AN ANTIBASIS RESULT FOR GRAPHS OF INFINITE BOREL CHROMATIC NUMBER

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Abstract. We answer in the negative a question posed by Kechris-Solecki-Todorcevic as to whether the shift graph on Baire space is minimal among graphs of indecomposably infinite Borel chromatic number. To do so, we use ergodic-theoretic techniques to construct a new graph amalgamating various properties of the shift actions of free groups. The resulting graph is incomparable with any graph induced by a function. We then generalize this construction and collect some of its useful properties.

1. Introduction

Recall that the *chromatic number*, written $\chi(G)$, of a (possibly infinite) graph $G = (V, E)$ with vertex set V and edge set E is defined to be the smallest cardinal κ for which there is a *coloring function* $c : V \to \kappa$ such that $c(x) \neq c(y)$ whenever $x \mathrel{E} y$. In the traditional set-theoretic framework of ZFC this notion is quite well studied, and invoking the Axiom of Choice in the form of a compactness argument establishes that such a graph has finite chromatic number if and only if there is a finite bound on the chromatic numbers of its finite induced subgraphs. In the absence of relatively strong choice principles, however, this equivalence fails drastically. For example, it is consistent with ZF+DC (assuming as usual the consistency of ZFC) that there is a graph G all of whose finite induced subgraphs have chromatic number at most two while $\chi(G) = 2^{\aleph_0}$, the cardinality of the continuum (an example, G_0, is discussed below).

One can clarify the dependency of these arguments on the Axiom of Choice, and shift the question into the descriptive set-theoretic milieu, by placing definability restrictions on the coloring functions. More precisely, one can fix a standard Borel space X and examine graphs G whose edge relation is realized as a symmetric, irreflexive subset of X^2. One may then consider the *Borel chromatic number*, defined as the least cardinality of a standard Borel space Y for which there is a Borel coloring function $c : X \to Y$ of G, with the usual requirement that G-adjacent vertices get different colors. This number was investigated by Kechris-Solecki-Todorcevic, who noticed that there are acyclic Borel graphs which have no Borel $\aleph_0$-coloring. Indeed, in [6, §6] such an example G_0 on $2^{\mathbb{N}}$ is produced with the following remarkable minimality property: an analytic graph G on a Hausdorff space X has uncountable Borel chromatic number iff there is a continuous homomorphism from G_0 to G. Moreover, in a model of ZF+DC in which all subsets of the reals have the property of Baire (see, e.g., [9, Theorem 7.16]), we have $\chi(G_0) = 2^{\aleph_0}$.

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There are many consequences of the existence of such a minimal object. It reduces the inherent complexity of the statement "G cannot be colored with countably many colors," since it is equivalent to the existential statement "there is a continuous homomorphism from G_0 to G." Moreover, recent work by the second author has isolated several classical descriptive set-theoretic dichotomies as consequences of precisely this form of graph minimality, suggesting that these graph coloring concerns are sufficiently flexible to encode seemingly unrelated set-theoretic notions.

Kechris-Solecki-Todorcevic have asked whether there is an analogous minimal graph among the collection of graphs with infinite Borel chromatic number. More precisely, they posed the question of whether the graph G_s on infinite sets of natural numbers, where two such sets are adjacent if one can be obtained from the other by removing its least element, is such a minimal graph [6, Question 8.1]. In addition to the benefits listed above, the existence of such a minimal graph would provide a nice analog to the ease of testing in `ZFC` whether a graph has infinite chromatic number.

In §2, we exhibit an acyclic, locally finite, Borel graph $G_{\mathbb{F}}$ of infinite Borel chromatic number with the property that there is no Borel homomorphism from any graph associated with a Borel function to $G_{\mathbb{F}}$. In particular, this gives a negative answer to [6, Question 8.1]. The graph $G_{\mathbb{F}}$ can be viewed as an amalgamation of graphs associated with shift actions of free groups. In §3, we discuss a more general way of amalgamating acyclic graphs into larger graphs, and catalog some useful properties of this amalgamation.

2. A graph incomparable with the shift

A *graph* on a set X is a symmetric, irreflexive set $G \subseteq X \times X$. The *degree* of a point x with respect to G is given by $\deg_G(x) = |\{y \in X : (x, y) \in G\}|$. The graph G is *locally countable* if every point has countable degree, and the graph G is *locally finite* if every point has finite degree. The *restriction* of G to a set $A \subseteq X$ is given by $G \mid A = G \cap (A \times A)$. We say that A is (G)-*independent* if $G \mid A = \emptyset$.

A κ-*coloring* of G is a function $c : X \to \kappa$ such that $c^{-1}(\{\alpha\})$ is independent for each $\alpha \in \kappa$. The *chromatic number* of G, $\chi(G)$, is the least cardinal κ for which there exists a κ-coloring of G. Analogously, the *Borel chromatic number*, $\chi_B(G)$, of a graph on a standard Borel space X is the least cardinality of a standard Borel space Y for which there is a Borel function $c : X \to Y$ with $c^{-1}(\{y\})$ independent for each $y \in Y$.

For a graph G, let E_G denote the equivalence relation generated by G. The classes of E_G are called the *connected components* of G, and G is *connected* if E_G has only one class. We say that G has *indecomposably infinite Borel chromatic number* if X cannot be partitioned into countably many Borel E_G-invariant sets on which G has finite Borel chromatic number.

We identify the space $[\mathbb{N}]^{\mathbb{N}}$ of infinite subsets of the natural numbers with the collection of strictly increasing sequences of natural numbers. The *unilateral shift* on $[\mathbb{N}]^{\mathbb{N}}$ is the function $s : [\mathbb{N}]^{\mathbb{N}} \to [\mathbb{N}]^{\mathbb{N}}$ given by

$$s(x)(i) = x(i + 1),$$

for all $x \in [\mathbb{N}]^{\mathbb{N}}$ and $i \in \mathbb{N}$. Let G_s denote the graph on $[\mathbb{N}]^{\mathbb{N}}$ given by

$$x \; G_s \; y \Leftrightarrow x = s(y) \text{ or } y = s(x).$$

Thus, G_s is an acyclic, locally finite, Borel graph on $[\mathbb{N}]^{\mathbb{N}}$. By a straightforward application of the Galvin-Prikry theorem, it is shown in [6, Example 3.2] that $\chi_B(G_s) = \aleph_0$, and it is therein conjectured that G_s is in some sense minimal among the collection of graphs of infinite Borel chromatic number.

If Γ is a countable group with generating set S (assumed not to contain the identity) and a is a free, Borel action of Γ on X, we define the graph $G(S,a)$ on X by

$$x\ G(S,a)\ y \Leftrightarrow \exists s \in S\ (x = s \cdot^a y \text{ or } y = s \cdot^a x).$$

To analyze such graphs, it will be useful to work with a measure. If G is a Borel graph on a standard Borel space X and μ is a Borel probability measure on X, the (μ-)*measurable chromatic number* of G, $\chi_\mu(G)$, is the least cardinality of a Polish space Y for which there is a (μ-)measurable function $c : X \to Y$ with $c^{-1}(y)$ a G-independent set for each $y \in Y$. To avoid trivial degeneracies, we assume from now on that (X,μ) is a standard probability space. We see immediately that $\chi_\mu(G) \leq \chi_B(G)$.

We denote by $\mathbb{F}_n$ $(n \geq 2)$ the free group on n generators, and by $\mathbb{F}_\infty$ the free group on $\aleph_0$-many generators. Fixing a set of free generators $\{\gamma_0, \gamma_1, \ldots\}$ for $\mathbb{F}_\infty$, we may canonically identify $\mathbb{F}_n$ with the subgroup of $\mathbb{F}_\infty$ generated by $\{\gamma_1, \ldots, \gamma_n\}$ (note that γ_0 is unused). Equip $2^{\mathbb{F}_\infty}$ with the $(1/2, 1/2)$ product measure μ_0, and denote by a_∞ the shift action of $\mathbb{F}_\infty$ on (X_0, μ_0), where $X_0 \subseteq 2^{\mathbb{F}_\infty}$ is the conull set on which this shift action is free. For each $n \geq 2$, let a_n denote the (free) action of $\mathbb{F}_n$ on (X_0,μ_0) obtained by restricting a_∞ to $\langle F_n \rangle$. Finally, let $G_n = G(F_n, a_n) \subseteq G(F_\infty, a_\infty)$, so that each G_n is an acyclic, Borel graph with respect to which every point has degree $2n$. Moreover, each E_{G_n} is ergodic since the shift action of $\mathbb{F}_\infty$ is mixing [5, Example 3.1].

From [6, Proposition 4.6] it follows that $\chi_B(G_n) \leq 2n+1$, and we will next sketch the argument that $\chi_{\mu_0}(G_n) \to \infty$ as $n \to \infty$. Let b_n denote the free part of the shift of $\mathbb{F}_n$ on $2^{\mathbb{F}_n}$, equipped with the usual product measure. Each a_n is isomorphic to the product $(b_n)^{\mathbb{N}}$ acting on $(2^{\mathbb{F}_n})^{\mathbb{N}}$. In turn, $(b_n)^{\mathbb{N}}$ is isomorphic to the shift action of $\mathbb{F}_n$ on $(2^{\mathbb{N}})^{\mathbb{F}_n}$. Bowen [1] has shown that this last shift action is weakly equivalent to b_n itself (for a definition and discussion of weak equivalence, see [4, Chapter 10]). In [2] and independently [7] it is shown that the graph associated with the action b_n (and a free set of generators) has an upper bound on the measure of an independent set tending towards zero as $n \to \infty$. Since this bound respects isomorphism and weak equivalence, we see by the above observations that $\chi_{\mu_0}(G_n) \to \infty$ as $n \to \infty$. More details of this argument may be found in [2, Corollary 4.2, Theorem 4.17, Remark 4.18].

We amalgamate these graphs into a single graph $G_{\mathbb{F}}$ on $X_{\mathbb{F}} = X_0 \times 2^{\mathbb{N}}$ (with the product measure $\mu_{\mathbb{F}} = \mu_0 \times (1/2, 1/2)^{\mathbb{N}}$) with countably infinite Borel and measurable chromatic numbers. Partition $2^{\mathbb{N}}$ into countably many Borel sets $A_2, A_3, \ldots$ of positive measure. Fix an aperiodic, ergodic, measure-preserving, Borel automorphism $\sigma : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$, and define $T : X_{\mathbb{F}} \to X_{\mathbb{F}}$ by $T(x,y) = (\gamma_0 \cdot x, \sigma(y))$. Finally, define $G_{\mathbb{F}}$ on $X_{\mathbb{F}}$ by

$$(x,y)\ G_{\mathbb{F}}\ (x',y') \Leftrightarrow \exists n \geq 2\ (xG_nx' \text{ and } y = y' \in A_n) \text{ or } (x,y) = T^{\pm 1}(x',y').$$

Here we are pasting continuum-many copies of G_n into each A_n, and using T to tie them together. We see that $G_{\mathbb{F}}$ is an acyclic, locally finite, Borel graph with $\chi_B(G_{\mathbb{F}}) = \chi_\mu(G_{\mathbb{F}}) = \aleph_0$. Moreover, $E_{G_{\mathbb{F}}}$ is an ergodic equivalence relation, so there is no way to partition $X_{\mathbb{F}}$ into countably many $E_{G_{\mathbb{F}}}$-invariant pieces on which $G_{\mathbb{F}}$ has finite $\mu_{\mathbb{F}}$-measurable chromatic number.

Of course, the same idea works in general to sew countably many acyclic graphs together using an ergodic automorphism, and acyclicity is preserved provided this automorphism is sufficiently "free" from the original graphs. We examine this idea in more detail in §3.

Recall that a *homomorphism* from a graph G on X to a graph H on Y is a function $\varphi : X \to Y$ such that

$$x \; G \; x' \Rightarrow \varphi(x) H \varphi(x').$$

With any function $f : X \to X$ we may associate the graph G_f on X given by

$$x \; G_f \; y \Leftrightarrow x \neq y \text{ and } (x = f(y) \text{ or } y = f(x)).$$

We abbreviate E_{G_f} by E_f.

Proposition 1. *Suppose that $f : X \to X$ is a Borel function. Then there is no Borel homomorphism from $G_{\mathbb{F}}$ to G_f.*

Proof. Suppose, towards a contradiction, that $\varphi : X_{\mathbb{F}} \to X$ is a Borel homomorphism from $G_{\mathbb{F}}$ to G_f. Denote by $(\mu_{\mathbb{F}})_*$ the push-forward of $\mu_{\mathbb{F}}$ by φ, i.e., the measure given by

$$(\mu_{\mathbb{F}})_*(A) = \mu_{\mathbb{F}}(\varphi^{-1}(A)),$$

for all Borel $A \subseteq X$. By [8, Theorem A] there is a $(\mu_{\mathbb{F}})_*$-measurable 3-coloring $c : X \to 3$. But then $c \circ \varphi$ is a $\mu_{\mathbb{F}}$-measurable 3-coloring of $G_{\mathbb{F}}$, the desired contradiction. □

Remark 2. The above argument in fact also rules out a Borel homomorphism from $G_{\mathbb{F}}$ to any acyclic Borel graph G whose associated equivalence relation E_G is measure hyperfinite.

Proposition 3. *Suppose that $f : X \to X$ is a Borel function. If there is a Borel homomorphism from G_f to $G_{\mathbb{F}}$, then $\chi_B(G_f) \leq 3$.*

Proof. Suppose that $\varphi : X \to X_{\mathbb{F}}$ is a Borel homomorphism from G_f to $G_{\mathbb{F}}$. We use φ to pull a particular partition of the edges of $G_{\mathbb{F}}$ back to G_f. Define subgraphs H_T, H of G_f by

$$\begin{aligned} x \; H_T \; y &\Leftrightarrow x \; G_f \; y \text{ and } \varphi(x) = T^{\pm 1}(\varphi(y)), \text{ and} \\ x \; H \; y &\Leftrightarrow x \; G_f \; y \text{ and } \operatorname{proj}_{2^{\mathbb{N}}}(\varphi(x)) = \operatorname{proj}_{2^{\mathbb{N}}}(\varphi(y)), \end{aligned}$$

where $\operatorname{proj}_{2^{\mathbb{N}}}$ denotes projection onto $2^{\mathbb{N}}$. Observe that $G_f = H_T \sqcup H$. Moreover, since T is a Borel automorphism, it follows from [6, Proposition 4.7] that H_T has Borel chromatic number at most 3. It therefore suffices to argue that $\chi_B(H)$ is finite, since $\chi_B(G_f) \leq 3\chi_B(H)$, thus [6, Theorem 5.1] would imply that $\chi_B(G_f) \leq 3$. We will in fact show that $\chi_B(H) \leq 3$ as well.

Lemma 4. *Suppose that G is a Borel graph on X with $G \subseteq G_f$. Then there is a Borel function $g : X \to X$ with $G = G_g$. In particular, if $\chi_B(G)$ is finite, then $\chi_B(G) \leq 3$.*

Proof. Note that each connected component of G contains at most one point x such that x and $f(x)$ are not G-related. Simply define $g : X \to X$ by

$$g(x) = \begin{cases} f(x) \text{ if } x \mathrel{G} f(x), \text{ and} \\ x \text{ otherwise,} \end{cases}$$

so that $G = G_g$. If also $\chi_B(G_g)$ is finite we have $\chi_B(G_g) \leq 3$ (see [6, Theorem 5.1]). □

For $n \geq 2$, put $B_n = \varphi^{-1}(X_0 \times A_n)$, where $A_n \subseteq 2^{\mathbb{N}}$ is as in the definition of $G_{\mathbb{F}}$. Each B_n is a union of connected components of H (since the only edges in $G_{\mathbb{F}}$ connecting distinct $X_0 \times A_n$, $X_0 \times A_m$ are those induced by T). Moreover, for each $n \geq 2$, the Borel chromatic number of $H \mid B_n$ is finite, since $(\varphi \times \varphi)[H] \mid (X_0 \times A_n)$ has degree bounded by $2n$ (again applying [6, Proposition 4.6]). By then applying Lemma 4 to H on each B_n, we see that $\chi_B(H) \leq 3$ as promised. □

Corollary 5. *There is neither a Borel homomorphism from G_s to $G_{\mathbb{F}}$ nor a Borel homomorphism from $G_{\mathbb{F}}$ to G_s.*

We now shift our attention to the existence of graphs below both G_s and $G_{\mathbb{F}}$. While it remains unknown whether there exists a graph of indecomposably infinite Borel chromatic number which homomorphs into both, we can rule out such graphs under additional injectivity assumptions on one of these homomorphisms. For graphs G on X and H on Y, we say that $\varphi : X \to Y$ is *locally injective* if for all $w \in X$ and all G-neighbors x_0, x_1 of w, $\varphi(x_0) = \varphi(x_1) \Rightarrow x_0 = x_1$.

Corollary 6. *Suppose that $f : X \to X$ is a Borel function. Then there is no Borel graph G of infinite Borel chromatic number which admits a locally injective Borel homomorphism to G_f and a Borel homomorphism to $G_{\mathbb{F}}$.*

Proof. Suppose, towards a contradiction, that there is such a graph G on X. Fix a locally injective Borel homomorphism φ from G to G_f, and define a function $g : X \to X$ by

$$g(x) = \begin{cases} y \text{ if } x \mathrel{G} y \text{ and } \varphi(y) = f \circ \varphi(x), \text{ and} \\ x \text{ otherwise.} \end{cases}$$

It is easy to see that $G = G_g$, and the result follows from Proposition 3. □

We close this section by considering the finer partial order of Borel reducibility among graphs. Given graphs G on X and H on Y, we say that $\varphi : X \to Y$ is a *reduction* of G to H if $x_0 \mathrel{G} x_1 \Leftrightarrow \varphi(x_0) \mathrel{H} \varphi(x_1)$. When we restrict our attention to reductions in the locally countable context, we can discard the assumption of local injectivity in the previous result.

Proposition 7. *There is no locally countable Borel graph G of infinite Borel chromatic number which admits both a Borel reduction to G_s and a Borel homomorphism to $G_{\mathbb{F}}$.*

Proof. Suppose, towards a contradiction, that there is such a graph G on X. Since G is locally countable, we may assume without loss of generality that every vertex has at least one G-neighbor. Fix a Borel reduction φ of G to G_s. We consider the Borel equivalence relation E on X defined by $x \mathrel{E} y \Leftrightarrow \varphi(x) = \varphi(y)$. Every equivalence class of E is countable, since $[x]_E \subseteq \{y \in X : x$ and y have the same set of neighbors}. As φ witnesses the smoothness of F, there is a Borel transversal A of F [5, Proposition 6.4]. Then $\varphi \mid A$ is an injective Borel reduction of $G \mid A$ to G_s. As $\varphi \mid A$ is a Borel homomorphism from $G \mid A$ to $G_{\mathbb{F}}$, Corollary 6 implies that $\chi_B(G \mid A) < \aleph_0$.

Fix $k \in \omega$ and a Borel coloring $c : A \to k$ of $G \mid A$. Define $c' : X \to k$ by $c'(x) = c(y)$, where y is the unique element of $A \cap [x]_F$. Then c' is a coloring of G, since the F-saturation of any G-independent set remains G-independent. Since the range of c' is finite, this contradicts our assumption that G has infinite Borel chromatic number. □

3. A more general construction

Suppose that E and F are equivalence relations on a measure space (X, μ). We say that E and F are *independent*, in symbols $E \perp F$, if there is no sequence $x_0, x_1, \ldots, x_n = x_0$ with $n > 1$, $x_i \neq x_j$ $(0 \leq i < j < n)$, and $x_0 E x_1 F x_2 E x_3 \ldots$ (i.e., there are no nontrivial cycles whose "edges" alternate between E and F). We say that E and F are *μ-independent* if they are independent after removing a μ-null set. We denote by $\mathrm{Aut}(X, \mu)$ the set of μ-preserving automorphisms of X, and say that the *support* of $T \in \mathrm{Aut}(X, \mu)$ is the set $\{x \in X : T(x) \neq x\}$.

Recall that $\mathrm{Aut}(X, \mu)$ is a Polish space once equipped with the weak topology (see [4, Chapter 1]). The basic open sets in the weak topology are determined by finite partitions $\{A_0, \ldots, A_n\}$, $\{B_0, \ldots, B_n\}$ of X into Borel sets and $\varepsilon > 0$, and have the form $\{T \in \mathrm{Aut}(X, \mu) : \mu(T(A_i) \triangle B_i) < \varepsilon\}$. Equivalently, each nonempty basic open set is determined by a partition $\mathcal{P}$ of X into countably many Borel sets, $T_0 \in \mathrm{Aut}(X, \mu)$, and $\varepsilon > 0$, having the form $\{T : \forall P \in \mathcal{P}\ \mu(T(P) \triangle T_0(P)) < \varepsilon\}$.

The following is a generalization of the main lemma (and the subsequent remark) in [10, III]. As usual, a measure μ is E-quasi-invariant if the E-saturation of a μ-null set remains μ-null.

Theorem 8. *Suppose that E is a countable Borel equivalence on a standard Borel space X and μ is an E-quasi-invariant standard Borel probability measure on X. Then the set of automorphisms T such that E and E_T are μ-independent is comeager in* $\mathrm{Aut}(X, \mu)$.

Before jumping into the proof of Theorem 8, we establish some lemmas in the more restrictive Borel context.

Lemma 9. *Suppose that $T_1, \ldots, T_k : X \to X$ are Borel automorphisms of a standard Borel space X and $Y \subseteq X$ is such that for all $x \in Y$ and $i, j \leq k$, $T_i(x) \neq T_j(x)$. Then there is a partition of Y into $N = k^2+1$ Borel parts $Y_1, \ldots, Y_N$ such that for each $n \leq N$ the collection $\{T_i(Y_n) : i \leq k\}$ forms a pairwise disjoint family.*

Proof. Consider the Borel graph G on Y given by

$$x \; G \; y \Leftrightarrow x \neq y \text{ and } \exists i, j \ (T_i(x) = T_j(y)).$$

The graph G has degree bounded by k^2 and thus admits a Borel coloring by $k^2+1 = N$ colors [6, Theorem 4.6]. We choose these colors as our sets Y_n. Suppose, towards a contradiction, that $T_i(Y_n) \cap T_j(Y_n) \neq \emptyset$, and fix $x, y \in Y_n$ with $T_i(x) = T_j(y)$. Since Y_n is G-independent, we must have $x = y$, contradicting the hypothesis of the lemma. □

Given a partition $\mathcal{P}$ of X and an automorphism $T : X \to X$, we say that T *factors over* $\mathcal{P}$ if $T(P) = P$ for all $P \in \mathcal{P}$.

Lemma 10. *Suppose that $\mathcal{P}$ is a partition of X into countably many Borel parts, and B is a Borel subset of X. Then there is a Borel involution $I : X \to X$ factoring over $\mathcal{P}$ such that* $\operatorname{supp}(I)$ *is a cocountable subset of B. Moreover, if μ is a nonatomic Borel probability measure on X then I may be chosen to preserve μ.*

Proof. Simply note that for each $P \in \mathcal{P}$ there is a Borel involution I_P with support contained in $B \cap P$ fixing at most one element of $B \cap P$ (such a fixed point is only necessary if $|B \cap P|$ is finite and odd). Set $I = \bigcup_{P \in \mathcal{P}} I_P$. To ensure that this is μ-preserving, for each $P \in \mathcal{P}$ with $\mu(B \cap P) > 0$, choose I_P to flip two subsets of $B \cap P$ of equal measure. □

For each $i \in \mathbb{N}$, let ι_i be a Borel involution of full support. We say that a formal word w in the alphabet $\{\iota_i : i \in \mathbb{N}\} \sqcup \{\tau, \tau^{-1}\}$ is *reduced* if no two elements of $\{\iota_i : i \in \mathbb{N}\}$ appear consecutively, nor do τ and τ^{-1} appear consecutively. With each reduced word w and Borel automorphism $T : X \to X$, we may associate a Borel automorphism T_w inductively by

$$\begin{aligned} T_\emptyset &= \mathrm{id}, \\ T_{\iota_i \frown w} &= \iota_i \circ T_w, \text{ for } i \in \mathbb{N}, \\ T_{\tau \frown w} &= T \circ T_w, \\ T_{\tau^{-1} \frown w} &= T^{-1} \circ T_w. \end{aligned}$$

Intuitively, we simply plug in T for the symbol τ. We say that v is a *subword* of w, written $v \sqsubseteq w$, if there exist possibly empty words w_0, w_1 such that $w = w_0 \frown v \frown w_1$; if $v \neq w$, we say that v is a *proper subword* of w, written $v \sqsubset w$. We say that v is an *initial subword* (resp., *terminal subword*) of w if there exists w_0 such that $w = v \frown w_0$ (resp., $w = w_0 \frown v$). With each reduced word w we may associate its *inverse* w^{-1} defined inductively so that ι_i is its own inverse, τ and τ^{-1} are inverses, and $(w_0 \frown w_1)^{-1} = w_1^{-1} \frown w_0^{-1}$.

If T is a bijection on X and $\mathcal{P}$ is a partition of X, let $T[\mathcal{P}]$ denote the partition $\{T(P) : P \in \mathcal{P}\}$. If $\mathcal{P}$, $\mathcal{P}'$ are two partitions of X, denote by $\mathcal{P} \vee \mathcal{P}'$ the partition $\{P \cap P' : P \in \mathcal{P}, P' \in \mathcal{P}'\}$. If $\mathcal{A}$ is a finite collection of subsets of X, we write $\mathcal{P}(\mathcal{A})$ to denote the finite partition generated by $\mathcal{A}$ (or, equivalently, the atoms of the algebra generated by $\mathcal{A}$).

Lemma 11. *Suppose that $\mathcal{P}$ is a partition of X into countably many Borel sets, w is a nonempty reduced word, and T is a Borel automorphism of X such that T_v has cocountable support for all nonempty proper subwords $v \sqsubset w$. Then there is a Borel automorphism S factoring over $\mathcal{P}$ such that $(T \circ S)_w$ has cocountable support. Moreover, if μ is a nonatomic Borel probability measure on X then S may be chosen to preserve μ.*

Proof. The conclusion is immediate if $w = \iota_i$ for some i. Replacing w by w^{-1} if necessary, we may assume $\tau \sqsubseteq w$, and write $w = w_0 \frown \tau \frown w_1$. Discarding a countable set, we may assume that T_v has full support for all nonempty proper subwords $v \sqsubset w$. By Lemma 9 there is a partition $\{Y_1, \ldots, Y_N\}$ of X into Borel sets such that for all distinct proper terminal subwords $v, v' \sqsubset w$ and all $n \leq N$, $T_v(Y_n) \cap T_{v'}(Y_n) = \emptyset$. We handle the sets Y_n one at a time, via the following fact.

Sublemma 12. *Suppose that T' is a Borel automorphism of X, $\mathcal{P}'$ is a partition of X into countably many Borel sets, and Y is a Borel subset of X such that $T'_v(Y) \cap T'_{v'}(Y) = \emptyset$ for all distinct proper terminal subwords $v, v' \sqsubset w$. Then there is a Borel involution I of X factoring over $\mathcal{P}'$ such that $(T' \circ I)_w$ has cocountable support. Moreover, if μ is a nonatomic Borel probability measure on X then I may be chosen to preserve μ.*

Proof. Set $Y^{\text{fix}} = Y \setminus \operatorname{supp}(T'_w)$ and choose (by Lemma 10) a Borel involution I which factors over $\mathcal{P}'$ such that $\operatorname{supp}(I)$ is a cocountable subset of $T'_{w_1}(Y^{\text{fix}})$. Then for all $y \in Y$, $(T' \circ I)_{w_1}(y) = T'_{w_1}(y)$, since w_1 does not begin with τ^{-1} and $\operatorname{supp}(I)$ is disjoint from $T'_v(Y)$ for all proper subwords $v \sqsubset w_1$. Analogously, for all $y \in T'_{\tau^\frown w_1}(Y)$, we have $(T' \circ I)_{w_0}(y) = T'_{w_0}(y)$. Now if $y \in Y \cap \operatorname{supp}(T'_w)$, we see $(T' \circ I)_w(y) = T'_{w_0} \circ T' \circ I \circ T'_{w_1}(y) = T'_{w_0} \circ T' \circ T'_{w_1}(y) = T'_w(y) \neq y$. On the other hand, for cocountably many $y \in Y^{\text{fix}}$ we have $(T' \circ I)_w(y) = T'_{w_0} \circ T' \circ I \circ T'_{w_1}(y) \neq T'_{w_0} \circ T' \circ T'_{w_1}(y) = T'_w(y) = y$, since $I \circ T'_{w_1}(y) = T'_{w_1}(y)$ for only countably many y. Thus, I is as required, and of course may be chosen to preserve μ (again by Lemma 10). □

We now iteratively apply the sublemma to each Y_n. First, we apply the sublemma to the automorphism T, the partition $\mathcal{P}_1 = \mathcal{P} \vee \mathcal{P}(\{T_v(Y_n) : v \sqsubseteq w \text{ and } n \leq N\})$, and the set Y_1 to obtain an involution I_1 with $Y_1 \setminus \operatorname{supp}((T \circ I_1)_w)$ countable. Apply Lemma 9 to partition $\operatorname{supp}((T \circ I_1)_w) \cap Y_1$ into Borel sets $Y_1^1, \dots Y_1^M$ such that $T_v(Y_1^M) \cap T_{v'}(Y_1^M) = \emptyset$ for all distinct terminal subwords $v, v' \sqsubseteq w$. Note that here we allow one of v, v' to equal w. Next, apply the sublemma to the automorphism $T_1 = T \circ I_1$, the partition $\mathcal{P}_2 = \mathcal{P}_1 \vee \mathcal{P}(\{(T_1)_v(Y_1^m) : v \sqsubseteq w \text{ and } m \leq M\})$, and the set Y_2 to obtain a second involution I_2. Note that the hypotheses of the sublemma are met, since for all proper terminal subwords $v \sqsubset w$ and $n \leq N$ we have $(T_1)_v(Y_n) = T_v(Y_n)$, as the involution I_1 factors over the partition $\mathcal{P}(\{T_v(Y_n) : v \sqsubseteq w \text{ and } n \leq N\})$.

In general, at stage n we set $T_{n-1} = T \circ I_1 \circ \dots \circ I_{n-1}$ and first apply Lemma 9 to partition $\operatorname{supp}((T_{n-1})_w) \cap Y_{n-1}$ into Borel sets $Y_{n-1}^1, \dots Y_{n-1}^M$ such that $T_v(Y_{n-1}^M) \cap T_{v'}(Y_{n-1}^M) = \emptyset$ for all distinct terminal subwords $v, v' \sqsubseteq w$. Subsequently, we apply the sublemma to the automorphism T_{n-1}, along with the partition $\mathcal{P}_n = \mathcal{P}_{n-1} \vee \mathcal{P}(\{(T_{n-1})_v(Y_{n-1}^m) : v \sqsubseteq w \text{ and } m \leq M\})$, and the set Y_n to obtain the next involution I_n. We claim that the automorphism $S = I_1 \circ \dots \circ I_N$ satisfies the conclusion of the lemma.

That is, we want to show that the set of x such that $(T \circ S)_w(x) = x$ is countable. It suffices to show that if $x \in Y_n$ and $(T_n)_w(x) \neq x$, then $(T \circ S)(x) \neq x$, where as before $T_n = T \circ I_1 \circ \dots \circ I_n$. Fix $m \leq M$ such that $x \in Y_n^m$. Since each of the involutions $I_{n+1}, \dots, I_N$ factors over $\mathcal{P}(\{(T_n)_v(Y_n^m) : v \sqsubseteq w\})$, we have $(T \circ S)_v(Y_n^m) = (T_n)_v(Y_n^m)$ for all $v \sqsubseteq w$. In particular, $(T \circ S)_w(Y_n^m) \cap Y_n^m = \emptyset$, and thus $(T \circ S)_w(x) \neq x$. □

Proof of Theorem 8. Suppose that E is a countable Borel equivalence on a standard Borel space X and μ is an E-quasi-invariant Borel probability measure on X. We want to show that the the set of automorphisms T such that E and E_T are μ-independent is comeager in $\operatorname{Aut}(X, \mu)$ (with respect to the weak topology).

We may (see [3]) write E as the union of the graphs of countably many Borel involutions. A straightforward maximality argument then allows us to write $E \setminus \{(x, x) : x \in X\}$ as the union of the graphs of countably many fixed-point-free

Borel involutions $\iota_0, \iota_1, \ldots$. Using our earlier notation, it is suffices to show that for all nonempty reduced words w in the alphabet $\{\iota_0, \iota_1, \ldots, \tau, \tau^{-1}\}$, the set $\{T : \mu(\mathrm{supp}(T_w)) = 1\}$ is comeager in $\mathrm{Aut}(X, \mu)$. We may assume by induction that for each proper subword $v \sqsubset w$ the set $\{T : \mu(\mathrm{supp}(T_v)) = 1\}$ is comeager.

We first check that $U_\varepsilon = \{T : \mu(\mathrm{supp}(T_w)) > 1 - \varepsilon)\}$ is open for each $\varepsilon > 0$. Fix $T_0 \in U_\varepsilon$. By Lemma 9 there is a partition $Y_1, \ldots Y_N$ of $\mathrm{supp}((T_0)_w)$ into finitely many sets such that $Y_n \cap (T_0)_w(Y_N) = \emptyset$ for each $n \leq N$. Set $\mathcal{P} = \mathcal{P}(\{(T_0)_v(y_n) : v \sqsubseteq w \text{ and } n \leq N\})$. Then, since each $(T_0)_v$ is μ-class preserving, for each $\delta > 0$ there is some $\varepsilon_\delta > 0$ such that whenever T satisfies $\mu(T(P) \mathbin{\triangle} T_0(P)) < \varepsilon_\delta$ for all $P \in \mathcal{P}$ then $\mu(Y_n \cap T_w(Y_n)) < \delta$. That is, choosing δ smaller than $\mu(\mathrm{supp}((T_0)_w)) - (1 - \varepsilon)$, the set $\{T : \forall P \in \mathcal{P}\ \mu(T(P) \mathbin{\triangle} T_0(P)) < \varepsilon_\delta\}$ is a neighborhood of T_0 contained in U_ε. Consequently, the set $\{T : \mu(\mathrm{supp}(T_w)) = 1\}$ is G_δ in $\mathrm{Aut}(X, \mu)$.

So all that remains is to check that the set $\{T : \mu(\mathrm{supp}(T_v)) = 1\}$ is dense. Fix an open set U and by the inductive assumption an automorphism $T_0 \in U \cap \{T : \forall v \sqsubset w\ (\mu(\mathrm{supp}(T_v)) = 1)\}$. Without loss of generality, we may assume there is a finite partition $\mathcal{P}$ and $\varepsilon > 0$ such that $U = \{T : \forall P \in \mathcal{P}\ (\mu(T(P) \mathbin{\triangle} T_0(P)) < \varepsilon)\}$. By Lemma 11, there is a μ-preserving automorphism S factoring over $\mathcal{P}$ such that $(T \circ S)_v$ has conull support for all $v \sqsubseteq w$. Since $T_0 \circ S(P) = T(P)$ for all $P \in \mathcal{P}$, we see $T_0 \circ S \in U \cap \{T : \forall v \sqsubseteq w\ (\mu(\mathrm{supp}(T_v)) = 1)\}$ as desired. □

Before generalizing the construction of $G_{\mathbb{F}}$ in the previous section, we digress a bit to discuss costs. The relevant notions are defined in [5, III].

Corollary 13. *Suppose that E is a μ-preserving, aperiodic, countable, Borel equivalence relation on a standard probability space (X, μ). Then there is a free, μ-preserving action of F_∞ whose associated orbit equivalence relation is independent of E. Consequently, for all $\alpha > 0$ there is a μ-preserving, countable, Borel equivalence relation $E_\alpha \supseteq E$ such that $C_\mu(E_\alpha) = C_\mu(E) + \alpha$. Moreover, if E is treeable then E_α may be taken to be treeable.*

Proof. Apply Theorem 8 in succession to obtain a sequence of automorphisms $T_0, T_1, \ldots \in \mathrm{Aut}(X, \mu)$ such that for each i we have $E_{T_i} \perp E \vee \bigvee_{j<i} E_{T_j}$. Then the set $\{T_i : i \in \mathbb{N}\}$ generates a free F_∞ action whose associated equivalence relation is independent of E.

To obtain a superequivalence relation increasing the cost of E by α, first write $\alpha = n + \varepsilon$ for some $n \in \mathbb{N}$ and $0 \leq \varepsilon < 1$. Fix a Borel set $A_\varepsilon \subseteq X$ with $\mu(A_\varepsilon) = \varepsilon$ and define T'_n by

$$T'_n(x) = \begin{cases} T_n(x) \text{ if } x \in A_\varepsilon, \\ x \text{ if } x \notin A_\varepsilon. \end{cases}$$

Then, by a fundamental result of Gaboriau's theory of costs (see, e.g., [5, Theorem 27.2]), the equivalence relation

$$E_\alpha = E \vee \bigvee_{i<n} E_{T_i} \vee E_{T'_n}$$

has cost $C_\mu(E) + \alpha$.

If E is treeable, it is evident that this E_α is also treeable by simply adding to the treeing of E the (symmetrized) graphs of $T_0, \ldots, T_{n-1}, T'_n$. □

Suppose now that we have a countable sequence of measure-preserving, Borel graphs $(G_n)_{n\in\mathbb{N}}$ on standard probability spaces (X_n, μ_n), respectively (i.e., for each

$n \in \mathbb{N}$, μ_n is E_{G_n}-invariant). We build another standard measure space (X, μ) by setting $X = \bigsqcup_{n \in \mathbb{N}} X_n$ and for Borel $A \subseteq X$,

$$\mu(A) = \sum_{n \in \mathbb{N}} 2^{-(n+1)} \mu_n(A \cap X_n).$$

We then may define a Borel graph $G_{\mathbb{N}}$ on X by

$$x \, G_{\mathbb{N}} \, y \Leftrightarrow \exists n \in \mathbb{N} \ (x, y \in X_n \text{ and } x \, G_n \, y).$$

Note then that μ is $E_{G_{\mathbb{N}}}$-invariant.

By Theorem 8, the generic $T \in \mathrm{Aut}(X, \mu)$ satisfies $E_T \perp E_{G_{\mathbb{N}}}$ (after, as usual, discarding a μ-null set). We may fix such a T which is ergodic, since of course the generic element of $\mathrm{Aut}(X, \mu)$ is ergodic [4, Theorem 2.6]. The *ergodic amalgamation of* (G_n) *by* T is the Borel graph

$$G_{(G_n),T} = G_{\mathbb{N}} \cup G_T$$

on X. When the particular choice of T is unimportant, we say G is an *ergodic amalgamation* of (G_n) if $G = G_{(G_n),T}$ for some T as above. Note of course that E_G is an ergodic equivalence relation. We collect here some useful facts about ergodic amalgamations.

Proposition 14. *Suppose that $(G_n)_{n \in \mathbb{N}}$ is a sequence of Borel, measure-preserving graphs on (X_n, μ_n), respectively, and that G on (X, μ) is an ergodic amalgamation of (G_n).*

(i) *If each G_n is acyclic, then G is acyclic. Moreover, the girth (the length of the shortest cycle) of G is equal to the least girth among the graphs G_n, and the clique number (the size of the largest set of vertices with all pairs adjacent) of G is equal to the largest clique number among the graphs G_n.*

(ii) *If each G_n is locally finite, then G is locally finite. Moreover, if every G_n has degree bounded by a fixed d, then G has degree bounded by $d + 2$.*

(iii) *The Borel chromatic number of G, $\chi_B(G)$, is bounded below by $\sup_n \chi_B(G_n)$ and above by $3 \sup_n \chi_B(G_n)$.*

(iv) *The measurable chromatic number of G, $\chi_\mu(G)$, is bounded below by $\sup_n \chi_{\mu_n}(G_n)$ and above by $3 \sup_n \chi_{\mu_n}(G_n)$.*

In particular, if G is an ergodic amalgamation of some sequence $(G_n)_{n \in \mathbb{N}}$ of graphs with measurable chromatic numbers tending towards infinity while each having bounded degree, and f is any Borel function on a Polish space such that G_f has infinite Borel chromatic number, the arguments in Proposition 1 and 3 ensure that there is no Borel homomorphism from G to G_f nor from G_f to G.

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