ as a Borel bijection $\varphi \colon A \to B$, with $A, B \subseteq X$ disjoint Borel sets and $xE(S,a)\varphi(x), \forall x \in A$. Then $X_M = A \cup B$ and so $\mu(A)$ is $1/2\mu(X_M)$. Thus $m(S,a)$ is equal to the supremum of $\mu(A)$ over all such $\varphi$.

It was shown in Conley-Kechris [CK, 4.2,4.3] that

$$a \prec b \Rightarrow i_\mu(S,a) \leq i_\mu(S,b), \chi_\mu^{ap}(S,a) \geq \chi_\mu^{ap}(S,b).$$

We note a similar fact about $m(S,a)$.

**Proposition 5.1** *Let $\Gamma$ be an infinite countable group and $S \subseteq \Gamma$ a finite set of generators. Then*

$$a \prec b \Rightarrow m(S,a) \leq m(S,b).$$

**Proof**. Let $\varphi\colon A \to B$ be a matching for $G(S,a)$. Then there are Borel decompositions $A = \bigsqcup_{i=1}^n A_n, B = \bigsqcup_{i=1}^n B_n$, and $s_1, \dots, s_n \in S^{\pm 1}$ with $\varphi|A_i = s_i^a|A_i, \varphi(A_i) = B_i$. Fix $\delta > 0$. Since $a \prec b$, for any $\epsilon > 0$, we can find a sequence $C_1, \dots, C_n$ of pairwise disjoint Borel sets such that for any $\gamma \in \{1\} \cup (S^{\pm 1})^2$, $|\mu(\gamma^a \cdot A_i \cap A_j) - \mu(\gamma^b \cdot C_i \cap C_j)| < \epsilon$, for $i \leq i, j \leq n$. Since $s_i^a \cdot A_i \cap A_j = \emptyset$, for all $1 \leq i,j \leq n$, and $s_i^a \cdot A_i \cap s_j^a \cdot A_j = \emptyset$, for all $1 \leq i \neq j \leq n$, it follows that $|\mu(A_i) - \mu(C_i)| < \epsilon, 1 \leq i \leq n, \mu(s_i^b \cdot C_i \cap C_j) < \epsilon, 1 \leq i,j \leq n$, and $\mu(s_i^b \cdot C_i \cap s_j^b \cdot C_j) < \epsilon, 1 \leq i \neq j \leq n$. By disjointifying and choosing $\epsilon$ very small compared to $\delta$, it is clear that we can find such pairwise disjoint $C_1, \dots, C_n$ with $s_i^b \cdot C_i \cap C_j = \emptyset, 1 \leq i,j \leq n, s_i^b \cdot C_i \cap s_j^b \cdot C_j = \emptyset, 1 \leq i \neq j \leq n$, and if $C = \bigsqcup_{i=1}^n C_i, D = \bigsqcup_{i=1}^n s_i^b \cdot C_i$, then $|\mu(C) - \mu(A)| < \delta$. Clearly $\psi\colon C \to D$ given by $\psi|C_i = s_i^b|C_i$ is a matching for $G(S,b)$ and $\mu(C) > \mu(A) - \delta$. Since $\delta$ was arbitrary this shows that $m(S,a) \leq m(S,b)$. $\dashv$

**(B)** The next result shows that, modulo weak equivalence, we can turn approximate parameters to exact ones.

**Theorem 5.2** *Let $\Gamma$ be an infinite countable group and $S \subseteq \Gamma$ a finite set of generators. Then for any $a \in \mathrm{FR}(\Gamma, X, \mu)$, there is $b \in \mathrm{FR}(\Gamma, X, \mu)$ such that*

*(i)* $a \sim_w b$ *and* $a \sqsubseteq b$,

*(ii)* $\chi_\mu^{ap}(S,a) = \chi_\mu^{ap}(S,b) = \chi_\mu(S,b)$,

*(iii)* $i_\mu(S,a) = i_\mu(S,b)$ *and* $i_\mu(S,b)$ *is attained,*

*(iv)* $m(S,a) = m(S,b)$ *and* $m(S,b)$ *is attained.*

**Proof**. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. The action $b$ will be an appropriate factor of the ultrapower $a_{\mathcal{U}}$.

Let $k = \chi^{ap}_{\mu}(S,a)$. This is finite by Kechris-Solecki-Todorcevic [KST, 4.6]. Let $i_\mu(S,a) = \iota \leq \frac{1}{2}$ and let $m(S,a) = m \leq \frac{1}{2}$. Then for each $n \geq 1$, find the following:

(a) A sequence $C^1_n, \dots, C^k_n$ of pairwise disjoint Borel sets such that $s^a \cdot C^i_n \cap C^i_n = \emptyset$, for $1 \leq i \leq k, s \in S^{\pm 1}$, and $\mu(\bigsqcup_{i=1}^k C^i_n) \geq 1 - \frac{1}{n}$.

(b) A Borel set $I_n$ such that $s^a \cdot I_n \cap I_n = \emptyset, s \in S^{\pm 1}$, and $\mu(I_n) \geq \iota - \frac{1}{n}$.

(c) A pairwise disjoint family of Borel sets $(A^s_n)_{s \in S^{\pm 1}}$, such that $s^a \cdot A^s_n \cap A^t_n = \emptyset, s,t \in S^{\pm 1}, s^a \cdot A^s_n \cap t^a \cdot A^t_n = \emptyset, s,t \in S^{\pm 1}, s \neq t$, and

$$\mu(\bigsqcup_{s \in S^{\pm 1}} A^s_n) \geq m - \tfrac{1}{n}.$$

Consider now the ultrapower action $a_{\mathcal{U}}$ on $(X_{\mathcal{U}}, \mu_{\mathcal{U}})$ and the sets $C^i = [(C^i_n)]_n \in \boldsymbol{B}^0_{\mathcal{U}}, 1 \leq i \leq k, I = [(I_n)]_{\mathcal{U}} \in B^0_{\mathcal{U}}$ and $A^s = [(A^s_n)]_{\mathcal{U}} \in B^0_{\mathcal{U}}, s \in S^{\pm 1}$. Viewed as elements of $\mathrm{MALG}_{\mu_{\mathcal{U}}}$ they satisfy:

(a′) $C^i \cap C^j = \emptyset, 1 \leq i \neq j \leq k, s^{a_{\mathcal{U}}} \cdot C^i \cap C^i = \emptyset, 1 \leq i \leq k, s \in S^{\pm 1}; \mu_{\mathcal{U}}(\bigsqcup_{i=1}^k C^i) = 1$,

(b′) $s^{a_{\mathcal{U}}} \cdot I \cap I = \emptyset, s \in S^{\pm 1}; \mu_{\mathcal{U}}(I) \geq \iota$,

(c′) $A^s \cap A^t = \emptyset, s \neq t, s,t \in S^{\pm 1}; s^{a_{\mathcal{U}}} \cdot A^s \cap A^t = \emptyset, s,t \in S^{\pm 1}; s^{a_{\mathcal{U}}} \cdot A^s \cap t^{a_{\mathcal{U}}} \cdot A^t = \emptyset, s \neq t, s,t \in S^{\pm 1}; \mu(\bigsqcup_{s \in S^{\pm 1}} A^s) \geq m$.

Let now $\boldsymbol{B}_0$ be a countable Boolean subalgebra of $\mathrm{MALG}_{\mu_{\mathcal{U}}}$ closed under the action $a_{\mathcal{U}}$, the functions $S_{\mathcal{U}}, T_{\mathcal{U}}$ of §2, **(B)**, §3, **(B)**, resp., and containing the algebra $\boldsymbol{C}_0$ of §4, **(B)** and also $C^i$ $(1 \leq i \leq k), I, A^s$ $(s \in S^{\pm 1})$. Let $\boldsymbol{B} = \sigma(\boldsymbol{B}_0)$ and let $b$ be the factor of $a_{\mathcal{U}}$ corresponding to $\boldsymbol{B}$. (We can of course assume that $b \in \mathrm{FR}(\Gamma, X, \mu)$.) Then by §4, **(B)** again, $a \sim_w b$ and $a \sqsubseteq b$. So, in particular, $\chi^{ap}_\mu(S,a) = \chi^{ap}_\mu(S,b) = k, i_\mu(S,a) = i_\mu(S,b) = \iota$ and $m(S,a) = m(S,b) = m$, since $a \sim_w b$. The sets $(C^i)_{i \leq k}$ give a measurable coloring of $G(S,b)|A$, for some $A$ with $\mu(A) = 1$ and we can clearly color in a measurable way $G(S,b)| \sim A$ by $\ell$ colors, where $\ell$ is the chromatic number of the Cayley graph $\mathrm{Cay}(\Gamma, S)$ of $\Gamma, S$. Since $\ell \leq k$ and the action is free, it follows that $\chi_\mu(S,b) \leq k$, so $\chi_\mu(S,b) = \chi^{ap}_\mu(S,b)$. Finally, (b′), (c′) show that $i_\mu(S,b) = \iota$ and $m(S,b) = m$ are attained. $\dashv$

## 6 Brooks' Theorem for group actions

**(A)** Brooks' Theorem for finite graphs asserts that for any finite graph $G$ with degree bounded by $d$, the chromatic number $\chi(G)$ is $\leq d$, unless $d = 2$ and $G$ contains an odd cycle or $d \geq 3$ and $G$ contains a complete subgraph (clique) with $d+1$ vertices (and the chromatic number is always $\leq d+1$). In Conley-Kechris [CK] the question of finding analogs of the Brooks bound for graphs of the form $G(S, a)$ is studied. Let $d = |S^{\pm 1}|$ be the degree of $\mathrm{Cay}(\Gamma, S)$. First note that by Kechris-Solecki-Todorcevic [KST, 4.8], $\chi_\mu(S, a) \leq d + 1$ (in fact this holds even for Borel instead of measurable colorings). A compactness argument using Brooks' Theorem also shows that $\chi(S, a) \leq d$, where $\chi(S, a)$ is the chromatic number of $G(S, a)$. It was shown in Conley-Kechris [CK, 2.19, 2.20] that for any infinite $\Gamma$, $\chi_\mu^{ap}(S, a) \leq d$, for any $a \in \mathrm{FR}(\Gamma, X, \mu)$, so one has a full "approximate" version of Brooks' Theorem. How about the full measurable Brooks bound $\chi_\mu(S, a) \leq d$? This is easily false for some action $a$ (e.g., the shift action), when $\Gamma = \mathbb{Z}$ or $\Gamma = (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ (with the usual sets of generators) and it was shown in Conley-Kechris [CK, 5.12] that when $\Gamma$ has finitely many ends and is not isomorphic to $\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$, then one indeed has the Brooks' bound $\chi_\mu(S, a) \leq d$, for *any* $a \in \mathrm{FR}(\Gamma, X, \mu)$ (in fact even for Borel as opposed to measurable colorings). It is unknown if this still holds for $\Gamma$ with infinitely many ends but 5.2 shows that one has the full analog of the Brooks bound up to weak equivalence for any group $\Gamma$.

**Theorem 6.1** *For any infinite group $\Gamma$ and finite set of generators $S$ with $d = |S^{\pm 1}|$, for any $a \in \mathrm{FR}(\Gamma, X, \mu)$, there is $b \in \mathrm{FR}(\Gamma, X, \mu)$, with $b \sim_w a$ and $\chi_\mu(S, b) \leq d$.*

This also leads to the solution of an open problem arising in probability concerning random colorings of Cayley graphs.

Let $\Gamma$ be an infinite group with a finite set of generators $S$. Let $k \geq 1$. Consider the compact space $k^\Gamma$ on which $\Gamma$ acts by shift: $\gamma \cdot p(\delta) = p(\gamma^{-1}\delta)$. The set $\mathrm{Col}(k, \Gamma, S)$ of colorings of $\mathrm{Cay}(\Gamma, S)$ with $k$ colors is a closed (thus compact) invariant subspace of $k^\Gamma$. An *invariant, random $k$-coloring* of the Cayley graph $\mathrm{Cay}(\Gamma, S)$ is an invariant probability Borel measure on the space $\mathrm{Col}(k, \Gamma, S)$. Let $d$ be the degree of $\mathrm{Cay}(\Gamma, S)$. In Aldous-Lyons [AL, 10.5] the question of existence of invariant, random $k$-colorings is discussed and mentioned that Schramm (unpublished, 1997) had shown that for any $\Gamma, S$ there is an invariant, random $(d+1)$-coloring (this also follows from the more general Kechris-Solecki-Todorcevic [KST, 4.8]). They also point out that Brooks' Theorem implies that there is an invariant, random $d$-coloring when $\Gamma$ is a sofic group (for the definition of sofic group, see, e.g., Pestov [P]). The question of whether this holds for arbitrary $\Gamma$ remained open. We show that 6.1 above provides a positive answer. First it will be useful to note the following fact:

**Proposition 6.2** *Let $\Gamma$ be an infinite group, $S$ a finite set of generators for $\Gamma$ and let $k \geq 1$. Then the following are equivalent:*

*(i) There is an invariant, random $k$-coloring,*

*(ii) There is $a \in \mathrm{FR}(\Gamma, X, \mu)$ with $\chi_\mu(S, a) \leq k$.*

**Proof**. (ii) $\Rightarrow$ (i). Let $c\colon X \to \{1, \dots, k\}$ be a measurable coloring of $G(S, a)$. Define $C\colon X \to k^\Gamma$ by $C(x)(\gamma) = c((\gamma^{-1})^a \cdot x)$. Then $C$ is a Borel map from $X$ to $\mathrm{Col}(k, \Gamma, S)$ that preserves the actions, so $C_*\mu$ is an invariant, random $k$-coloring.

(i) $\Rightarrow$ (ii). Let $\rho$ be an invariant, random $k$-coloring. Consider the action of $\Gamma$ on $Y = \mathrm{Col}(k, \Gamma, S)$ (by shift). Fix also a free action $b \in \mathrm{FR}(\Gamma, Z, \nu)$ (for some $(Z, \nu)$). Let $X = Y \times Z, \mu = \rho \times \nu$. Then $\Gamma$ acts freely, preserving $\mu$ on $X$ by $\gamma \cdot (y, z) = (\gamma \cdot y, \gamma \cdot z)$. Call this action $a$. We claim that $\chi_\mu(S, a) \leq k$. For this let $c\colon X \to \{1, \dots, k\}$ be defined by $c((y, z)) = y(1)$ (recall that $y \in \mathrm{Col}(k, \Gamma, S)$, so $y\colon \Gamma \to \{1, \dots, k\}$ is a coloring of $\mathrm{Cay}(\Gamma, S)$). It is easy to check that this a measurable $k$-coloring of $G(S, a)$. $\dashv$

**Remark 6.3** From the proof of (ii) $\Rightarrow$ (i) in 6.2, it is clear that if $a \in \mathrm{FR}(\Gamma, X, \mu)$ has $\chi_\mu(S, a) \leq k$, then there is an invariant, random $k$-coloring which is a factor of $a$.

We now have

**Corollary 6.4** *Let $\Gamma$ be an infinite group and $S$ a finite set of generators. Let $d = |S^{\pm 1}|$. Then there is an invariant, random $d$-coloring. Moreover, for each $a \in \mathrm{FR}(\Gamma, X, \mu)$ there is such a coloring which is weakly contained in $a$.*

**Proof**. This is immediate from 6.1 and 6.3. $\dashv$

Lyons and Schramm (unpublished, 1997) raised the question (see Lyons-Nazarov [LN,§5]) of whether there is, for any $\Gamma, S$, an invariant, random $\chi$-coloring, where $\chi = \chi(\mathrm{Cay}(\Gamma, S))$ is the chromatic number of the Cayley graph. It is pointed out in this paper that the answer is affirmative for amenable groups (as there is an invariant measure for the action of $\Gamma$ on $\mathrm{Col}(\chi, \Gamma, S)$ by amenability) but the general question is open.

**Remark 6.5** One cannot in general strengthen the last statement in 6.4 to: For each $a \in \mathrm{FR}(\Gamma, X, \mu)$, there is an invariant, random $d$-coloring which is a factor of $a$. Indeed, this fails for $\Gamma = \mathbb{Z}$ or $\Gamma = (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ (with the usual set of generators $S$ for which $d = 2$) and $a$ the shift action of $\Gamma$ on $2^\Gamma$, since then the shift action of $\Gamma$ on $\mathrm{Col}(2, \Gamma, S)$ with this random coloring would be mixing and then as in (i) $\Rightarrow$ (ii) of 6.2, by taking $b$ to be also mixing, one could have a mixing action $a \in \mathrm{FR}(\Gamma, X, \mu)$ for which there is a measurable 2-coloring, which easily gives a contradiction. On the other hand, it follows from the result in [CK, 5.12] that was mentioned earlier, that for any $\Gamma$ with finitely many ends, except for $\Gamma = \mathbb{Z}$ or $\Gamma = (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$, one indeed has for any $a \in \mathrm{FR}(\Gamma, X, \mu)$ an invariant, random $d$-coloring which is a factor of the action $a$. We do not know if this holds for groups with infinitely many ends.

**(B)** Let $\Gamma, S$ be as before and let $\mathrm{Aut}_{\Gamma,S} = \mathrm{Aut}(\mathrm{Cay}(\Gamma, S))$ be the automorphism group of the Cayley graph with the pointwise convergence topology. Thus $\mathrm{Aut}_{\Gamma,S}$ is Polish and locally compact. The group $\mathrm{Aut}_{\Gamma,S}$ acts continuously on $\mathrm{Col}(k, \Gamma, S)$ by: $\varphi \cdot c(\gamma) = c(\varphi^{-1}(\gamma))$. Clearly $\Gamma$ can be viewed as a closed subgroup of $\mathrm{Aut}_{\Gamma,S}$ identifying $\gamma \in \Gamma$ with the (left-)translation automorphism $\delta \mapsto \gamma\delta$. It will be notationally convenient below to denote this translation automorphism by $\langle\gamma\rangle$. One can now consider a stronger notion of an invariant, random $k$-coloring by asking that the measure is now invariant under $\mathrm{Aut}_{\Gamma,S}$ instead of $\Gamma$ (i.e. $\langle\Gamma\rangle$). To distinguish the two notions let us call the stronger one a $\mathrm{Aut}_{\Gamma,S}$*-invariant, random $k$-coloring*. We now note that the existence of an invariant, random $k$-coloring is equivalent to the existence of an $\mathrm{Aut}_{\Gamma,S}$-invariant, random $k$-coloring. In fact it follows from the following more general fact (applied to the special case of the action of $\mathrm{Aut}_{\Gamma,S}$ on $\mathrm{Col}(k, \Gamma, S)$).

**Proposition 6.6** *Let* $\mathrm{Aut}_{\Gamma,S}$ *be as before and assume* $\mathrm{Aut}_{\Gamma,S}$ *acts continuously on a compact, metrizable space* $X$*. Then there exists a* $\Gamma$*-invariant Borel probability measure on* $X$ *iff there is a* $\mathrm{Aut}_{\Gamma,S}$*-invariant Borel probability measure on* $X$*.*

**Proof**. Denote by $R = R_{\Gamma,S} = \mathrm{Aut}_1(\mathrm{Cay}(\Gamma, S))$ the subgroup of $G = \mathrm{Aut}_{\Gamma,S}$ consisting of all $\varphi \in G$ with $\varphi(1) = 1$ (we view this as the rotation group of $\mathrm{Cay}(\Gamma, S)$ around 1).

Note that $R$ is compact and $R \cap \Gamma = \{1\}$. Moreover, $G = \Gamma R = R\Gamma$, since if $\varphi \in G$, then $\varphi = \langle\gamma\rangle r$, where $\gamma = \varphi(1)$ and $r = \langle\gamma\rangle^{-1}\varphi$. So

$R$ is a transversal for the (left-) cosets of $\Gamma$, thus $G/\Gamma$ is compact (in the quotient topology), i.e., $\Gamma$ is a co-compact lattice in $G$. It follows that $G/\Gamma$ is amenable in the sense of Greenleaf [Gr] and Eymard [Ey] and so by Eymard [Ey, p.12], 6.6 follows. (We would like to thank the referee for bringing to our attention the connection of 6.6 with the Greenleaf-Eymard concept of amenable quotient.) For the convenience of the reader, we will give this proof in detail below. Some of the notation we establish will be also used later on.

First note that (since $\Gamma$ is a lattice) $G$ is unimodular, i.e., there is a left and right invariant Haar measure on $G$ (see, e.g., Einsiedler-Ward [EW, 9.20]), so fix such a Haar measure $\eta$. Since $R$ is compact (and $G = \Gamma R$), $\infty > \eta(R) > 0$ and we normalize $\eta$ so that $\eta(R) = 1$. Then $\rho = \eta | R$ is the Haar measure of $R$.

Every $\varphi \in G$ can be written as

$$\varphi = \langle\gamma\rangle r = r'\langle\gamma'\rangle$$

for unique $\gamma, \gamma' \in \Gamma, r, r' \in R$. Here $\gamma = \varphi(1), r = \langle\gamma\rangle^{-1}\varphi$ and $\gamma' = (\varphi^{-1}(1))^{-1}, r' = \varphi\langle\gamma'\rangle^{-1} = \varphi\langle\varphi^{-1}(1)\rangle$. This gives a map $\alpha \colon \Gamma \times R \to R$ defined by $\alpha(\gamma, r) = r'$, where $\langle\gamma\rangle r = r'\langle\gamma'\rangle$. Thus

$$\alpha(\gamma, r) = \langle\gamma\rangle r \langle r^{-1}(\gamma^{-1})\rangle.$$

One can now easily verify that this is a continuous action of $\Gamma$ on $R$ and we will write

$$\gamma \cdot r = \alpha(\gamma, r) = \langle\gamma\rangle r \langle r^{-1}(\gamma^{-1})\rangle.$$

(If we identify $R$ with the quotient $G/\Gamma$, then this action is just the canonical action of $\Gamma$ on $G/\Gamma$.)

Moreover this action preserves the Haar measure $\rho$. Indeed, fix $\gamma \in \Gamma$ and put $p_\gamma(r) = \gamma \cdot r$. We will show that $p_\gamma \colon R \to R$ preserves $\rho$. For $\delta \in \Gamma$, let $R_\delta = \{r \in R \colon r^{-1}(\gamma^{-1}) = \delta\}$. Then $R = \bigsqcup_{\delta\in\Gamma} R_\delta$ and $p_\gamma(r) = \langle\gamma\rangle r\langle\delta\rangle$ for $r \in R_\delta$, thus $p_\gamma | R_\delta$ preserves $\eta$ and so $p_\gamma$ preserves $\rho$.

Assume now that $\mu_\Gamma$ is a Borel probability measure on $X$ which is $\Gamma$-invariant. We will show that there is a Borel probability measure $\mu_G$ on $X$ which is $G$-invariant. Define

$$\mu_G = \int_R (r \cdot \mu_\Gamma) dr,$$

where the integral is over the Haar measure $\rho$ on $R$, i.e., for each continuous $f \in C(X)$,

$$\mu_G(f) = \int_R (r \cdot \mu_\Gamma)(f) dr,$$

with $r \cdot \mu_\Gamma(f) = \mu_\Gamma(r^{-1} \cdot f), r^{-1} \cdot f(x) = f(r \cdot x)$. (As usual we put $\sigma(f) = \int f d\sigma$.) We will verify that $\mu_G$ is $G$-invariant.

Let $F \colon X \to X$ be a homeomorphism. For $\sigma$ a Borel probability measure on $X$, let $F \cdot \sigma = F_* \sigma$ be the measure defined by

$$F \cdot \sigma(f) = \sigma(f \circ F),$$

for $f \in C(X)$. Then we have

$$F \cdot \mu_G = \int_R F \cdot (r \cdot \mu_\Gamma) dr,$$

because for $f \in C(X)$,

$$\begin{aligned} F \cdot \mu_G(f) &= \mu_G(f \circ F) \\ &= \int (r \cdot \mu_\Gamma)(f \circ F) dr \\ &= \int F \cdot (r \cdot \mu_\Gamma) dr. \end{aligned}$$

We first check that $\mu_G$ is $R$-invariant. Indeed if $s \in R$,

$$\begin{aligned} s \cdot \mu_G &= \int s \cdot (r \cdot \mu_\Gamma) dr \\ &= \int (sr) \cdot \mu_\Gamma dr \\ &= \int (r \cdot \mu_\Gamma) dr \\ &= \mu_G \end{aligned}$$

by the invariance of Haar measure.

Finally we verify that $\mu_G$ is $\Gamma$-invariant (which completes the proof that

$\mu_G$ is $G$-invariant as $G = \Gamma R$). Indeed, in the preceding notation

$$\begin{aligned}\langle\gamma\rangle \cdot \mu_G &= \int \langle\gamma\rangle \cdot (r \cdot \mu_\Gamma) dr \\ &= \int (\langle\gamma\rangle r) \cdot \mu_\Gamma dr \\ &= \int (\gamma \cdot r) \cdot (\langle\gamma'\rangle \cdot \mu_\Gamma) dr \\ &= \int (\gamma \cdot r) \cdot \mu_\Gamma dr\end{aligned}$$

(as $\langle\gamma'\rangle \cdot \mu_\Gamma = \mu_\Gamma$ for any $\gamma' \in \Gamma$).

But we have seen before that $r \mapsto \gamma \cdot r$ preserves the Haar measure of $R$, so

$$\begin{aligned}\langle\gamma\rangle \cdot \mu_G &= \int (\gamma \cdot r) \cdot \mu_\Gamma dr \\ &= \int (r \cdot \mu_\Gamma) dr \\ &= \mu_G\end{aligned}$$

$\dashv$

**(C)** As was discussed in 6.5, for any $\Gamma, S$ with finitely many ends, except $\Gamma = \mathbb{Z}$ or $\Gamma = (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$, and any $a \in \mathrm{FR}(\Gamma, X, \mu)$, there is an invariant, random $d$-coloring, where $d = |S^{\pm 1}|$, which is a factor of $a$. This is of particular interest in the case where $a$ is the shift action $s_\Gamma$ of $\Gamma$ on $[0,1]^\Gamma$ (with the usual product measure). In that case $\mathrm{Aut}_{\Gamma,S} = \mathrm{Aut}(\mathrm{Cay}(\Gamma, S))$ also acts via shift on $[0,1]^\Gamma$ via $\varphi \cdot p(\gamma) = p(\varphi^{-1}(\gamma))$ and one can ask whether there is actually a $\mathrm{Aut}_{\Gamma,S}$-invariant, random $d$-coloring, which is a factor of the shift action of $\mathrm{Aut}_{\Gamma,S}$ on $[0,1]^\Gamma$. We indeed have:

**Theorem 6.7** *Let $\Gamma$ be an infinite countable group, $S$ a finite set of generators, and let $d = |S^{\pm 1}|$. If $\Gamma$ has finitely many ends but is not isomorphic to $\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$, and $\mathrm{Aut}_{\Gamma,S} = \mathrm{Aut}(\mathrm{Cay}(\Gamma, S))$, there is a $\mathrm{Aut}_{\Gamma,S}$-invariant, random $d$-coloring which is a factor of the shift action of $\mathrm{Aut}_{\Gamma,S}$ on $[0,1]^\Gamma$.*

**Proof**. Put again $G = \mathrm{Aut}_{\Gamma,S}$. Let $X$ be the free part of the action of $G$ on $[0,1]^\Gamma$, i.e.,

$$X = \{x \in [0,1]^\Gamma \colon \forall \varphi \in G \setminus \{1\} (\varphi \cdot x \neq x)\},$$

(where $\varphi \cdot x$ is the action of $G$ on $[0,1]^\Gamma$).

If $\mu$ is the product measure on $[0,1]^\Gamma$, then $\mu(X) = 1$, since $X \supseteq \{x \in [0,1]^\Gamma \colon x \text{ is } 1-1\} = X_0$ and $\mu(X_0) = 1$. Moreover $X$ is a $G$-invariant Borel subset of $[0,1]^\Gamma$.

Since $R = \mathrm{Aut}_1(\mathrm{Cay}(\Gamma, S))$ is compact, $E_R^X$, the equivalence relation induced by $R$ on $X$, admits a Borel selector and

$$X_R = X/R = \{R \cdot x \colon x \in X\}$$

is a standard Borel space. Define the following Borel graph $E$ on $X_R$

$$(R \cdot x)E(R \cdot y) \Leftrightarrow \exists s \in S^{\pm 1}(\langle s \rangle R \cdot x \cap R \cdot y \neq \emptyset).$$

**Lemma 6.8** *If $(R \cdot x)E(R \cdot y)$, then*

$$(x_1, x_2) \in M_{R \cdot x, R \cdot y} \Leftrightarrow x_1 \in R \cdot x \;\&\; x_2 \in R \cdot y \;\&\; \exists s \in S^{\pm 1}(\langle s \rangle \cdot x_1 = x_2),$$

*(is the graph of) a bijection between $R \cdot x, R \cdot y$ consisting of edges of the graph $G(S, s_\Gamma)$, i.e., it is a matching.*

**Proof**. Fix $x_1^0 \in R \cdot x_1, x_2^0 \in R \cdot x_2$ and $s_0 \in S^{\pm 1}$ with $\langle s_0 \rangle \cdot x_1^0 = x_2^0$.

First we check that $M_{R \cdot x, R \cdot y}$ is a matching. Let $(x_1, x_2), (x_1, x_2') \in M_{R \cdot x, R \cdot y}$ and let $\langle s \rangle \cdot x_1 = x_2, \langle s' \rangle \cdot x_1 = x_2'$, for some $s, s' \in S^{\pm 1}$, and $r \cdot x_2 = x_2'$, for some $r \in R$. Then $r\langle s \rangle \cdot x_1 = \langle s' \rangle \cdot x_1$, so $r\langle s \rangle = \langle s' \rangle$, thus $r \in \Gamma$, so $r = 1$ and $x_2 = x_2'$. Similarly $(x_1, x_2), (x_1', x_2) \in M_{R \cdot x, R \cdot y}$ implies that $x_1 = x_1'$.

Next we verify that for every $x_1 \in R \cdot x$, there is an $x_2 \in R \cdot y$ with $(x_1, x_2) \in M_{R \cdot x, R \cdot y}$. Let $r_1 \in R$ be such that $r_1 \cdot x_1 = x_1^0$, so $\langle s_0 \rangle r_1 \cdot x_1 = x_2^0$. Now

$$\begin{aligned}\langle s_0 \rangle r_1 &= \big(\langle s_0 \rangle r_1 \langle r_1^{-1}(s_0^{-1}) \rangle\big) \langle r_1^{-1}(s_0^{-1}) \rangle^{-1} \\ &= r_2^{-1} \langle s' \rangle,\end{aligned}$$

where $r_2 \in R$ and $s' \in S^{\pm 1}$. Thus $r_2^{-1}\langle s' \rangle \cdot x_1 = x_2^0$, so $\langle s' \rangle \cdot x_1 = r_2 \cdot x_2^0 = x_2 \in R \cdot y$ and $(x_1, x_2) \in M_{R \cdot x, R \cdot y}$. Similarly for every $x_2 \in R \cdot y$ there is $x_1 \in R \cdot x$ with $(x_1, x_2) \in M_{R \cdot x, R \cdot y}$, and the proof is complete. $\dashv$

**Lemma 6.9** *Let $x \in X$. Then the map*

$$\gamma \mapsto R \cdot (\langle \gamma \rangle^{-1} \cdot x)$$

*is an isomorphism of* $\mathrm{Cay}(\Gamma, S)$ *with the connected component of $R \cdot x$ in $E$.*

**Proof**. Let $\gamma \in \Gamma$ and let $s_1, \ldots, s_k \in S^{\pm 1}$ be such that $\gamma^{-1} = s_n \ldots s_1$. Then $(R \cdot x) E (R \cdot (\langle s_1 \rangle \cdot x)) E \ldots E(R \cdot (\langle \gamma \rangle^{-1} \cdot x))$, so $R \cdot (\langle \gamma \rangle^{-1} \cdot x)$ is in the connected component of $R \cdot x$. Conversely assume that $R \cdot y$ is in the connected component of $R \cdot x$ and say $(R \cdot x) E (R \cdot x_1) E (R \cdot x_2) E \ldots E(R \cdot x_{n-1}) E (R \cdot y)$. By Lemma 6.8, there are $s_1, \ldots s_n \in S^{\pm 1}$ and $x'_1, \ldots, x'_n$ such that $\langle s_1 \rangle \cdot x = x'_1 \in R \cdot x_1, \langle s_2 \rangle \cdot x'_1 = x'_2 \in R \cdot x_2, \ldots, \langle s_n \rangle \cdot x'_{n-1} = x'_n \in R \cdot y$. Let $\gamma^{-1} = s_n \ldots s_1$. Then $x'_n = \langle \gamma \rangle^{-1} \cdot x \in R \cdot y$, so $R \cdot (\langle \gamma \rangle^{-1} \cdot x) = R \cdot y$. Thus $\gamma \mapsto R \cdot (\langle \gamma \rangle^{-1} \cdot x)$ maps $\Gamma$ onto the connected component of $R \cdot x$.

We next check that $\gamma \mapsto R \cdot (\langle \gamma \rangle^{-1} \cdot x)$ is 1-1. Indeed if $R \cdot (\langle \gamma \rangle^{-1} \cdot x) = R \cdot (\langle \delta^{-1} \rangle \cdot x)$, then $r \langle \gamma \rangle^{-1} \cdot x = \langle \delta \rangle^{-1} \cdot x$ for some $r \in R$, so as before $r = 1$ and $\gamma = \delta$.

Finally let $(\gamma, \gamma s)$ be an edge in the Cayley graph of $\Gamma, S$. Then clearly $R \cdot (\langle \gamma \rangle^{-1} \cdot x) E R \cdot \langle \gamma s \rangle^{-1} \cdot x) = R \cdot (\langle s \rangle^{-1} \langle \gamma \rangle^{-1} \cdot x)$. Conversely assume that $R \cdot (\langle \gamma \rangle^{-1} \cdot x) E R \cdot (\langle \delta \rangle^{-1} \cdot x)$, so that, by 6.8 again, there are $s \in S^{\pm 1}, r \in R$ with $\langle s \rangle \langle \gamma \rangle^{-1} \cdot x = r \langle \delta \rangle^{-1} \cdot x$, i.e., $\langle s \rangle \langle \gamma \rangle^{-1} = r \langle \delta \rangle^{-1}$. Then $r = 1$ and $\gamma s^{-1} = \delta$, so $(\gamma, \delta)$ is an edge in the Cayley graph. $\dashv$

The following will be needed in the next section, so we record it here.

Let $\pi \colon X \to X_R$ be the projection function: $\pi(x) = R \cdot x$. Let $\nu = \pi_* \mu$ be the image of $\mu$.

**Lemma 6.10** *$E$ preserves the measure $\nu$.*

**Proof**. Let $\varphi \colon A \to B$ be a Borel bijection with $A, B$ Borel subsets of $X_R$ and $\mathrm{graph}(\varphi) \subseteq E$. We will show that $\nu(A) = \nu(B)$.

We have $\nu(A) = \mu(\bigcup_{R \cdot x \in A} R \cdot x)$ and similarly for $B$. If $\varphi(R \cdot x) = R \cdot y$, then $M_{R \cdot x, R \cdot y}$ gives a Borel bijection of $R \cdot x, R \cdot y$ whose graph consists of edges of $G(S, s_\Gamma)$ and $\bigcup_{R \cdot x \in A} M_{R \cdot x, R \cdot y}$ gives the graph of a Borel bijection of $\bigcup_{R \cdot x \in A} R \cdot x$ with $\bigcup_{R \cdot x \in B} R \cdot x$, therefore $\nu(A) = \nu(B)$. $\dashv$

We now complete the proof of the proposition. Consider the graph $(X_R, E)$. By 6.9, it is a Borel graph whose connected components are isomorphic to Cayley graphs of degree $d = |S^{\pm 1}|$ that have finitely many ends. So by Conley-Kechris [CK, 5.1, 5.7, 5.11] and Lemma 6.9, $(X_R, E)$ has a Borel $d$-coloring. $C_R \colon X_R \to \{1, \ldots, d\}$. Define now $C \colon X \to \{1, \ldots, d\}$ by

$$C(x) = C_R(R \cdot x)$$

Then clearly $C$ is a Borel $d$-coloring of $G(S, a)$. We use this as usual to define a random $d$-coloring of the Cayley graph. Define

$$\psi \colon X \to \mathrm{Col}(d, \Gamma, S)$$

by

$$\psi(x)(\gamma) = C(\langle\gamma\rangle^{-1} \cdot x).$$

and consider the measure $\psi_* \mu$ on $\mathrm{Col}(d, \Gamma, S)$. This will be $G$-invariant provided that $\psi$ preserves the $G$-action, which we now verify.

First it is clear that $\psi$ preserves the $\Gamma$-action. It is therefore enough to check that it preserves the $R$-action, i.e., $\psi(r \cdot x) = r \cdot \psi(x)$ for each $x \in X, r \in R$. Let $\gamma \in \Gamma$ in order to check that $\psi(r \cdot x)(\gamma) = (r \cdot \psi(x))(\gamma)$ or $C(\langle\gamma\rangle^{-1} r \cdot x) = \psi(x)(r^{-1}(\gamma)) = C(\langle r^{-1}(\gamma)\rangle^{-1} \cdot x)$. But recall that

$$\langle\gamma\rangle^{-1} r = (\langle\gamma\rangle^{-1} r \langle r^{-1}(\gamma)\rangle)\langle r^{-1}(\gamma)\rangle^{-1},$$

so $\langle\gamma\rangle^{-1} r = r' \langle r^{-1}(\gamma)\rangle^{-1}$, for some $r' \in R$, therefore $R \cdot (\langle\gamma\rangle^{-1} r \cdot x) = R \cdot (\langle r^{-1}(\gamma)\rangle^{-1} \cdot x)$ and since $C(y)$ depends only on $R \cdot y$, this completes the proof. $\dashv$

**(D)** Fix an infinite group $\Gamma$ and a finite set of generators $S$, let $G = \mathrm{Aut}_{\Gamma,S} = \mathrm{Aut}(\mathrm{Cay}(\Gamma, S))$ and let $R = R_{\Gamma,S} = \mathrm{Aut}_1(\mathrm{Cay}(\Gamma, S))$ as in the proof of 6.6. Then the action $\gamma \cdot r$ of $\Gamma$ on $R$ defined there is an action by measure preserving homeomorphisms on the compact, metrizable group $R$. Provided that $\Gamma, S$ have the property that $R$ is uncountable, this may provide an interesting example of an action of $\Gamma$.

For instance let $\Gamma = \mathbb{F}_2$, the free group with two generators, and let $S = \{a, b\}$ be a set of free generators. Then it is not hard to see that the action of $\Gamma$ on $R$ is free (with respect to the Haar measure $\rho$ on $R$). Indeed, let $\Gamma_n = \{w \in \Gamma \colon |w| = n\}$ (where $|w|$ denotes word length in the generators $a, b$) and for $w, v \in \Gamma_n$, let $N_{w,v} = \{r \in R \colon r(w) = v\}$. If $v \neq v' \in \Gamma_n$, then $N_{w,v} \cap N_{w,v'} = \emptyset$ and since $R$ acts transitively on $\Gamma_n$, there is $r \in R$ with $rv' = v'$, so $rN_{w,v} = N_{w,v'}$ and thus $\rho(N_{w,v}) = N_{w,v'}$. So

$$\rho(N_{w,v}) = \frac{1}{|\Gamma_n|}$$

for $w, v \in \Gamma_n$.

Let now $\gamma \in \Gamma \setminus \{1\}$ and assume that $r \in R$ is such that $\gamma^{-1} \cdot r = \langle\gamma\rangle^{-1} r \langle r^{-1}(\gamma)\rangle = r$ or $\langle\gamma\rangle r = r\langle r^{-1}(\gamma)\rangle$, so for all $\delta \in \Gamma$, $\gamma r(\delta) = r(r^{-1}(\gamma)\delta)$ or $r^{-1}(\gamma)\delta = r^{-1}(\gamma r(\delta))$ and letting $r(\delta) = \epsilon$, we have $r^{-1}(\gamma) r^{-1}(\epsilon) = r^{-1}(\gamma\epsilon)$. Since $\epsilon$ was arbitrary in $\Gamma$, this shows that $r^{-1}(\gamma^n) = (r^{-1}(\gamma))^n, \forall n \geq 1$. It is thus enough to show that for each $\gamma \in \Gamma \setminus \{1\}, \{r \in R \colon \forall n \geq 1 (r(\gamma^n) = (r(\gamma))^n)\}$ is null. Let $|\gamma^n| = a_n \to \infty$. Then if $\gamma \in \Gamma$, $\{r \in R \colon r(\gamma^n) =$

$(r(\gamma))^n\} \subseteq \bigcup_{\epsilon\in\Gamma_k}\{r \in R \colon r(\gamma^n) = \epsilon^n\}$, so $\rho(\{r \in R \colon r(\gamma^n) = (r(\gamma))^n\}) \leq \sum_{\epsilon\in\Gamma_k} \rho(N_{\gamma^n,\epsilon^n}) \to 0$ as $n \to \infty$. Thus $\{r \in R \colon \forall n \geq 1(r(\gamma^n) = (r(\gamma))^n)\}$ is null.

# 7 Matchings

**(A)** Let $\Gamma$ be an infinite group and $S$ a finite set of generators for $\Gamma$. For $a \in \mathrm{FR}(\Gamma, X, \mu)$, recall that $m(S, a)$ is the matching number of $a$, defined in §5. If $m(S, a) = \frac{1}{2}$ and the supremum in the definition of $m(S, a)$ is attained, we say that $G(S, a)$ admits an a.e. perfect matching.

Abért, Csoka, Lippner and Terpa [ACLT] have shown that the Cayley graph $\mathrm{Cay}(\Gamma, S)$ admits a perfect matching.

Let $E_{\Gamma,S}$ be the set of edges of the Cayley graph $\mathrm{Cay}(\Gamma, S)$ and consider the space $2^{E_{\Gamma,S}}$, which we can view as the space of all $A \subseteq E_{\Gamma,S}$. Denote by

$$M(\Gamma, S)$$

the closed subspace consisting of all $M \subseteq E_{\Gamma,S}$ that are perfect matchings of the Cayley graph. The group $\mathrm{Aut}_{\Gamma,S} = \mathrm{Aut}(\mathrm{Cay}(\Gamma, S))$ acts on $2^{E_{\Gamma,S}}$ by shift: $\varphi \cdot x(\gamma, \delta) = x(\varphi^{-1}(\gamma), \varphi^{-1}(\delta))$ and so does the subgroup $\Gamma \leq \mathrm{Aut}_{\Gamma,S}$. Clearly $M(\Gamma, S)$ is invariant under this action.

A $\mathrm{Aut}_{\Gamma,S}$-*invariant, random perfect matching* of the Cayley graph is a shift invariant probability Borel measure on $M(\Gamma, S)$. If such a measure is only invariant under the shift action by $\Gamma$, we call it an *invariant, random perfect matching.*

Lyons and Nazarov [LN] considered the question of the existence of invariant, random perfect matchings which are factors of the shift of $\Gamma$ on $[0, 1]^\Gamma$ and showed the following result.

**Theorem 7.1** (Lyons-Nazarov [LN, 2.4]) *Let $\Gamma$ be a non-amenable group, $S$ a finite set of generators for $\Gamma$ and assume that $\mathrm{Cay}(\Gamma, S)$ is bipartite (i.e., has no odd cycles). Then there is a $\mathrm{Aut}_{\Gamma,S}$-invariant, random perfect matching, which is a factor of the shift action of $\mathrm{Aut}_{\Gamma,S}$ on $[0, 1]^\Gamma$.*

Let us next note some facts that follow from earlier considerations in this paper.

**Proposition 7.2** *Let $\Gamma$ be an infinite group and $S$ a finite set of generators for $\Gamma$. Then the following are equivalent:*

*(i) There is an invariant, random perfect matching.*

*(ii) There is $a \in \mathrm{FR}(\Gamma, X, \mu)$ such that $G(S, a)$ admits an a.e. perfect matching.*

*(iii) There is a sequence $a_n \in \mathrm{FR}(\Gamma, X, \mu)$ with $m(S, a_n) \to \frac{1}{2}$.*

**Proof**. As in 6.2 and 5.2. ⊣

**Proposition 7.3** *For $\Gamma, S$ as in 7.2., if $a \in \mathrm{FR}(\Gamma, X, \mu)$ is such that the matching number $m(S, a) = \frac{1}{2}$, then there is $b \in \mathrm{FR}(\Gamma, X, \mu)$ with $b \sim_w a$ and $G(S, b)$ admitting an a.e. perfect matching, and there is an invariant, random perfect matching weakly contained in $a$.*

**Proof**. As in 5.2 and the proof of 6.2. ⊣

**Proposition 7.4** *Let $\Gamma, S, \mathrm{Aut}_{\Gamma,S}$ be as before. Then there is an invariant, random perfect matching iff there is an $\mathrm{Aut}_{\Gamma,S}$-invariant, random perfect matching.*

**Proof**. By 6.6. ⊣

We now have

**Proposition 7.5** *Let $\Gamma$ be an infinite group and $S$ a finite set of generators.*

*(i) If $\Gamma$ is amenable or if $S$ has an element of infinite order, then for any $a \in \mathrm{FR}(\Gamma, X, \mu), m(S, a) = \frac{1}{2}$.*

*(ii) If $S$ has an element of even order, then for any $a \in \mathrm{FR}(\Gamma, X, \mu), G(S, a)$ admits an a.e. perfect matching.*

**Proof**. i) When $\Gamma$ is amenable, this follows from the result of Abért, Csoka, Lippner and Terpa [ACLT] that $\mathrm{Cay}(\Gamma, S)$ admits a perfect matching, using also the quasi-tiling machinery of Ornstein-Weiss [OW], as in Conley-Kechris [CK, 4.10, 4.11]. The second case follows immediately from Rokhlin's Lemma.

ii) This is obvious. ⊣

We do not know if $m(S, a) = \frac{1}{2}$ holds for *every* $\Gamma, S, a \in \mathrm{FR}(\Gamma, X, \mu)$. By 7.5 the only problematic case is when $S$ consists of elements of odd order and $\Gamma$ is not amenable. We will see however that the answer is affirmative for the group $\Gamma = (\mathbb{Z}/3\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z})$ and the usual set of generators $S = \{s, t\}$ with $s^3 = t^3 = 1$. In fact we have the following stronger result:

**Theorem 7.6** *Let $\Gamma = (\mathbb{Z}/3\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z})$ with the usual set of generators $S = \{s, t\}$, with $s^3 = t^3 = 1$. Then for any $a \in \mathrm{FR}(\Gamma, X, \mu)$, $G(S, a)$ admits an a.e. perfect matching.*

**Proof.** Suppose that $M$ is a matching for some graph $G = (X, E)$. Recall that an $(M)$-*augmenting path* in $G$ is a path $x_0, x_1, \cdots, x_{2k+1}$ ($k \in \mathbb{N}$) such that $x_0, x_{2k+1} \notin X_M$, the edges of the form $(x_{2i+1}, x_{2i+2})$ are in $M$, and the edges of the form $(x_{2i}, x_{2i+1})$ are not in $M$.

We will in fact show more generally that any $\mu$-preserving graph $G = (X, E)$ on $(X, \mu)$ whose connected components are isomorphic to the Cayley graph $\mathrm{Cay}(\Gamma, S)$ admits a $\mu$-a.e. matching.

Elek-Lippner [EL2] establishes that for any Borel matching $M$ of $G$ and any $k$, there is a Borel matching $M'$ of $G$ such that $X_M \subseteq X_{M'}$ and $M'$ has no augmenting paths of length $< k$.

**Lemma 7.7** *Suppose that $M_n$ is a Borel matching contained in $G$ with no augmenting paths of length less than $4n$. Then $\mu(X_{M_n}) > 1 - 2^{-n}$.*

**Proof.** Fix $x \neq y$ in $X \setminus X_{M_n}$ so that $d_G(x, y) = k$ is least possible. We first show that $k > 2n$. Let $x = x_0 E x_1 E \cdots E x_{k-1} E x_k = y$ be the unique $G$-path from $x$ to $y$ of length $k$. Since $x_1, \ldots, x_{k-1}$ are in $X_{M_n}$ by the minimality assumption, we may fix edges $m_1, \ldots, m_{k-1}$ in $M_n$ with each $x_i$ incident with $m_i$ (note that $m_i$ may equal $m_{i+1}$). For each $x_i$, let $z_i$ denote the vertex incident with $m_i$ not equal to $x_i$. Also let $e_i = (x_i, x_{i+1})$, for $i < k$.

There is a unique augmenting path form $x$ to $y$ with vertex set $\{x, y\} \cup \{x_i : 1 \leq i < k\} \cup \{z_i : 1 \leq i < k\}$ defined as follows: Say that $x_i$ ($1 \leq i < k$) is of type 0 if $m_i$ is either $e_i$ or $e_{i-1}$. Say that $x_i$ is of type R(ight) if $(x_i, z_i, x_{i+1})$ is a triangle in the Cayley graph and $x_i$ is of type L(eft) if $(x_i, z_i, x_{i-1})$ is a triangle in the Cayley graph. Note that every $x_i$ is in exactly one of these types and that it is not possible to have $x_i$ which is of type R but $x_{i+1}$ is of type L. Our augmenting path is obtained by keeping all $m_i$ that happen to be in the original path from $x$ to $y$ and replacing $e_i$ for $x_i$ of type R by $(x_i, z_i), (z_i, x_{i+1})$ and $e_{i-1}$ for $x_i$ of type L by $(x_{i-1}, z_i), (z_i, x_i)$. This augmenting path has length at most $2k - 1$. But by assumption $M_n$ has no augmenting paths of length less than $4n$, which implies $k > 2n$.

In other words, if $x, y$ are distinct elements of $X \setminus X_{M_n}$, then $B_n(x)$ and $B_n(y)$ are disjoint, where $B_n(x)$ denotes the distance $n$ ball centered at $x$. Since $|B_n(x)| > 2^n$, we have $\mu(X \setminus X_{M_n}) < 2^{-n}$ as required. $\dashv$

The lemma on its own shows that the matching number for $G$ is $1/2$. To show that the supremum is attained, we use the result of Elek-Lippner [EL2] mentioned earlier to find a sequence of Borel matchings $(M_n)$ with $X_{M_n} \subseteq X_{M_{n+1}}$ and with $M_n$ having no augmenting paths of length less than $4n$. Then we use the argument in Lyons-Nazarov [LN] to show that $M$ defined by

$$(x, y) \in M \Leftrightarrow \exists m \forall n \geq m \ (x, y) \in M_n$$

is a Borel matching with $\mu(X_M) = 1$. $\dashv$

We also do not know if for every $\Gamma, S$, there is an invariant, random perfect matching (a question brought to our attention by Abért and also Lyons).

**(B)** We recall also the following result of Lyons-Nazarov [LN]:

**Theorem 7.8** (Lyons-Nazarov [LN, 2.6]) *Let $(X, \mu)$ be a non-atomic, standard measure space and $G = (X, E)$ a Borel locally countable graph which is bipartite and measure preserving (i.e., the equivalence relation it generates is measure preserving). If $G$ is* expansive*, i.e., there is $c > 1$ such that for each Borel independent set $A \subseteq X, \mu(A') \geq c\mu(A)$, where $A' = \{x \colon \exists y Ex(y \in A)\}$, then $G$ admits an a.e. perfect matching.*

We note that, using the argument in 6.7, one can show that Theorem 7.8 implies Theorem 7.1.

**Proof that 7.8 $\Rightarrow$ 7.1**. Using the notation of the proof of 6.7, we first show that the graph $E$ defined there satisfies the hypotheses of 7.8.

**Lemma 7.9** *$(X_R, E)$ is bipartite.*

**Proof**. By 6.9. $\dashv$

**Lemma 7.10** *$(X_R, E)$ is strictly expanding.*

**Proof**. Let $A \subseteq X_R$ be an independent Borel set and $A' = \{x \in X_R \colon \exists y \in A(xEy)\}$. Since the group $\Gamma$ is not amenable, the graph $G(S, s_\Gamma)$, where $s_\Gamma$ is the shift action of $\Gamma$ on $[0,1]^\Gamma$ is strictly expanding, so let $c > 1$ be the constant witnessing that. We will show that $\nu(A') \geq c\nu(A)$. This is immediate since $\bigcup_{R \cdot x \in A} R \cdot x$ is independent in $G(S, s_\Gamma)$ and $(\bigcup_{R \cdot x \in A} R \cdot x)' = \bigcup_{R \cdot x \in A'} R \cdot x$. $\dashv$

Thus by 7.8, there is an a.e. perfect matching for $(X_R, E)$ which we denote by $M_R$. Using 6.8 this gives an a.e. perfect matching $M$ for $G(S, s_\Gamma)$ defined by

$$(x, y) \in M \Leftrightarrow (R \cdot x, R \cdot y) \in M_R \;\&\; (x, y) \in M_{R \cdot x, R \cdot y}.$$

Define now

$$\varphi\colon [0,1]^\Gamma \to M(\Gamma, S)$$

by

$$(\gamma, \gamma s) \in \varphi(x) \Leftrightarrow (\langle\gamma\rangle^{-1} \cdot x, \langle s\rangle^{-1}\langle\gamma\rangle^{-1} \cdot x) \in M,$$

for $s \in S^{\pm 1}$. It is enough to show that $\varphi$ preserves the $\mathrm{Aut}_{\Gamma,S}$-action.

First we check that $\varphi(\langle\delta\rangle \cdot x) = \delta \cdot \varphi(x)$ for $\delta \in \Gamma$. Indeed $(\gamma, \gamma s) \in \varphi(\langle\delta\rangle \cdot x) \Leftrightarrow (\langle\gamma\rangle^{-1}\langle\delta\rangle \cdot x, \langle s\rangle^{-1}\langle\gamma\rangle^{-1}\langle\delta\rangle \cdot x) \in M \Leftrightarrow (\delta^{-1}\gamma, \delta^{-1}\gamma s) \in \varphi(x) \Leftrightarrow (\gamma, \gamma s) \in \delta \cdot \varphi(x)$.

Finally we verify that $\varphi(r \cdot x) = r \cdot \varphi(x)$, for $r \in R$, i.e., $(\gamma, \gamma s) \in \varphi(r \cdot x) \Leftrightarrow (\gamma, \gamma s) \in r \cdot \varphi(x)$. Now

$$(\gamma, \gamma s) \in \varphi(r \cdot x) \Leftrightarrow (\langle\gamma\rangle^{-1} r \cdot x, \langle s\rangle^{-1}\langle\gamma\rangle^{-1} r \cdot x) \in M$$

and

$$\begin{aligned}(\gamma, \gamma s) \in r \cdot \varphi(x) &\Leftrightarrow (r^{-1}(\gamma), r^{-1}(\gamma s)) \in \varphi(x) \\ &\Leftrightarrow \big(\langle r^{-1}(\gamma)\rangle^{-1} \cdot x, \langle s'\rangle^{-1}\langle r^{-1}(\gamma)\rangle^{-1} \cdot x\big) \in M,\end{aligned}$$

where $r^{-1}(\gamma s) = r^{-1}(\gamma)s'$, for some $s' \in S^{\pm 1}$. Now $\langle\gamma\rangle^{-1} r = p\langle\gamma'\rangle$, for some $p \in R$ and $\gamma' = (r^{-1}(\gamma))^{-1}$. We have therefore to show that

$$(p\langle\gamma'\rangle \cdot x, \langle s\rangle^{-1} p\langle\gamma'\rangle \cdot x) \in M \Leftrightarrow (\langle\gamma'\rangle \cdot x, \langle s'\rangle^{-1}\langle\gamma'\rangle \cdot x) \in M.$$

Clearly $p\langle\gamma'\rangle \cdot x, \langle\gamma'\rangle \cdot x$ belong to the same $R$-orbit, so it is enough to show that $p' = \langle s\rangle^{-1} p\langle s'\rangle \in R$. Because then $\langle s\rangle^{-1} p\langle\gamma'\rangle \cdot x = p'\langle s'\rangle^{-1}\langle\gamma'\rangle \cdot x$ and thus $R \cdot (p\langle\gamma'\rangle \cdot x) = R \cdot (\langle\gamma'\rangle \cdot x) = A, R \cdot (\langle s\rangle^{-1} p\langle\gamma'\rangle \cdot x) = R \cdot (\langle s'\rangle^{-1}\langle\gamma'\rangle \cdot x) = B$ and $(p\langle\gamma'\rangle \cdot x, \langle s\rangle^{-1} p\langle\gamma'\rangle \cdot x) \in M \Leftrightarrow (p\langle\gamma'\rangle \cdot x, \langle s\rangle^{-1} p\langle\gamma'\rangle \cdot x) \in M_{A,B} \Leftrightarrow (\langle\gamma'\rangle \cdot x, \langle s'\rangle^{-1}\langle\gamma'\rangle \cdot x) \in M_{A,B}$ (by 6.8). Now $p' \in \mathrm{Aut}_{\Gamma,S}$ and $p'(1) = s^{-1}p(s') = s^{-1}\big((\langle\gamma\rangle^{-1} r\langle\gamma'\rangle^{-1})\langle s'\rangle\big) = s^{-1}\big(\langle\gamma\rangle^{-1} r((\gamma')^{-1}s')\big) = s^{-1}\gamma^{-1} r(r^{-1}(\gamma)s') = s^{-1}\gamma^{-1} r(r^{-1}(\gamma s)) = s^{-1}\gamma^{-1}\gamma s = 1$, so $p' \in R$. $\dashv$

# 8 Independence numbers

Let $\Gamma$ be an infinite group and $S$ a finite set of generators. Consider the set

$$I(\Gamma, S) = \{i_\mu(S, a) \colon a \in \mathrm{FR}(\Gamma, X, \mu)\}$$

of independence numbers of actions of $\Gamma$. It was shown in Conley-Kechris [CK, §4, **(C)**] that $I(\Gamma, S)$ is a closed interval $[i_\mu(S, s_\Gamma), i_\mu(S, a^{\mathrm{erg}}_{\Gamma,\infty})]$, where $s_\Gamma$ is the shift action of $\Gamma$ on $[0,1]^\Gamma$ and $a^{\mathrm{erg}}_{\Gamma,\infty}$ is the maximum, in the sense of weak containment, free ergodic action. Let

$$I^{\mathrm{erg}}(\Gamma, S) = \{i_\mu(S, a) \colon a \in \mathrm{FR}(\Gamma, X, \mu),\ a \text{ ergodic}\}.$$

The question of understanding the nature of $I^{\mathrm{erg}}(\Gamma, S)$ was raised in Conley-Kechris [CK, §4, **(C)**]. We prove here the following result:

**Theorem 8.1** *Let $\Gamma$ be an infinite group and $S$ a finite set of generators. If $\Gamma$ has property* (T)*, then $I^{\mathrm{erg}}(\Gamma, S)$ is a closed set.*

**Proof**. Since $\Gamma$ has property (T), fix finite $Q \subseteq \Gamma$ and $\epsilon > 0$ with the following property: If $a \in A(\Gamma, X, \mu)$ and there is a Borel set $A \subseteq X$ with

$$\forall \gamma \in Q(\mu(\gamma^a \cdot A \Delta A) < \epsilon \mu(A)(1 - \mu(A))),$$

then $a$ is not ergodic (see, e.g., Kechris [Ke2, 12.6]).

Let now $\iota_n \in I^{\mathrm{erg}}(\Gamma, S), \iota_n \to \iota$, in order to show that $\iota \in I^{\mathrm{erg}}(\Gamma, S)$. Let $a_n \in \mathrm{FR}(\Gamma, X, \mu)$ be ergodic with $\iota_\mu(S, a_n) = \iota_n$. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$ and consider the action $a = \prod_n a_n/\mathcal{U}$ on $(X_\mathcal{U}, \mu_\mathcal{U})$. Then it is clear that there is no non-trivial $\Gamma$-invariant element in the measure algebra $\mathrm{MALG}_{\mu_\mathcal{U}}$. Because if $A = [(A_n)]_\mathcal{U}$ were $\Gamma$-invariant, with $\mu_\mathcal{U}(A) = \delta, 0 < \delta < 1$, then $\mu_\mathcal{U}(\gamma^a \cdot A \Delta A) = 0, \forall \gamma \in A$, so $\lim_{n \to \mathcal{U}} \mu(\gamma^{a_n} \cdot A_n \Delta A_n) = 0$ and $\mu(A_n) \to \delta$, so for some $n$, and all $\gamma \in Q, \mu(\gamma^{a_n} \cdot A_n \Delta A) < \epsilon \mu(A_n)\mu(1 - \mu(A_n))$, thus $a_n$ is not ergodic, a contradiction.

Fix also independent sets $A_n \subseteq X$ for $a_n$ with $|\mu(A_n) - \iota_n| < \frac{1}{n}$. Let $A = [(A_n)]_\mathcal{U}$. Then $A$ is independent for $a$ modulo null sets (i.e., $s^a \cdot A \cap A$ is $\mu_\mathcal{U}$-null, $\forall s \in S^{\pm 1}$) and $\mu_\mathcal{U}(A) = \iota$. Consider now the factor $b$ of $a$ corresponding to the $\sigma$-algebra $\boldsymbol{B} = \sigma(\boldsymbol{B}_0)$, where $\boldsymbol{B}_0$ is a countable Boolean subalgebra of $\mathrm{MALG}_{\mu_\mathcal{U}}$ closed under $a$, the functions $S_\mathcal{U}, T_\mathcal{U}$ of §2, **(B)**, §3, **(B)**, resp., and containing $A$. We can view $b$ as an element of $\mathrm{FR}(\Gamma, X, \mu)$. First note that $b$ is ergodic, since $\mathrm{MALG}_{\mu_\mathcal{U}}$ and thus $\boldsymbol{B}$ has no $\Gamma$-invariant non-trivial sets.

We now claim that $\iota_\mu(S,b) = \iota$, which completes the proof. Since $A \in \boldsymbol{B}$, it is clear that $\iota_\mu(S,b) \geq \mu_\mathcal{U}(A) = \iota$. So assume that $\iota_\mu(S,b) > \iota$ towards a contradiction, and let $B \in \mathrm{MALG}_{\mu_\mathcal{U}}$ be such that $s^a \cdot B \cap B = \emptyset$, $\forall s \in S^{\pm 1}$, and $\mu_\mathcal{U}(B) = \kappa > \iota$. We can assume of course that $B = [(B_n)]_\mathcal{U} \in \boldsymbol{B}^0_\mathcal{U}$, so $\lim_{n\to\mathcal{U}} \mu(B_n) = \kappa$ and $\lim_{n\to\mathcal{U}} \mu(s^{a_n} \cdot B_n \cap B_n) = 0, \forall s \in S^{\pm 1}$. Let $C_n = B_n \setminus s^{a_n} \cdot B_n$, so that $s^{a_n} \cdot C_n \cap C_n = \emptyset$ and $\mu(C_n) = \mu(B_n) - \mu(s^{a_n} \cdot B_n \cap B_n)$, thus $\lim_{n\to\mathcal{U}} \mu(C_n) = \lim_{n\to\mathcal{U}} \mu(B_n) = \kappa > \iota$. Since $\iota_n \to \iota$, for all large enough $n$, $\iota_n < \frac{\iota+\kappa}{2}$ and thus for some $U \in \mathcal{U}$, and any $n \in U, \mu(C_n) > \frac{\iota+\kappa}{2}$ but $\iota_\mu(S, a_n) = \iota_n < \frac{\iota+\kappa}{2}$. Since $C_n$ is an independent set for $a_n$, this gives a contradiction. ⊣

Similar arguments show that the set of matching numbers $m(S,a), a \in \mathrm{FR}(\Gamma, X, \mu)$, is the interval $[m(S, s_\Gamma), m(S, a^{\mathrm{erg}}_{\Gamma,\infty})]$, and the set of matching numbers of the ergodic, free actions is a closed set, if $\Gamma$ has property (T).

Finally, we have the following result, which shows that for certain groups (and sets of generators) the set $I^{\mathrm{erg}}(\Gamma, S)$ is infinite.

**Theorem 8.2** *Suppose that $\Gamma$ is a group with finite generating set $S$, and that $a, b$ are elements of $\mathrm{FR}(\Gamma, X, \mu)$ with $i_\mu(S,a) < 1/2$ and $\chi^{\mathrm{ap}}_\mu(S,b) = 2$. Then the set of independence numbers of free, ergodic actions of $\mathbb{Z} * \Gamma$ (with respect to the natural generating set $\{z\} \cup S$) intersects the interval $(i_\mu(S,a), 1/2)$ in an infinite set.*

**Proof**. Fix $\epsilon > 0$ and $n \in \mathbb{N}$. We may find disjoint $G(S,b)$-independent sets $B_0, B_1 \subseteq X$ witnessing approximate 2-colorability of $b$ (i.e., $\mu(B_i) \geq 1/2 - \epsilon$). Let $U$ be the set of automorphisms in $\mathrm{Aut}(X,\mu)$ which come within $\epsilon$ of flipping $B_0$ and $B_1$, i.e.,

$$U = \{T \in \mathrm{Aut}(X,\mu) : \mu(T(B_0) \triangle B_1) < \epsilon \text{ and } \mu(T(B_1) \triangle B_0) < \epsilon\}.$$

Clearly $U$ is open, non-empty in the weak topology of $\mathrm{Aut}(X,\mu)$. The collection of aperiodic, weakly mixing automorphisms is comeager with respect to this topology, see, e.g., [Ke2]. Also comeager is the collection of automorphisms orthogonal to $E_a \vee E_b$ (where $T$ is *orthogonal* to an equivalence relation $F$ if there is no nontrivial injective sequence of the form $x_0, T^{z_0}(x_0), x_1, T^{z_1}(x_1), \ldots, x_n, T^{z_n}(x_n) = x_0$ with each $T^{z_i}(x_i) F x_{i+1}$); see e.g., [CM]. So we may then fix an aperiodic, weakly mixing automorphism $T \in U$ which is orthogonal to $E_a \vee E_b$. Define now an action $a_{n,\epsilon}$ of $\mathbb{Z} * \Gamma$

(with generating set $\{z\} \cup S$) on $(n+1) \times X$ (with the product measure $\nu = c \times \mu$, with $c$ the normalized counting measure on $n+1$) by

$$\begin{aligned} z \cdot (i, x) &= (i + 1 \bmod n+1, T(x)), \\ \gamma \cdot (0, x) &= (0, \gamma^a(x)), \\ \gamma \cdot (i, x) &= (i, \gamma^b(x)), \text{ if } 1 \le i < n+1. \end{aligned}$$

This action is ergodic as the map $(i, x) \mapsto (i + 1 \bmod n+1, T(x))$ is ergodic by weak mixing of $T$. It is also free by the orthogonality of $T$ to $E_a \vee E_b$. We compute bounds for the independence number of the graph resulting from this action.

First consider the independent set

$$A_{n,\epsilon} = \{1, 2, \dots, n\} \times (B_0 \setminus T(B_0)).$$

We have $\nu(A_{n,\epsilon}) \ge (n/(n+1))((1/2 - \epsilon) - \epsilon) = (n/2(n+1))(1 - 4\epsilon)$.

Next suppose that $A \subseteq (n+1) \times X$ is an arbitrary independent set. By considering the graph's restriction to each $\{i\} \times X$, we certainly have $\nu(A) \le i_\mu(S, a)/(n+1) + n i_\mu(S, b)/(n+1) < 1/2$, so

$$(n/2(n+1))(1 - 4\epsilon) \le i_\nu(a_{n,\epsilon}) = i_\nu(\{z\} \cup S, a_{n,\epsilon}) < 1/2.$$

We may then recursively build sequences $(n_m)$, $(\epsilon_m)$ so that the values $i_\nu(a_{n_m,\epsilon_m})$ are strictly increasing with $m$ and in the interval $(i_\mu(S, a), 1/2)$, completing the proof. $\dashv$

Examples of $(\Gamma, S)$ for which such $a, b$ exist include all non-amenable $\Gamma$ and $S$ for which $\mathrm{Cay}(\Gamma, S)$ is bipartite; see [CK, 4.6, 4.14].

# 9 Sofic actions

**(A)** Recall that a group $G$ is *sofic* if for every finite $F \subseteq G$ and $\epsilon > 0$, there is $n \ge 1$ and $\pi \colon F \to S_n$ (= the symmetric group on $n = \{0, \dots, n-1\}$) such that (denoting by $\mathrm{id}_X$ the identity map on a set $X$):

(i) $1 \in F \Rightarrow \pi(1) = \mathrm{id}_n$,

(ii) $\gamma, \delta, \gamma\delta \in F \Rightarrow \mu_n(\{m \colon \pi(\gamma)\pi(\delta)(m) \ne \pi(\gamma\delta)(m)\}) < \epsilon$,

(iii) $\gamma \in F \setminus \{1\} \Rightarrow \mu_n(\{m \colon \pi(\gamma)(m) = m\}) < \epsilon$, where $\mu_n$ is the normalized counting measure on $n$.

Elek-Lippner [EL1] have introduced a notion of soficity for equivalence relations. We give an alternative definition due to Ozawa [O].

Let $(X,\mu)$ be a standard measure space and $E$ a measure preserving, countable Borel equivalence relation on $X$. We let

$$\begin{gathered}[[E]] = \{\varphi\colon \varphi \text{ is a Borel bijection } \varphi\colon A \to B, \\ \text{where } A, B \text{ are Borel subsets of } X \text{ and} \\ xE\varphi(x),\ \mu\text{-a.e. } (x \in A)\}.\end{gathered}$$

We identify $\varphi, \psi$ as above if their domains are equal modulo null sets and they agree a.e. on their domains. We define the *uniform metric* on $[[E]]$ by

$$\delta_X(\varphi,\psi) = \mu(\{x\colon \varphi(x) \neq \psi(x)\}),$$

where

$$\varphi(x) \neq \psi(x)$$

means that

$$x \in \mathrm{dom}(\varphi)\Delta\mathrm{dom}(\psi)$$

or

$$x \in \mathrm{dom}(\varphi) \cap \mathrm{dom}(\psi) \ \&\ \varphi(x) \neq \psi(x).$$

If $\varphi\colon A \to B$ we put $\mathrm{dom}(\varphi) = A$, $\mathrm{rng}(\varphi) = B$. If $\varphi\colon A \to B, \psi\colon C \to D$ are in $[[E]]$, we denote by $\varphi\psi$ their composition with $\mathrm{dom}(\varphi\psi) = C \cap \psi^{-1}(A \cap D)$ and $\varphi\psi(x) = \varphi(\psi(x))$ for $x \in \mathrm{dom}(\varphi\psi)$. If $(\varphi_i)_{i\in I}$, $I$ countable, is a pairwise disjoint family of elements of $[[E]]$, i.e., $\mathrm{dom}(\varphi_i), i \in I$, are pairwise disjoint and $\mathrm{rng}(\varphi_i), i \in I$, are pairwise disjoint, then $\bigsqcup_{i\in I} \varphi_i \in [[E]]$, is the union of the $\varphi_i, i \in I$. If $\varphi\colon A \to B$ is in $[[E]]$, we denote by $\varphi^{-1}\colon B \to A$ the inverse function, which is also in $[[E]]$. We also denote by $\emptyset$ the empty function. Finally, if $X = n$ and $\mu = \mu_n$ is the normalized counting measure, we let $[[n]]$ be the set of all injections between subsets of $n$ (thus $[[n]] = [[E]]$, where $E = n \times n$) and we let $\delta_n$ be the corresponding uniform (or Hamming) metric on $[[n]]$, so that $\delta_n(\varphi,\psi) = \frac{1}{n}|\{m\colon \varphi(m) \neq \psi(m)\}|$.

**Definition 9.1** *A measure preserving countable Borel equivalence relation $E$ on a non-atomic standard measure space $(X,\mu)$ is* sofic *if for each finite $F \subseteq [[E]]$ and each $\epsilon > 0$, there is $n \geq 1$ and $\pi\colon F \to [[n]]$ such that*

*i)* $\mathrm{id}_X \in F \Rightarrow \pi(\mathrm{id}_X) = \mathrm{id}_n$, $\emptyset \in F \Rightarrow \pi(\emptyset) = \emptyset$, *ii)* $\varphi, \psi, \varphi\psi \in F \Rightarrow \delta_n(\pi(\varphi\psi), \pi(\varphi)\pi(\psi)) < \epsilon$,

*iii)* $\varphi \in F \Rightarrow \left|\mu(\{x\colon \varphi(x) = x\}) - \mu_n(\{m\colon \pi(\varphi)(m) = m\})\right| < \epsilon$ .

We do not know if this definition is equivalent to the one in which $[[E]]$ is replaced by the *full group* $[E] = \{\varphi \in [[E]]\colon \mu(\mathrm{dom}(\varphi)) = 1\}$ and $[[n]]$ by $S_n$ or even if it is equivalent to the soficity of the full group.

The following two facts, brought to our attention in a seminar talk by Adrian Ioana, can be proved by routine but somewhat cumbersome calculations.

**Proposition 9.2** *There is an absolute constant $K > 1$ (e.g., $K = 10,000$ is good enough) such that the following holds:*

*Let $F, \epsilon, n, \pi$ satisfy 9.1 i)–iii) and moreover ($\theta \in F \Rightarrow \theta^{-1}, \mathrm{id}_{\mathrm{dom}(\theta)} \in F$). Let $\varphi, \psi \in F$ be such that $F$ also contains $\varphi\psi, \varphi^{-1}\psi$ and $\mathrm{id}_A$, for any $A$ in the Boolean algebra generated by the domains of $\varphi, \psi, \varphi\psi, \varphi^{-1}\psi$ and their inverses. Then $\delta_X(\varphi, \psi) < \epsilon \Rightarrow \delta_n(\pi(\varphi), \pi(\psi)) < K\epsilon$.*

**Proposition 9.3** *Let $E$ be a measure preserving countable Borel equivalence relation on a non-atomic standard measure space $(X, \mu)$. Suppose $F_0 \subseteq F_1 \subseteq \cdots \subseteq [[E]]$ are increasing finite subsets of $[[E]]$ with $\emptyset, \mathrm{id}_X \in F_0$ and, letting $\bigoplus F_m = \{\bigsqcup_{i=1}^k \varphi_i\colon \varphi_i \in F_m\}$, $\bigcup_m(\bigoplus F_m)$ is dense in $[[E]]$. Suppose that $G_m$ are finite subsets of $[[E]]$ with $F_m \subseteq G_m$ and*

1. *$\varphi, \psi \in F_m \Rightarrow \varphi\psi, \varphi^{-1}\psi \in G_m$,*

2. *if $\varphi, \psi \in F_m$, then $\mathrm{id}_A \in G_m$, for any $A$ in the Boolean algebra generated by the domains of $\varphi, \psi, \varphi\psi, \varphi^{-1}\psi$ and their inverses,*

3. *$\varphi, \psi \in F_m \Rightarrow \varphi \wedge \psi \in G_m$, where $\varphi \wedge \psi$ is the restriction of $\varphi$ (equivalently $\psi$) to $\mathrm{dom}(\varphi) \cap \mathrm{dom}(\psi) \cap \{x\colon \varphi(x) = \psi(x)\}$.*

*Finally, suppose that for every $m$ and every $\epsilon > 0$ there is an $n$ and $\pi : G_m \to [[n]]$ that satisfies the properties in the definition of soficity. Then $E$ is sofic.*

We next define sofic actions. For $(X, \mu)$ a non-atomic, standard measure space and $\Gamma$ a countable group, for each $a \in A(\Gamma, X, \mu)$, denote by $E_a$ the induced equivalence relation (defined modulo null sets)

$$xE_a y \Leftrightarrow \exists \gamma \in \Gamma(\gamma^a \cdot x = y).$$

**Definition 9.4** *An action $a \in A(\Gamma, X, \mu)$ is* sofic *if $E_a$ is sofic.*

Let now $\boldsymbol{A}_0$ be any countable Boolean subalgebra of $\mathrm{MALG}_\mu$ closed under an action $a \in \mathrm{FR}(\Gamma, X, \mu)$ and generating $\mathrm{MALG}_\mu$. Let $\Gamma = \{\gamma_n \colon n \in \mathbb{N}\}$, and let $(A_m)_{m\in\mathbb{N}}$ enumerate the elements of $\boldsymbol{A}_0$. Let $(\varphi_i^a)_{i\in\mathbb{N}}$ enumerate the family of elements of $[[E_a]]$ of the form $\gamma_n^a|A_m, n, m \in \mathbb{N}$. Then by 9.3 we have the following criterion. (Notice that if $F_m = \{\varphi_0^a, \dots, \varphi_m^a\} \cup \{\emptyset, \mathrm{id}_X\}$, then there is $G_m \subseteq \{\varphi_0^a, \varphi_1^a, \dots\}$, with $F_m, G_m, E_a$ satisfying the conditions in 9.3.)

**Proposition 9.5** *The action $a \in \mathrm{FR}(\Gamma, X, \mu)$ is sofic provided that for each $m$ and $\epsilon > 0$, 9.1 holds for $F = \{\varphi_0^a, \dots, \varphi_m^a\}$ and $\epsilon$.*

We now have the following fact.

**Proposition 9.6** *Let $(X, \mu)$ be a non-atomic standard measure space. Let $a_n \in A(\Gamma, X, \mu)$ be sofic actions and $a_n \to a$, where $a \in \mathrm{FR}(\Gamma, X, \mu)$. Then $a$ is sofic. In particular, if $a \in \mathrm{FR}(\Gamma, X, \mu), b \in A(\Gamma, X, \mu)$, $b$ is sofic and $a \prec b$, then $a$ is sofic.*

**Proof**. Fix a countable Boolean algebra $\boldsymbol{A}_0$ which generates $\mathrm{MALG}_\mu$ and is closed under $a$. Let $(\gamma_n), (A_m), (\varphi_i^a)$ be as before for the action $a$, so that $(\varphi_i^a)$ enumerates all $\gamma_n^a|A_m$. For $m, \epsilon > 0$ we want to verify 9.1 for $F = \{\varphi_0^a, \dots, \varphi_m^a\}, \epsilon > 0$. Say, for $i \leq m, \varphi_i^a = \delta_i^a|B_i$, where $\delta_i \in \Gamma, B_i \in \boldsymbol{A}_0$. Note that $\delta_i$ is uniquely determined, by the freeness of the action $a$, if $B_i \neq \emptyset$.

Fix $i \leq m$ with $\delta_i \neq 1$. Since the action $a$ is free, as in the proof of 3.3, we can write $B_i = \bigsqcup_{k=1}^\infty B_{i,k}$, where $\delta_i^a \cdot B_{i,k} \cap B_{i,k} = \emptyset$, for all $k$. Choose $n_i$ so large that $\mu(B_i \setminus \bigcup_{k=1}^{n_i} B_{i,k}) < \epsilon/4$. Since $a_n \to a$, we can find an $N_i$ so large that for all $N \geq N_i$ and all $k \leq n_i$ we have

$$|\mu(\delta_i^{a_N} \cdot B_{i,k} \cap B_{i,k}) - \mu(\delta_i^a \cdot B_{i,k} \cap B_{i,k})| < \frac{\epsilon}{4n_i}.$$

Since $\mu(\delta_i^a \cdot B_{i,k} \cap B_{i,k}) = 0$, this says that

$$\mu(\delta_i^{a_N} \cdot B_{i,k} \cap B_{i,k}) < \frac{\epsilon}{4n_i}.$$

If now $x \in B_{i,k}$ and $\delta_i^{a_N} \cdot x = x$, we have $x \in \delta_i^{a_N} \cdot B_{i,k} \cap B_{i,k}$. Thus

$$\mu(\{x \in B_i \colon \delta_i^{a_N} \cdot x = x\}) < \mu(B_i \setminus \bigcup_{k=1}^{n_i} B_{i,k}) + \sum_{k=1}^{n_i} \mu(\delta_i^{a_N} \cdot B_{i,k} \cap B_{i,k}) < \epsilon/2.$$

Choose $N$ larger than all $N_i$ $(i \leq m, \delta_i \neq 1)$ and large enough so that $\mu((B_j \cap (\delta_j^{-1})^{a_N} \cdot B_i) \Delta (B_j \cap (\delta_j^{-1})^a \cdot B_i)) < \frac{\epsilon}{2K}$, for $i, j \leq m$, and let $\psi_i = \delta_i^{a_N} | B_i, i \leq m$. Let then $F \subseteq [[E_{a_N}]]$ be such that $(\theta \in F \Rightarrow \theta^{-1}, \mathrm{id}_{\mathrm{dom}(\theta)} \in F)$ and moreover $F$ contains the maps $\psi_i, \psi_i \psi_j, \psi_i^{-1} \psi_j, i, j \leq m$, and $\mathrm{id}_A$, for any $A$ in the Boolean algebra generated by the domains of these functions and their inverses. Let then $\pi : F \to [[n]]$ satisfy 9.1 with $\frac{\epsilon}{2K}$. Put $\pi(\varphi_i^a) = \pi_N(\psi_i)$. We will show that this satisfies i)-iii) of 9.1. It is clear that i) holds.

For iii): Given $\varphi_i, 1 \leq i \leq m$, note that $\mu(\{x \colon \varphi_i^a(x) = x\}) = \mu(B_i) = \mu(\{x \colon \psi_i(x) = x\})$, if $\delta_i = 1$, and $\mu(\{x \colon \varphi_i^a(x) = x\}) = 0$, if $\delta_i \neq 1$, while in this case $\mu(\{x \colon \psi_i(x) = x\}) = \mu(\{x \in B_i \colon \delta_i^{a_N} \cdot x = x\}) < \epsilon/2$. Thus $|\mu(\{x \colon \varphi_i^a(x) = x\}) - \mu(\{x \colon \psi_i(x) = x\})| < \epsilon/2$ and so iii) holds.

For ii): Assume $i, j \leq m$ and for some $k \leq m, \varphi_i^a \varphi_j^a = \varphi_k^a$. Assume also first that $B_k \neq \emptyset$. Then

$$\begin{aligned} \varphi_i^a \varphi_j^a &= \delta_i^a \delta_j^a | \big(B_j \cap (\delta_j^{-1})^a \cdot B_i\big) \\ &= (\delta_i \delta_j)^a | \big(B_j \cap (\delta_j^{-1})^a \cdot B_i\big) \\ &= \delta_k^a | B_k, \end{aligned}$$

so $\delta_k = \delta_i \delta_j$ and $B_k = B_j \cap (\delta_j^{-1})^a \cdot B_i$. Then $\psi_i = \delta_i^{a_N} | B_i, \psi_j = \delta_j^{a_N} | B_j, \psi_i \psi_j = \delta_i^{a_N} \delta_j^{a_N} | B_j \cap (\delta_j^{-1})^{a_N} \cdot B_i, \psi_k = (\delta_i \delta_j)^{a_N} | B_j \cap (\delta_j^{-1})^a \cdot B_i$. Therefore $\delta_X(\psi_i \psi_j, \psi_k) < \frac{\epsilon}{2K}$. Then, by 9.2, $\delta_n(\pi_N(\psi_i \psi_j), \pi(\psi_k)) < \frac{\epsilon}{2}$. Therefore

$$\begin{aligned} &\delta_n(\pi(\varphi_i^a \varphi_j^a), \pi(\varphi_i^a) \pi(\varphi_j^a)) \\ &\qquad = \delta_n(\pi(\varphi_k), \pi(\varphi_i) \pi(\varphi_j)) \\ &\qquad = \delta_n(\pi_N(\psi_k), \pi_N(\psi_i) \pi_N(\psi_j)) \\ &\qquad \leq \delta_n(\pi_N(\psi_k), \pi_N(\psi_i \psi_j)) + \delta_n(\pi_N(\psi_i \psi_j), \pi_N(\psi_i) \pi_n(\psi_j)) \\ &\qquad < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon \end{aligned}$$

and the proof is complete.

In the case $B_k = \emptyset$, we consider two subcases:

(1) One of $\varphi_i^a, \varphi_j^a$ is $\emptyset$. Then one of $\psi_i, \psi_j$ is $\emptyset$ and $\psi_i \psi_j = \psi_k = \emptyset$ and thus $\delta_X(\psi_i \psi_j, \psi_k) < \frac{\epsilon}{2K}$.

(2) Both $\varphi_i^a, \varphi_j^a$ are not $\emptyset$. Then as before $\psi_i = \delta_i^{a_N} | B_i, \psi_j = \delta_j^{a_N} | B_j, \psi_i \psi_j = \delta_i^{a_N} \delta_j^{a_N} | B_j \cap (\delta_j^{-1})^{a_N} \cdot B_i$ but $\mu(B_j \cap (\delta_j^{-1})^a \cdot B_i) = 0$ and $\psi_k = \emptyset$. Since $\mu(B_j \cap (\delta_j^{-1})^{a_N} \cdot B_i) < \frac{\epsilon}{2K}$, we still have $\delta_X(\psi_i \psi_j, \psi_k) < \frac{\epsilon}{2K}$.

So in either subcase we are done as before. $\dashv$

**(B)** Consider now a sofic group $\Gamma$ and fix an increasing sequence $1 \in F_0 \subseteq F_1 \subseteq \ldots$ of finite subsets of $\Gamma$ with $\bigcup_n F_n = \Gamma$. For each $n$, let $X_n$ be a finite set of cardinality $\geq n$ with the normalized counting measure $\mu_n$ such that there is a map $\pi_n \colon F_n \to S_{X_n}$ (= the permutation group of $X_n$) such that

i) $\pi_n(1) = \mathrm{id}_{X_n}$,

ii) $\gamma, \delta, \gamma\delta \in F_n \Rightarrow \mu_n(\{x \colon \pi(\gamma)\pi(\delta)(x) \neq \pi(\gamma\delta)(x)\}) < \frac{1}{n}$,

iii) $\gamma \in F_n \setminus \{1\} \Rightarrow \mu_n(\{x \colon \pi(\gamma)(x) = x\}) < \frac{1}{n}$.

Define then $a_n \colon \Gamma \times X \to X$ by

$$a_n(\gamma, x) = \pi_n(\gamma)(x)$$

Then abbreviating $a_n(\gamma, x)$ by $\gamma \cdot_n x$ we have

i) $1 \cdot_n x = x$,

ii) $\gamma, \delta, \gamma\delta \in F_n \Rightarrow \mu_n(\{x \colon \gamma\delta \cdot_n x \neq \gamma \cdot_n (\delta \cdot_n x)\}) < \frac{1}{n}$,

iii) $\gamma \in F_n \setminus \{1\} \Rightarrow \mu_n(\{x \colon \gamma \cdot_n x = x\}) < \frac{1}{n}$.

So we can view $a_n$ as an "approximate free action" of $\Gamma$ on $X_n$.

Fix now a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and let $X_{\mathcal{U}} = (\prod_n X_n)/\mathcal{U}$ and $\mu_{\mathcal{U}}$ the corresponding measure on the $\sigma$-algebra $\boldsymbol{B}_{\mathcal{U}}$ of $X_{\mathcal{U}}$. By 2.5 this is non-atomic. As in §3, we can also define an action $a_{\mathcal{U}}$ of $\Gamma$ on $X_{\mathcal{U}}$ by

$$\gamma^{a_{\mathcal{U}}} \cdot [(x_n)]_{\mathcal{U}} = [(\gamma \cdot_n x_n)]_{\mathcal{U}}$$

(note that $\gamma \cdot_n x_n$ is well-defined for $\mathcal{U}$-almost all $n$). This action is measure preserving and, by iii) above, it is free, i.e., for $\gamma \in \Gamma \setminus \{1\}$, $\mu_{\mathcal{U}}(\{x \in X_{\mathcal{U}} \colon \gamma^{a_{\mathcal{U}}} \cdot x \neq x\}) = 0$ (see 3.2). So let $\boldsymbol{B}_0$ be a countable subalgebra of $\mathrm{MALG}_{\mu_{\mathcal{U}}}$ closed under the action $a_{\mathcal{U}}$, the function $S_{\mathcal{U}}$ of §2, **(B)** and $T_{\mathcal{U}}$ of §3, **(C)**. Let $\boldsymbol{B} = \sigma(\boldsymbol{B}_0)$ and let $b$ be the factor corresponding to $\boldsymbol{B}$. Then $b \in \mathrm{FR}(\Gamma, X, \mu)$, for a non-atomic standard measure space $(X, \mu)$.

We use this construction to give another proof of the following result:

**Theorem 9.7** (Elek-Lippner [EL1]). *Let $\Gamma$ be an infinite sofic group and let $s_\Gamma$ be the shift action of $\Gamma$ on $[0,1]^\Gamma$. Then $s_\Gamma$ is sofic.*

**Proof**. Consider the factor $b$ as in the preceding discussion. By Abért-Weiss [AW], $s_\Gamma \prec b$, thus using 9.6, it is enough to show that $b$ is sofic. Using 9.5, it is clearly enough to show the following: For any $\gamma_1, \dots, \gamma_k \in \Gamma, [(A_n^1)]_{\mathcal{U}}, \dots, [(A_n^k)]_{\mathcal{U}} \in \boldsymbol{B}_0$ and $\epsilon > 0$, letting $\varphi_i = \gamma_i^{a_{\mathcal{U}}}|[(A_n^i)]_{\mathcal{U}}$, there is $n$ and a map $\pi \colon \{\varphi_i \colon i \leq k\} \to [[X_n]]$ (the set of injections between subsets of $X_n$) such that

i) $\varphi_i = \mathrm{id}_X \Rightarrow \pi(\varphi_i) = \mathrm{id}_{X_n}$, $\varphi_i = \emptyset \Rightarrow \pi(\varphi_i) = \emptyset$,

ii) If $i, j, \ell \leq k$ and $\varphi_i\varphi_j = \varphi_\ell$, then $\mu_n(\{x \colon \pi(\varphi_i)\pi(\varphi_j)(x) \neq \pi(\varphi_\ell)(x)\}) < \epsilon$,

iii) $|\mu_{\mathcal{U}}(\{x \colon \varphi_i(x) = x\}) - \mu_n(\{x \colon \pi(\varphi_i)(x) = x\})| < \epsilon$.

Since $a_{\mathcal{U}}$ is free, note that $\varphi_i = \gamma_i^{a_{\mathcal{U}}}|[(A_n^i)]_{\mathcal{U}}$ uniquely determines $\gamma_i$, if $[(A_n^i)]_{\mathcal{U}} \neq \emptyset$. Choose now $n \in \mathbb{N}$ so that:

a) $\mu_n(\{x \colon \gamma_\ell \cdot_n x \neq \gamma_i \cdot_n (\gamma_j \cdot_n x)\}) < \frac{\epsilon}{2}$, if $\gamma_\ell = \gamma_i\gamma_j$ $(i, j, \ell \leq k)$,

b) $\mu_n(\{x \colon \gamma_i \cdot_n x = x\}) < \epsilon$, if $\gamma_i \neq 1$,

c) $\mu_n(A_n^\ell \Delta (A_n^j \cap \gamma_j^{-1} \cdot_n A_n^i)) < \frac{\epsilon}{2}$, if $\varphi_i\varphi_j = \varphi_\ell$ $(i, j, \ell \leq k)$,

d) $|\mu_{\mathcal{U}}([A_n^i]_{\mathcal{U}}) - \mu_n(A_n^i)| < \epsilon/2$ $(i \leq k)$.

Note that c) is possible since $[(A_n^\ell)]_{\mathcal{U}}$ is the domain of $\varphi_\ell$, while $[(A_n^j)]_{\mathcal{U}} \cap (\gamma_j^{-1})^{a_{\mathcal{U}}} \cdot [(A_n^i)]_{\mathcal{U}}$ is the domain of $\varphi_i\varphi_j$, thus $0 = \mu_{\mathcal{U}}\big([(A_n^\ell)]_{\mathcal{U}} \Delta ([(A_n^j)]_{\mathcal{U}} \cap (\gamma_j^{-1})^{a_{\mathcal{U}}} \cdot [(A_n^i)]_{\mathcal{U}})\big) = \lim_{n \to \mathcal{U}} \mu_n(A_n^\ell \Delta (A_n^j \cap \gamma_j^{-1} \cdot_n A_n^i))$. Now define

1) $\pi(\varphi_i) = \mathrm{id}_{X_n}$, if $\varphi_i = \mathrm{id}_X$; $\pi(\varphi_i) = \emptyset$, if $\varphi_i = \emptyset$,

2) $\pi(\varphi_i) = \gamma_i^{a_n}|A_n^i$, otherwise,

where as usual $\gamma_i^{a_n}(x) = a_n(\gamma_i, x)$. We claim that this works. Clearly i) is satisfied. Also iii) is satisfied. Indeed if $\gamma_i \neq 1$, $\mu_{\mathcal{U}}(\{x \colon \varphi_i(x) = x\}) = 0$ and $\mu_n(\{x \colon \pi(\varphi_i)(x) = x\}) \leq \mu_n(\{x \colon \gamma_i \cdot_n x = x\}) < \epsilon$. If $\gamma_i = 1$, then $\mu_{\mathcal{U}}(\{x \colon \varphi_i(x) = x\}) = \mu_{\mathcal{U}}([A_n^i]_{\mathcal{U}})$ and $\mu_n(\{x \colon \pi(\varphi_i)(x) = x\}) = \mu_n(A_n^i)$, so $|\mu_{\mathcal{U}}(\{x \colon \varphi_i(x) = x\}) - \mu_n(\{x \colon \pi(\varphi_i)(x) = x\}| < \epsilon$. Finally for ii), assume $\varphi_i\varphi_j = \varphi_\ell$ $(i, j, \ell \leq k)$. Consider first the case when $\varphi_\ell \neq \emptyset$ (and thus $\varphi_i, \varphi_j$ are not $\emptyset$). Then $\gamma_i\gamma_j = \gamma_\ell$ and so

$$\mu_n(\{x \colon \gamma_\ell \cdot_n x \neq \gamma_i \cdot_n (\gamma_j \cdot_n x)\}) < \frac{\epsilon}{2},$$

thus

$$\mu_n(\{x\colon \pi(\varphi_\ell)(x) \neq \pi(\varphi_i)\pi(\varphi_j)(x)\}) \leq$$
$$\mu_n(A_n^\ell \Delta(A_n^j \cap \gamma_j^{-1} \cdot_n A_n^i)) + \mu_n(\{x\colon \gamma_\ell \cdot_n x \neq \gamma_i \cdot_n (\gamma_j \cdot_n x)\}) < \epsilon.$$

The case when $\varphi_\ell = \emptyset$ can be handled as in the proof of ii) in 9.6 (case $B_k = \emptyset$). $\dashv$

**(C)** It is a well-known problem whether every countable group is sofic. Elek-Lippner [EL1] also raised the question of whether every measure preserving, countable Borel equivalence relation on a standard measure space is sofic. They also ask the question of whether *every* free action $a \in \mathrm{FR}(\Gamma, X, \mu)$ of a sofic group $\Gamma$ is sofic. They show that all treeable equivalence relations are sofic and thus every strongly treeable group (i.e., one for which all free actions are treeable) has the property that all its free actions are sofic. These groups include the amenable and the free groups. Another class of groups with all free actions sofic is the class MD discussed in Kechris [Ke3]. A group $\Gamma$ is in MD if it is residually finite and its finite actions (i.e., actions that factor through an action of a finite group) are dense in $A(\Gamma, X, \mu)$. These include residually finite amenable groups, free groups, and (Bowen) surface groups, and lattices in $\mathrm{SO}(3, 1)$. Moreover MD is closed under subgroups and finite index extensions.

To see that every free action of a group in MD is sofic, note that by Kechris [Ke3, 4.8] if $a \in \mathrm{FR}(\Gamma, X, \mu)$, then $a \prec \iota_\Gamma \times p_\Gamma$, where $\iota_\Gamma$ is the trivial action of $\Gamma$ on $(X, \mu)$ and $p_\Gamma$ the translation action of $\Gamma$ on its profinite completion on $\hat{\Gamma}$. It is easy to check that $\iota_\Gamma \times p_\Gamma$ is sofic and thus $a$ is sofic by 9.6. (Alternatively we can use 9.6 and the fact that every finite action is sofic.)

We note that the fact that every free group $\Gamma$ has MD and thus every free action of $\Gamma$ is sofic can be used to give an alternative proof of the result of Elek-Lippner [EL1] that every measure preserving, treeable equivalence relation is sofic. Indeed it is a known fact that if $E$ is such an equivalence relation on $(X, \mu)$, then there is $a \in \mathrm{FR}(\mathbb{F}_\infty, X, \mu)$ such that $E \subseteq E_a$. This follows for example by the method of proof of Conley-Miller [CM, Prop. 8] or by using [CM, Prop 9], that shows that $E \subseteq F$ where $F$ is treeable of infinite cost, and then using Hjorth's result (see [KM, 28.5]) that $F$ is induced by a free action of $\mathbb{F}_\infty$. Since $E_a$ is sofic and $[[E]] \subseteq [[E_a]]$, it immediately follows that $E$ is sofic.

We do not know if every measure preserving treeable equivalence relation $E$ is contained in some $E_a$, where $a \in \mathrm{FR}(\mathbb{F}_2, X, \mu)$.

**Remark.** For arbitrary amenable groups $\Gamma$, one can use an appropriate Følner sequence to construct a free action $a_{\mathcal{U}}$ on an ultrapower of finite sets as in §9, **(B)**. Then using an argument as in Kamae [Ka], one can see that *every* measure preserving action of $\Gamma$ is a factor of this ultrapower (and thus as in 9.7 again every such action is sofic).

# 10 Concluding remarks

There are sometimes alternative approaches to proving some of the results in this paper using weak limits in appropriate spaces of measures instead of ultrapowers.

One approach is to replace the space of actions $A(\Gamma, X, \mu)$ by a space of invariant measures for the shift action of $\Gamma$ on $[0,1]^\Gamma$ as in Glasner-King [GK].

Let $(X, \mu)$ be a non-atomic, standard measure space. Without loss of generality, we can assume that $X = [0,1], \mu = \lambda =$ Lebesgue measure on [0,1]. Denote by $\mathrm{SIM}_\mu(\Gamma)$ the compact (in the weak*-topology) convex set of probability Borel measures $\nu$ on $[0,1]^\Gamma$ which are invariant under the shift action $s_\Gamma$, such that the marginal $(\pi_1)_*\nu = \mu$ (where $\pi_1 \colon [0,1]^\Gamma \to [0,1]$ is defined by $\pi_1(x) = x(1)$). For $a \in A(\Gamma, X, \mu)$ let $\varphi^a \colon [0,1] \to [0,1]^\Gamma$ be the map $\varphi^a(x)(\gamma) = (\gamma^{-1})^a \cdot x$, and let $(\varphi^a)_*\mu = \mu_a \in \mathrm{SIM}_\mu(\Gamma)$. Then $\Phi(a) = \mu_a$ is a homeomorphism of $A(\Gamma, X, \mu)$ with a dense, $G_\delta$ subset of $\mathrm{SIM}_\mu(\Gamma)$ (see [GK]).

One can use this representation of actions to give another proof of Corollary 4.5.

If $a_n \in A(\Gamma, X, \mu), n \in \mathbb{N}$, is given, consider $\mu_n = \mu_{a_n} \in \mathrm{SIM}_\mu(\Gamma)$ as above. Then there is a subsequence $n_0 < n_1 < n_2 < \dots$ such that $\mu_{n_i} \to \mu_\infty \in \mathrm{SIM}_\mu(\Gamma)$ (convergence is in the weak*-topology of measures). Then $\mu_\infty$ is non-atomic, so we can find $a_\infty \in A(\Gamma, X, \mu)$ such that $a_\infty$ on $(X, \mu)$ is isomorphic to $s_\Gamma$ on $([0,1]^\Gamma, \mu_\infty)$. One can then show (as in the proof of (1) $\Rightarrow$ (3) in 4.3) that there are $b_{n_i} \cong a_{n_i}, b_{n_i} \in A(\Gamma, X, \mu)$ such that $b_{n_i} \to a_\infty$. (Similarly if we let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$ and $\mu^{\mathcal{U}} = \lim_{n \to \mathcal{U}} \mu_n$ and $a^{\mathcal{U}}$ in $A(\Gamma, X, \mu)$ is isomorphic to $s_\Gamma$ on $([0,1]^\Gamma, \mu^{\mathcal{U}})$, then there are $b_n \in A(\Gamma, X, \mu), b_n \cong a_n$ with $\lim_{n \to \mathcal{U}} b_n = a^{\mathcal{U}}$.)

For other results, related to graph combinatorics, one needs to work with shift-invariant measures on other spaces. Let $\Gamma$ be an infinite group with a finite set of generators $S$. We have already introduced in §6 the compact space $\mathrm{Col}(k,\Gamma,S)$ of $k$-colorings of $\mathrm{Cay}(\Gamma,S)$ and in §7 the compact space $M(\Gamma,S)$ of perfect matchings of $\mathrm{Cay}(\Gamma,S)$. On each one of these we have a canonical shift action of $\Gamma$ and we denote by $\mathrm{INV}_{\mathrm{Col}}(\Gamma,S)$, $\mathrm{INV}_M(\Gamma,S)$ the corresponding compact spaces of invariant, Borel probability measures (i.e., the spaces of invariant, random $k$-colorings and invariant, random perfect matchings, resp.). Similarly, identifying elements of $2^\Gamma$ with subsets of $\Gamma$, we can form the space $\mathrm{Ind}(\Gamma,S)$ of all independent in $\mathrm{Cay}(\Gamma,S)$ subsets of $\Gamma$. This is again a closed subspace of $2^\Gamma$ which is shift invariant and we denote by $\mathrm{INV}_{\mathrm{Ind}}(\Gamma,S)$ the compact space of invariant, Borel measures on $\mathrm{Ind}(\Gamma,S)$, which we can call *invariant, random independent sets*.

If $a \in \mathrm{FR}(\Gamma,X,\mu)$ and $A \subseteq X$ is a Borel independent set for $G(S,a)$, then we define the map

$$I_A\colon X \to \mathrm{Ind}(\Gamma,S),$$

given by

$$\gamma \in I_A(x) \Leftrightarrow (\gamma^{-1})^a \cdot x \in A.$$

This preserves the $\Gamma$-actions, so $(I_A)_*\mu = \nu \in \mathrm{INV}_{\mathrm{Ind}}(\Gamma,S)$. Moreover $\nu(\{B \in \mathrm{Ind}(\Gamma,S)\colon 1 \in B\}) = \mu(A)$. If $i_\mu(S,a) = \iota$ and $A_n \subseteq X$ are Borel independent sets with $\mu(A_n) \to \iota$, let $\nu_n = (I_{A_n})_*\mu$. Then the shift action $a_n$ on $(\mathrm{Ind}(\Gamma,S),\mu_n)$ may not be free but one can still define independent sets for this action as being those $C$ such that $s^{a_n} \cdot C \cap C = \emptyset$ (modulo null sets) and also the independence number $\iota_{\nu_n}(s,a_n)$ as before. We can also assume, by going to a subsequence, that $\nu_n \to \nu_\infty$. Denote by $a_\infty$ the shift action for $(\mathrm{Ind}(\Gamma,S),\nu_\infty)$. Then $\{B \in \mathrm{Ind}(\Gamma,S)\colon 1 \in B\}$ is independent for $a_n$ and $a_\infty$, so $\iota_{\nu_\infty}(S,a_n) \geq \iota$. But also $\iota_{\nu_n}(S,a_n) \leq \iota_\mu(S,a)$ and from this, it follows by a simple approximation argument that $\iota_{\nu_\infty}(S,a_\infty) \leq \iota$, so $\iota_{\nu_\infty}(S,a_\infty) = \iota$ and the sup is attained. This gives a weaker version of 5.2 (iii). Although one can check that $a_\infty \prec a$, it is not clear that $a_\infty$ is free and moreover we do not necessarily have that $a \sqsubseteq a_\infty$. This would be remedied if we could replace $a_\infty$ by $a_\infty \times a$, but it is not clear what the independence number of this product is. This leads to the following question: Let $a, b \in \mathrm{FR}(\Gamma,X,\mu)$ and consider $a \times b \in \mathrm{FR}(\Gamma,X^2,\mu^2)$. It is clear that $\iota_{\mu^2}(a \times b) \geq \max\{\iota_\mu(a),\iota_\mu(b)\}$. Do we have equality here?

Similar arguments can be given to prove weaker versions of 5.2 (iii), (iv).

However a weak limit argument as above (but for the space of colorings) can give an alternative proof of 6.4 using the "approximate" version of Brooks' Theorem in Conley-Kechris [CK] (this was pointed out to us by Lyons). Indeed let $a \in \mathrm{FR}(\Gamma, S, \mu), d = |S^{\pm 1}|$. By Conley-Kechris [CK, 2.9] and Kechris-Solecki-Todorcevic [KST, 4.8], there is $k > d$ and for each $n$, a Borel coloring $c_n \colon X \to \{1, \dots, k\}$ such that $\mu(c_n^{-1}(\{d+1, \dots, k\})) < \frac{1}{n}$. Let as usual $C_n \colon X \to \mathrm{Col}(k, \Gamma, S)$ be defined by $C_n(x)(\gamma) = c_n((\gamma^{-1})^a \cdot x)$. Let $(C_n)_*\mu = \nu_n$. Then $\nu_n(\{c \in \mathrm{Col}(k, \Gamma, S) \colon c(1) > d\}) = \mu(C_n^{-1}(\{d + 1, \dots, k\})) < \frac{1}{n}$. By going to a subsequence we can assume that $\nu_n \to \nu$, an invariant, random $k$-coloring. Now $\nu(\{c \in \mathrm{Col}(k, \Gamma, S) \colon c(1) > d\}) = 0$, thus $\nu$ concentrates on $\mathrm{Col}(d, \Gamma, S)$ and thus is an invariant, random $d$-coloring. Moreover it is not hard to check that it is weakly contained in $a$.

A similar argument can be used to show that for every $\Gamma, S$ except possibly non-amenable $\Gamma$ with $S$ consisting of elements of odd order, there is an invariant, random perfect matching (see 7.5).

Finally one can obtain by using weak limits in $\mathrm{INV}_{\mathrm{Ind}}(\Gamma, S)$ and the result in Glasner-Weiss [GW], that if $\Gamma$ has property (T) and $c_n \in I^{\mathrm{erg}}(\Gamma, S)$, $\iota_{\mu_n}(\Gamma, S) \to \iota$, then there is a measure $\nu \in \mathrm{INV}_{\mathrm{Ind}}(\Gamma, S)$ such that the shift action is ergodic relative to $\nu$ and has independence number equal to $\iota$, but it is not clear that this action is free.

# References


[ACLT] M. Abért, E. Csoka, G. Lippner and T. Terpai, On perfect matchings in infinite Cayley graphs, *in preparation*.

[AE] M. Abért and G. Elek, The space of actions, partition metric and combinatorial rigidity, *arXiv*:1108.2147v1.

[AW] M. Abért and B. Weiss, Bernoulli actions are weakly contained in any free action, *arXiv*:1103.1063v2.

[AL] D. Aldous and R. Lyons, Processes on unimodular random networks, *Elect. J. Prob.*, **12** (2007), 1454-1508.

[CK] C.T. Conley and A.S. Kechris, Measurable chromatic and independence numbers for ergodic graphs and group actions, *Group, Geometry and Dynamics*, to appear (available at www.math.caltech.edu/people/kechris.html).

[CM] C.T. Conley and B.D. Miller, An antibasis result for graphs of infinite Borel chromatic number, *preprint*, 2010 (available at: www.logic.univie.ac.at/∼conleyc8).

[EW] M. Einsiedler and T. Ward, *Ergodic theory with a view towards number theory*, Graduate Texts in Math., **259**. Springer, 2011.

[EL1] G. Elek and G. Lippner, Sofic equivalence relations, *J. Funct. Anal.*, **258** (2010), 1692–1708.

[EL2] G. Elek and G. Lippner, Borel oracles. An analytical approach to constant-time algorithms, *Proc. Amer. Math. Soc.*, **138 (8)** (2010), 2939–2947.

[ES] G. Elek and B. Szegedy, Limits of hypergaphs, removal and regularity lemmas. A non-standard approach, *arXiv*:0705.2179v1.

[Ey] P. Eymard, *Moyennes invariantes et represéntations unitaires*, Lecture Notes in Math., **300**, Springer-Verlag, 1972.

[GK] E. Glasner and J. King, A zero-one law for dynamical properties, *Contemp. Math.*, **215** (1998), 231–242.

[GW] E. Glasner and B. Weiss, Kazhdan's property T and the geometry of the collection of invariant measures, *Geom. Funct. Anal.*, **7** (1997), 917–935.

[Gr] F.P. Greenleaf, Amenable actions of locally compact groups, *J. Funct. Anal.*, **4** (1969), 295–315.

[Ka] T. Kamae, A simple proof of the ergodic theorem using nonstandard analysis, *Israel J. Math.*, **42(4)** (1982), 284–290.

[Ke1] A.S. Kechris, *Classical Descriptive Set Theory*, Springer, 1995.

[Ke2] A.S. Kechris, *Global Aspects of Ergodic Group Actions*, Amer. Math. Soc., 2010.

[Ke3] A.S. Kechris, Weak containment in the space of actions of a free group, *Israel J. Math.*, to appear (available at http://www.math.caltech.edu/people/kechris.html).

[KM] A.S. Kechris and B.D. Miller, *Topics in Orbit Equivalence*, Springer, 2004.

[KST] A.S. Kechris, S. Solecki, and S. Todorcevic, Borel chromatic numbers, *Adv. Math.*, **141(1)** (1999), 1–44.

[LN] R. Lyons and F. Nazarov, Perfect matchings as IID factors on non-amenable groups, *Europ. J. Comb.*, **32** (2011), 1115–1125.

[OW] D. Ornstein and B. Weiss, Entropy and isomorphism theorems for actions of amenable groups, *J. Anal. Math.*, **48** (1987), 1–14.

[O] N. Ozawa, Hyperlinearity, sofic groups and applications to group theory, *handwritten note*, 2009 (available at www.ms.u-tokyo.ac.jp/∼narutaka/publications.html).

[P] V. Pestov, Hyperlinear and sofic groups: a brief guide, *Bull. Symb. Logic.*, **14** (2008), 449–480.

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