# Lecture 17

[Kh-T] : If $Y^d \subseteq X^n$ proj. var., $\alpha, \beta$ nef on $X$, then $(\alpha^{d-i}\beta^i \cap [Y])_i$ log-conc.

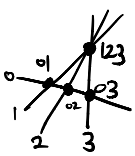
$M$ a matroid of rank $r$ on $E = \{0, \cdots, n\}$ (assume loopless).

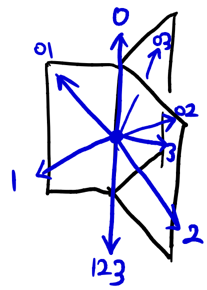
$\Sigma_M$ Bergman fan : cones $\longleftrightarrow$ chains of (nonempty, proper) flats

$\hookrightarrow$ is a <u>balanced fan</u>: $\Delta_M \in MW_{r-1}(\Sigma_{A_n})$ by $\Delta_M(\sigma) \mapsto \begin{cases} 1 & \sigma \in \Sigma_M \\ 0 & \sigma \notin \Sigma_M \end{cases}$

the Bergman class.

<u>E.g.</u> $U_{1,1} \oplus U_{2,3}$




<u>Ques.</u>
Does $(\alpha^{r-1-i}\beta^i \cap \Delta_M)_i$ have similar behavior as if $\Delta_M = [Y]$ for $Y \subseteq X_{A_n}$ subvariety?

Let $\Sigma$ be a simplicial fan that is subfan of a proj. fan $\overline{\Sigma}$, $\mathrm{lin}(\overline{\Sigma}) = 0$ and of pure dim $= d$ in $N_{\mathbb{R}}$ (where $\dim N_{\mathbb{R}} = n$). (i.e. a <u>quasiproj.</u> fan)

Let $A^\bullet(\Sigma) = \mathbb{R}[x_\rho \mid \rho \in \Sigma(1)]/\sim$ , $MW_\bullet(\Sigma)$ as usual BUT w/ $\mathbb{R}$-coeff.

$\mathcal{K}_\Sigma := \{ D \in A^1(\Sigma) \mid D = \sum_{\rho \in \Sigma(1)} a_\rho x_\rho$ st $\exists$ ample $\sum_{\rho \in \overline{\Sigma}(1)} a'_\rho x_\rho$ w/ $a'_\rho = a_\rho$ for $\rho \in \Sigma(1) \}$

<u>Defn</u> A pair $(\Sigma, \Delta_\Sigma)$ for $\Delta_\Sigma \in MW_d(\Sigma)$ satisfies the Kähler package if:

(PD) $A^\bullet(\Sigma)$ is a Poincaré duality alg. of dim. $d$ with $\int_\Sigma : A^d(\Sigma) \to \mathbb{R}$ via $\xi \mapsto \xi \cap \Delta_M$.
<u>i.e.</u> $A^\bullet = \bigoplus_{i=0}^{d} A^i$ st $A^0 = \mathbb{R}$ and $\int : A^d \xrightarrow{\sim} \mathbb{R}$ with $A^{d-i} \times A^i \to \mathbb{R}$ perfect.

(HL) For $\forall\ 0 \le k \le \frac{d}{2}$ and any $\ell \in \mathcal{K}_\Sigma$,
$$\times \ell^{d-2k} : A^k \longrightarrow A^{d-k}, \quad \xi \mapsto \xi \cdot \ell^{d-2k}$$ is an isom.

(HR) For $\forall\ 0 \le k \le \frac{d}{2}$ and any $\ell \in \mathcal{K}_\Sigma$,
$$A^k \times A^k \longrightarrow \mathbb{R}, \quad (\eta, \xi) \mapsto (-1)^k \int \eta \xi \ell^{d-2k}$$
is positive definite on $\ker(\times \ell^{d-2k+1})$.

<u>Rem</u> [Cattani '08] HL, HR in <u>all deg.s</u> $\iff$ mixed HL & HR.
$\hookrightarrow$ necessary! (See prev. exer. on Lorentzian polynom.)

**Lem** (a) $(A^\bullet, \int_A) \twoheadrightarrow (B^\bullet, \int_B)$ of P.D. alg. of same dim. is an isom.

(b) $f \in A^k$ homog. in P.D. alg of dim $n$. Then $A^\bullet/\mathrm{ann}(f) = \{a \in A^\bullet \mid af = 0\}$ is a Poincaré duality alg. of dim $n-k$ w/ $\int_{new}(\cdot) = \int_A (\cdot) f$.

**N.B.** (1) If $Y \subset X$ smth proj. var. $H^\bullet(X)/\mathrm{ann}[Y]$ = ring whose elts are $\xi \cap [Y]$, and mult. is $(\eta \cap [Y])(\xi \cap [Y]) = (\eta\xi) \cap [Y]$.

$H^\bullet(X)/\mathrm{ann}[Y]$ is image of $H^\bullet(X) \to H^\bullet(Y)$ iff the image has (PD)

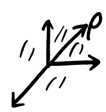
If $A^\bullet(\Sigma)$ has (PD), then $MW_d(\Sigma) \simeq \mathbb{R}\Delta_\Sigma$, (Converse: [Adiprasito]) and in particular,

$$\begin{array}{ccc} A^\bullet(\overline{\Sigma}) & \twoheadrightarrow & A^\bullet(\Sigma) \\ \downarrow & \swarrow \simeq & \\ A^\bullet(\overline{\Sigma})/\mathrm{ann}\,\Delta_\Sigma & & \end{array}, \quad x_\sigma \mapsto \begin{cases} x_\sigma & \sigma \in \Sigma \\ 0 & \sigma \notin \Sigma \end{cases}$$

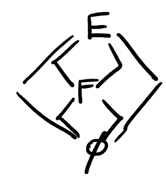
and $\int_\Sigma : A^d(\Sigma) \to \mathbb{R}$, $\xi \cap \Delta_\Sigma$.

(2) $\mathrm{star}(\sigma, \Sigma) :=$ fan in $N_\mathbb{R}/\mathrm{span}(\sigma)$ whose cones are images of $\tau \succeq \sigma$

$X_{\mathrm{star}(\sigma,\Sigma)} \simeq V(\sigma)$ the orbit-closure corresp. to $\sigma$ in $X_\Sigma$.

E.g. 

$\mathrm{star}(\rho, \Sigma) = \longleftarrow\bullet\longrightarrow \iff \mathbb{P}^1$

E.g. $\mathrm{star}(\rho_F, \Sigma_M)$ for a flat $F$ is $\Sigma_{M|F} \times \Sigma_{M/F}$ 


$A^\bullet(\Sigma) \twoheadrightarrow A^\bullet(\Sigma)/\mathrm{ann}(x_\rho) \simeq A^\bullet(\mathrm{star}(\rho, \Sigma))$

**Defn** For a balanced fan $\Sigma$, it is Lefschetz if it satisfies the Kähler package and every $\mathrm{star}(\sigma, \Sigma)$ is Lefschetz.

**Thm** [Ardila-Denham-Huh '20] If two quasiproj $\Sigma_1$ & $\Sigma_2$ have the same support, then $\Sigma_1$ is Lefschetz iff $\Sigma_2$ is. (Trop. Hodge thry)

**Thm** [Adiprasito-Huh-Katz '18] $\Sigma_M$ is Lefschetz.