WEIGHTED V@R AND ITS PROPERTIES

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Abstract. The paper deals with the study of the coherent risk measure, which we call *Weighted V@R*. It is a risk measure of the form

$$\rho_\mu(X) = \int_{[0,1]} \mathrm{TV@R}_\lambda(X)\mu(d\lambda),$$

where μ is a probability measure on $[0,1]$ and TV@R stands for Tail V@R.

After investigating some basic properties of this risk measure, we apply the obtained results to the financial problems of pricing, optimization, and capital allocation. It turns out that, under some regularity conditions on μ, Weighted V@R possesses some nice properties that are not shared by Tail V@R. To put it briefly, Weighted V@R is "smoother" than Tail V@R. This allows one to say that weighted V@R is one of the most (or maybe the most) important classes of coherent risk measures.

Key words and phrases. Capital allocation problem, coherent risk measure, determining set, distorted measure, minimal extreme measure, optimization based on coherent risk measures, option-based model, pricing based on coherent risk measures, strict diversification, Tail V@R, Weighted V@R.

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1 Introduction

The theory of coherent risk measures is a very new, important, and rapidly evolving branch of the modern financial mathematics. This concept was introduced by Artzner, Delbaen, Eber, and Heath [2], [3]. Since then, many papers on the topic have followed; surveys of the modern state of the theory are given in [14], [16; Ch. 4], [24], and [27].

A very important class of coherent risk measures is given by *Tail V@R* (the terms *Average V@R*, *Conditional V@R*, and *expected shortfall* are also used). Tail V@R of order $\lambda \in [0,1]$ is a map $\rho_\lambda : L^\infty \to \mathbb{R}$ (we have a fixed probability space $(\Omega, \mathcal{F}, \mathsf{P})$) defined by

$$\rho_\lambda(X) = -\inf_{\mathsf{Q}\in\mathcal{D}_\lambda} \mathsf{E}_\mathsf{Q} X,$$

where $\mathcal{D}_\lambda$ is the set of probability measures Q that are absolutely continuous with respect to P with $\frac{d\mathsf{Q}}{d\mathsf{P}} \le \lambda^{-1}$. The importance of Tail V@R is seen from a result of Kusuoka [20], who proved that ρ_λ is the smallest law invariant coherent risk measure that dominates V@R$_\lambda$ (we recall the precise formulation in Section 2). This suggests an opinion that Tail V@R is one of the best coherent risk measures.

However, Tail V@R has the following obvious drawback. It depends only on the tail of the distribution of X to the left of its λ-quantile, while the form of the distribution to the right of this quantile does not affect Tail V@R. A natural way to eliminate this drawback is to consider weighted average of Tail V@Rs with different λ. This is a risk measure of the form

$$\rho_\mu(X) = \int_{[0,1]} \rho_\lambda(X)\mu(d\lambda), \tag{1.1}$$

where μ is a probability measure on $[0,1]$. We call this risk measure *Weighted V@R* and its study is the goal of this paper.

This risk measure has already been investigated in the literature. Kusuoka [20] proved that any law invariant comonotonic coherent risk measure is of this form. In the same paper, Kusuoka proved that any law invariant risk measure has the form $\sup_{\mu\in\mathfrak{M}} \rho_\mu$, where $\mathfrak{M}$ is a set of probability measures on $[0,1]$ (we recall the precise formulations in Section 2). Furthermore, according to the representation theorem for coherent risk measures that was proved for general probability spaces by Delbaen [13], ρ_μ can be represented as

$$\rho_\mu(X) = -\inf_{\mathsf{Q}\in\mathcal{D}} \mathsf{E}_\mathsf{Q} X, \quad X \in L^\infty \tag{1.2}$$

with some set $\mathcal{D}$ of probability measures that are absolutely continuous with respect to P. Clearly, such a set $\mathcal{D}$ is not unique, but there exists the largest set $\mathcal{D}_\mu$, for which representation (1.2) is true. (It has the form $\mathcal{D}_\mu = \{\mathsf{Q} \ll \mathsf{P} : \mathsf{E}_\mathsf{Q} X \ge -\rho_\mu(X) \text{ for any } X \in L^\infty\}$.) Carlier and Dana [7] provided a representation of $\mathcal{D}_\mu$ (we recall it in Section 4).

When risk measures are applied to financial problems, it is almost necessary to extend them to the space L^0 of all random variables. In Section 2 of the present paper, we extend Weighted V@R to L^0 simply by formula (1.2) with L^∞ replaced by L^0. Here we understand the expectation $\mathsf{E}_\mathsf{Q} X$ as $\mathsf{E}_\mathsf{Q} X^+ - \mathsf{E}_\mathsf{Q} X^-$ with the convention $\infty - \infty = -\infty$ (so that $\mathsf{E}_\mathsf{Q} X$ is well defined for any Q and X). This way to extend risk measures from L^∞ to L^0 was proposed in [10]. Throughout the whole paper, we deal with Weighted V@R on L^0.

Section 3 contains two representations of Weighted V@R. One of them extends representation (1.1) from L^∞ to L^0, while the other one establishes the connection between Weighted V@R and *distorted measures* that have been used in insurance for 15 years (see [15]). Let us remark that the corresponding L^∞-representation is well known (see [16; Th. 4.64] or [24; Th. 1.51]).

In Section 4, we provide two representations of $\mathcal{D}_\mu$, both of which have different forms compared to the representation in [7]. These representations are needed for the applications to pricing, optimization, and capital allocation considered in the next sections.

The main result of Section 5 is that

$$\rho_\mu(X+Y) < \rho_\mu(X) + \rho_\mu(Y)$$

provided that $\operatorname{supp}\mu = [0,1]$ and X, Y are not comonotone (in particular, the latter condition is satisfied if the distribution of (X,Y) has a joint density). We call this the *strict diversification property*. This property is very important from the viewpoint of financial mathematics because it leads to the uniqueness of a solution of various optimization problems based on coherent risk measures.

In [10], we introduced the notion of an *extreme measure*. The class of extreme measures for a coherent risk measure ρ and a random variable X is defined as

$$\mathcal{X}_\rho(X) = \{\mathsf{Q} \in \mathcal{D} : \mathsf{E}_\mathsf{Q} X = -\rho(X)\},$$

where $\mathcal{D}$ is the largest set, for which $\rho(X) = -\inf_{\mathsf{Q}\in\mathcal{D}} \mathsf{E}_\mathsf{Q} X$. This notion turned out to be very convenient and important. In particular, solutions of several optimality pricing problems, solutions of equilibrium pricing problem, and solution of the capital allocation problem are expressed through extreme measures. Moreover, the *risk contribution* introduced in [10] is expressed through extreme measures. In general, the set $\mathcal{X}_\rho(X)$ can contain more than one point. However, as shown in Section 6, for $\rho = \rho_\mu$ with $\mu(\{0\}) = 0$, there exists a unique element of $\mathcal{X}_\rho$ that is the smallest in the convex order. We call it the *minimal extreme measure*. This notion is of importance for financial mathematics as it allows one to select a (unique) distinguished solution of the problems like capital allocation or optimality pricing, which possesses some nice properties. We call it the *central solution*.

One of the most important global problems of the modern financial mathematics is to narrow the No Arbitrage price intervals of contingent claims as they are known to be unacceptably wide in most incomplete models (see, for example, the discussion in [1; Sect. 5]). Several ways to do that have been proposed in the literature. One of them consists in considering actively traded derivatives as basic assets. In particular, a popular model is based on treating as basic assets European call options on a fixed asset with a fixed maturity and different strike prices. The corresponding model was first studied by Breeden and Litzenberger [6] and Banz and Miller [4]. A literature review of this model is given in [17]. Let us also mention the paper [9; Sect. 6], in which this model was analyzed from the general viewpoint of fundamental theorems of asset pricing.

Recently another (very promising) way to narrow fair price intervals has been proposed. It is known as the *No Good Deals* pricing. This technique was first considered by Cochrane and Saá-Requejo [12] and Bernardo and Ledoit [5]. An important feature of this theory is that there exists no canonical definition of a good deal (in particular, [12] and [5] employ different definitions). Carr, Geman, and Madan [8] and Jaschke and Küchler [18] proposed variants of No Good Deals pricing based on coherent risk measures. These techniques were further developed in [10] and [26].

In Section 7, we combine the two ways of narrowing fair price intervals described above. Namely, we apply the No Good Deals pricing technique from [10] to the model with European options as basic assets. The risk measure employed is Weighted V@R. This leads to the "double reduction" of fair price intervals.

Section 8 is devoted to another financial application of Weighted V@R. In [10; Sect. 3], we considered a problem of optimizing Risk-Adjusted Return on Capital (RAROC) based on coherent risk measures. This is the problem of maximizing the ratio

$$\mathrm{RAROC}(X) = \frac{\mathsf{E}_{\mathsf{P}} X}{\rho(X)},$$

where ρ is a coherent risk measure. In this paper, we present a solution of this problem in a complete model for the case, where ρ is Weighted V@R. It shows that in most natural situations the optimal strategy consists in buying a binary option with the payoff $I\left(\frac{d\mathsf{Q}}{d\mathsf{P}} \le c_*\right)$, where Q is the unique risk-neutral measure and c_* is the optimal threshold explicitly calculated in Section 8 (see Figure 4).

2 Basic Definitions

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space. It will be convenient for us to deal not with coherent risk measures, but with their opposites called *coherent utility functions* (this enables one to get rid of numerous minus signs).

Definition 2.1. A *coherent utility function on* L^∞ is a map $u : L^\infty \to \mathbb{R}$ with the properties:

(a) (Superadditivity) $u(X+Y) \ge u(X) + u(Y)$;
(b) (Monotonicity) If $X \le Y$, then $u(X) \le u(Y)$;
(c) (Positive homogeneity) $u(\lambda X) = \lambda u(X)$ for $\lambda \in \mathbb{R}_+$;
(d) (Translation invariance) $u(X+m) = u(X) + m$ for $m \in \mathbb{R}$;
(e) (Fatou property) If $|X_n| \le 1$, $X_n \xrightarrow{\mathsf{P}} X$, then $u(X) \ge \limsup_n u(X_n)$.

The corresponding *coherent risk measure* is $\rho(X) = -u(X)$.

The theorem below was established in [3] for the case of a finite Ω (in this case the axiom (e) is not needed) and in [13] for the general case. We denote by $\mathcal{P}$ the set of probability measures on $\mathcal{F}$ that are absolutely continuous with respect to P. Throughout the paper, we identify measures from $\mathcal{P}$ (these are typically denoted by Q) with their densities with respect to P (these are typically denoted by Z).

Theorem 2.2. *A function u satisfies conditions* (a)–(e) *if and only if there exists a nonempty set $\mathcal{D} \subseteq \mathcal{P}$ such that*

$$u(X) = \inf_{\mathsf{Q} \in \mathcal{D}} \mathsf{E}_{\mathsf{Q}} X, \quad X \in L^\infty. \tag{2.1}$$

Now, we use representation (2.1) to extend coherent utility functions to L^0.

Definition 2.3. A *coherent utility function on* L^0 is a map $u : L^0 \to [-\infty, \infty]$ defined as

$$u(X) = \inf_{\mathsf{Q} \in \mathcal{D}} \mathsf{E}_{\mathsf{Q}} X, \quad X \in L^0, \tag{2.2}$$

where $\mathcal{D} \subseteq \mathcal{P}$ and $\mathsf{E}_{\mathsf{Q}} X$ is understood as $\mathsf{E}_{\mathsf{Q}} X^+ - \mathsf{E}_{\mathsf{Q}} X^-$ with the convention $\infty - \infty = -\infty$.

Clearly, a set $\mathcal{D}$, for which representations (2.1) and (2.2) are true, is not unique. However, there exists the largest such set given by $\{\mathsf{Q} \in \mathcal{P} : \mathsf{E}_{\mathsf{Q}} X \ge u(X) \text{ for any } X\}$.

Definition 2.4. We will call the largest set, for which (2.1) (resp., (2.2)) is true, the *determining set of* u.

`Remarks`. `(i)` Clearly, the determining set is convex. For coherent utility functions on L^∞, it is also L^1-closed. However, for coherent utility functions on L^0, it is not necessarily L^1-closed. As an example, take a positive unbounded random variable X_0 such that $\mathsf{P}(X_0 = 0) > 0$ and consider $\mathcal{D}_0 = \{\mathsf{Q} \in \mathcal{P} : \mathsf{E_Q} X_0 = 1\}$. Clearly, the determining set $\mathcal{D}$ of the coherent utility function $u(X) = \inf_{\mathsf{Q}\in\mathcal{D}_0} \mathsf{E_Q} X$ belongs to the set $\{\mathsf{Q} \in \mathcal{P} : \mathsf{E_Q} X_0 \ge 1\}$. On the other hand, the L^1-closure of $\mathcal{D}_0$ contains a measure Q_0 concentrated on $\{X_0 = 0\}$.

`(ii)` Let $\mathcal{D}$ be an L^1-closed convex subset of $\mathcal{P}$. Define a coherent utility function u by (2.1) or (2.2). Then $\mathcal{D}$ is the determining set of u. Indeed, assume that the determining set $\mathcal{D}_0$ is greater than $\mathcal{D}$, i.e. there exists $\mathsf{Q}_0 \in \mathcal{D}_0 \setminus \mathcal{D}$. Then, by the Hahn-Banach theorem, we can find $X_0 \in L^\infty$ such that $\mathsf{E}_{\mathsf{Q}_0} X_0 < \inf_{\mathsf{Q}\in\mathcal{D}} \mathsf{E_Q} X$, which is a contradiction.

Now, we recall some basic facts related to Tail V@R. The next definition applies both to L^∞ and to L^0.

Definition 2.5. *Tail V@R* is the risk measure corresponding to the coherent utility function

$$u_\lambda(X) = \inf_{\mathsf{Q}\in\mathcal{D}_\lambda} \mathsf{E_Q} X,$$

where $\lambda \in [0,1]$ and

$$\mathcal{D}_\lambda = \left\{\mathsf{Q} \in \mathcal{P} : \frac{d\mathsf{Q}}{d\mathsf{P}} \le \lambda^{-1}\right\}.$$

Clearly, $u_0(X) = \operatorname{essinf}_\omega X(\omega)$. The following well-known proposition provides two representations of Tail V@R with $\lambda > 0$. Throughout the paper, we denote by $q_\lambda(X)$ the right λ-quantile of X, i.e. $q_\lambda(X) = \inf\{x : \mathsf{P}(X \le x) > \lambda\}$ (we use the convention $\inf\emptyset = +\infty$).

Proposition 2.6. **(i)** *Let* $\lambda \in (0,1]$, $X \in L^0$. *Then, for any* $Z^* \in \mathcal{D}_\lambda$ *such that*

$$Z^* = \begin{cases} \lambda^{-1} & \text{on } \{X < q_\lambda(X)\}, \\ 0 & \text{on } \{X > q_\lambda(X)\}, \end{cases} \tag{2.3}$$

we have $u_\lambda(X) = \mathsf{E_P} XZ^*$. *Conversely, if* $u_\lambda(X) > -\infty$, *then any* $Z^* \in \mathcal{D}_\lambda$ *such that* $u_\lambda(X) = \mathsf{E_P} XZ^*$, *should satisfy* (2.3).

(ii) *Let* $\lambda \in (0,1]$, $X \in L^0$. *Then*

$$u_\lambda(X) = \lambda^{-1} \int_{(-\infty, q_\lambda(X))} x\mathsf{Q}(dx) + c q_\lambda(X), \tag{2.4}$$

where $\mathsf{Q} = \mathrm{Law}_{\mathsf{P}} X$ *and* $c = 1 - \lambda^{-1}\mathsf{Q}((-\infty, q_\lambda(X)))$.

`Proof`. **(i)** We will assume that $\mathsf{E_P} X^- < \infty$ and $\lambda \in (0,1)$ (the other cases are analyzed trivially). Without loss of generality, $q_\lambda(X) = 0$. Then, for any $Z \in \mathcal{D}_\lambda$,

$$XZ - XZ^* = X(Z - \lambda^{-1})I(X < 0) + XZI(X > 0) \ge 0.$$

Furthermore, the a.s. inequality here is possible only if Z satisfies (2.3).

(ii) This is an immediate consequence of (i). □

The importance of Tail V@R is seen from a result of Kusuoka [20], which is stated below (its proof can also be found in [16; Th. 4.61] or [24; Th. 1.48]). Recall that a coherent utility function u is called *law invariant* if $u(X)$ depends only on the distribution of X. Note that, due to (2.4), u_λ is law invariant, and hence, u_μ on L^∞ is also law invariant.

Theorem 2.7. *Assume that $(\Omega, \mathcal{F}, \mathsf{P})$ is atomless. Let $\lambda \in [0,1]$ and u be a law invariant coherent utility function on L^∞ such that $u \le q_\lambda$. Then $u \le u_\lambda$.*

We now introduce the basic object of the paper.

Definition 2.8. (i) *Weighted V@R on L^∞* is the risk measure corresponding to the coherent utility function

$$u_\mu(X) = \int_{[0,1]} u_\lambda(X)\mu(d\lambda), \quad X \in L^\infty,$$

where μ is a probability measure on $[0,1]$.

(ii) *Weighted V@R on L^0* is the risk measure corresponding to the coherent utility function

$$u_\mu(X) = \inf_{\mathsf{Q}\in\mathcal{D}_\mu} \mathsf{E}_\mathsf{Q} X, \quad X \in L^0,$$

where $\mathcal{D}_\mu$ is the determining set of u_μ on L^∞.

Thus, we have used the following scheme to define u_μ on L^0:

$$u_\lambda \text{ on } L^\infty \quad\longrightarrow\quad u_\mu \text{ on } L^\infty \quad\longrightarrow\quad \mathcal{D}_\mu \quad\longrightarrow\quad u_\mu \text{ on } L^0.$$

Let us now recall two results of Kusuoka [20] (the proofs can also be found in [16; Cor. 4.58, Th. 4.87] or [24; Cor. 1.45, Th. 1.58]), which show the importance of Weighted V@R in view of the law invariance property. Recall that random variables X and Y are *comonotone* if $(X(\omega_2) - X(\omega_1))(Y(\omega_2) - Y(\omega_1)) \ge 0$ for $\mathsf{P}\times\mathsf{P}$-a.e. ω_1, ω_2; a coherent utility function u is *comonotonic* if $u(X+Y) = u(X) + u(Y)$ whenever X and Y are comonotone.

Theorem 2.9. (i) *On an atomless probability space, a coherent utility function u on L^∞ is law invariant and comonotonic if and only if it has the form $u = u_\mu$ with some probability measure μ.*

(ii) *On an atomless probability space, a coherent utility function u on L^∞ is law invariant if and only if it has the form $u = \inf_{\mu\in\mathfrak{M}} u_\mu$ with some collection $\mathfrak{M}$ of probability measures on $[0,1]$.*

3 Representation of Weighted V@R

For $X \in L^0$ and $\lambda \in (0,1]$, we set

$$Z_\lambda^*(X) = \begin{cases} \lambda^{-1} & \text{on } \{X < q_\lambda(X)\}, \\ c & \text{on } \{X = q_\lambda(X)\}, \\ 0 & \text{on } \{X > q_\lambda(X)\}, \end{cases} \tag{3.1}$$

where $c \in [0, \lambda^{-1}]$ is such that $\mathsf{E}_\mathsf{P} Z_\lambda^* = 1$.

Lemma 3.1. *Let $\lambda \in (0,1]$, $X \in L^0$, and f be an increasing function. Then $u_\lambda(f(X)) = \mathsf{E}_\mathsf{P} f(X) Z_\lambda^*(X)$.*

`Proof`. Without loss of generality, $q_\lambda(X) = 0$ and $f(0) = 0$. Then, for any $Z \in \mathcal{D}_\lambda$, we can write

$$f(X)Z - f(X)Z_\lambda^*(X) = f(X)(Z - \lambda^{-1})I(X<0) + f(X)ZI(X>0) \ge 0.$$

Theorem 3.2. **(i)** *Suppose that* $\mu(\{0\}) = 0$. *Then, for* $X \in L^0$, *we have* $u_\mu(X) = \mathsf{E}_\mathsf{P} X Z^*(X)$, *where* $Z^*(X) = \int_{(0,1]} Z^*_\lambda(X)\mu(d\lambda)$.

(ii) *We have*

$$u_\mu(X) = \int_{[0,1]} u_\lambda(X)\mu(d\lambda), \quad X \in L^0, \tag{3.2}$$

where $\int_{[0,1]} f(\lambda)\mu(d\lambda)$ *is understood as* $\int_{[0,1]} f^+(\lambda)\mu(d\lambda) - \int_{[0,1]} f^-(\lambda)\mu(d\lambda)$ *with the convention* $\infty - \infty = -\infty$.

Proof. **(i)** Any $Z \in \mathcal{D}_\mu$ can be represented as $\int_{(0,1]} Z_\lambda \mu(d\lambda)$ with $Z_\lambda \in \mathcal{D}_\lambda$ (see Theorem 4.4 below). Due to Lemma 3.1,

$$\begin{aligned}\mathsf{E}_\mathsf{P}(m \vee X \wedge n)Z &= \int_{(0,1]} [\mathsf{E}_\mathsf{P}(m \vee X \wedge n)Z_\lambda]\mu(d\lambda)\\ &\geq \int_{(0,1]} [\mathsf{E}_\mathsf{P}(m \vee X \wedge n)Z^*_\lambda(X)]\mu(d\lambda) = \mathsf{E}_\mathsf{P}(m \vee X \wedge n)Z^*(X), \quad m, n \in \mathbb{N}.\end{aligned}$$

Obviously,

$$\mathsf{E}_\mathsf{P} X Z^*(X) = \lim_{n\to\infty} \lim_{m\to-\infty} \mathsf{E}_\mathsf{P}(m \vee X \wedge n)Z^*(X) \tag{3.3}$$

and the same is true for $Z^*(X)$ replaced by Z. Thus, $\mathsf{E}_\mathsf{P} XZ \geq \mathsf{E}_\mathsf{P} XZ^*(X)$, so that $u_\mu(X) = \mathsf{E}_\mathsf{P} XZ^*(X)$.

(ii) Suppose first that $\mu(\{0\}) = 0$. Due to Lemma 3.1,

$$\mathsf{E}_\mathsf{P}(m \vee X \wedge n)Z^*(X) = \int_{(0,1]} u_\lambda(m \vee X \wedge n)\mu(d\lambda), \quad m, n \in \mathbb{N}.$$

Obviously,

$$\int_{(0,1]} u_\lambda(X)\mu(d\lambda) = \lim_{n\to\infty} \lim_{m\to-\infty} \int_{(0,1]} u_\lambda(m \vee X \wedge n)\mu(d\lambda).$$

Combining this with (3.3), we get

$$u_\mu(X) = \mathsf{E}_\mathsf{P} XZ^*(X) = \int_{(0,1]} u_\lambda(X)\mu(d\lambda).$$

Now, let $\mu(\{0\}) = \alpha > 0$. Then $\mu = \alpha\delta_0 + (1-\alpha)\widetilde{\mu}$ and it follows from Theorem 4.4 that $\mathcal{D}_\mu = \alpha\mathcal{D}_{\delta_0} + (1-\alpha)\mathcal{D}_{\widetilde{\mu}}$. If X is not bounded below, then, clearly, both sides of (3.2) are equal to $-\infty$. If X is bounded below, then $u_\mu(X) = \alpha u_0(X) + (1-\alpha)u_{\widetilde{\mu}}(X)$ and equality (3.2) for μ follows from (3.2) for $\widetilde{\mu}$, which was proved above. □

In order to provide another representation of Weighted V@R, let us consider the function

$$\Psi_\mu(x) = \begin{cases} \mu(\{0\}) + \int_0^x \int_{(y,1]} \lambda^{-1}\mu(d\lambda)dy, & x \in (0,1],\\ 0, & x = 0.\end{cases} \tag{3.4}$$

Clearly, Ψ_μ is concave, increasing, $\Psi_\mu(0) = 0$, and $\Psi_\mu(1) = 1$. Conversely, any such function can be represented in the form (3.4) with some probability measure μ (for details, see [16; Lem. 4.63] or [24; Lem. 1.50]).

Theorem 3.3. *For* $X \in L^0$,

$$u_\mu(X) = -\int_{-\infty}^0 \Psi_\mu(F(x))dx + \int_0^\infty (1 - \Psi_\mu(F(x)))dx,$$

where F *is the distribution function of* X, *and we use the convention* $\infty - \infty = -\infty$.

`Proof`. It is seen from (3.2) that

$$u_\mu(X)=\lim_{n\to\infty}\lim_{m\to-\infty}u_\mu(m\vee X\wedge n).$$

For bounded X, the statement of the theorem is known (see [16; Th. 4.64] or [24; Th. 1.51]), so that

$$u_\mu(m\vee X\wedge n)=-\int_{-\infty}^0\Psi_\mu(F_{mn}(x))dx+\int_0^\infty(1-\Psi_\mu(F_{mn}(x)))dx,$$

where F_{mn} is the distribution function of $m\vee X\wedge n$. In order to complete the proof, it is sufficient to note that

$$\begin{aligned}&-\int_{-\infty}^0\Psi_\mu(F(x))dx+\int_0^\infty(1-\Psi_\mu(F(x)))dx\\&=\lim_{n\to\infty}\lim_{m\to-\infty}\Bigl(-\int_{-\infty}^0\Psi_\mu(F_{mn}(x))dx+\int_0^\infty(1-\Psi_\mu(F_{mn}(x)))dx\Bigr).\end{aligned}$$

`Remark`. Some important regularity properties of μ can be expressed in terms of Ψ_μ. For example, $\Psi_\mu(0+)=0$ if and only if $\mu(\{0\})=0$ (this condition will be important in Sections 6 and 7; note also that this condition is equivalent to the lower semi-continuity of u_μ on L^∞); Ψ_μ is strictly concave if and only if $\operatorname{supp}\mu=[0,1]$ (this condition will be important in Sections 5 and 8).

4 Representation of the Determining Set

We begin with two auxiliary lemmas. The notation $\mu\preccurlyeq\nu$ means that ν dominates μ in the monotone order, i.e. $\mu((-\infty,x])\ge\nu((-\infty,x])$ for any x.

Lemma 4.1. *If $\mu\preccurlyeq\nu$, then $\mathcal{D}_\mu\supseteq\mathcal{D}_\nu$.*

`Proof`. There exist random variables ξ, η on some filtered probability space $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathsf{P}})$ such that $\operatorname{Law}\xi=\mu$, $\operatorname{Law}\eta=\nu$, and $\xi\le\eta$ a.s. (see [25; § 1.A]). We can write $u_\mu(X)=\mathsf{E}_{\widetilde{\mathsf{P}}}\varphi(\xi)$, $u_\nu(X)=\mathsf{E}_{\widetilde{\mathsf{P}}}\varphi(\eta)$, where $\varphi(\lambda)=u_\lambda(X)$. As φ is increasing, $u_\mu\le u_\nu$. Clearly, this implies that $\mathcal{D}_\mu\supseteq\mathcal{D}_\nu$. □

Lemma 4.2. *If μ_n tend to μ weakly and $\mu_n\preccurlyeq\mu$, then $\mathcal{D}_\mu=\bigcap_n\mathcal{D}_{\mu_n}$.*

`Proof`. Suppose that there exists $\mathsf{Q}_0\in\bigcap_n\mathcal{D}_{\mu_n}\setminus\mathcal{D}_\mu$. As $\mathcal{D}_\mu$ is L^1-closed, we can apply the Hahn-Banach theorem, which yields $X_0\in L^\infty$ such that $\mathsf{E}_{\mathsf{Q}_0}X_0<\inf_{\mathsf{Q}\in\mathcal{D}_\mu}\mathsf{E}_\mathsf{Q}X_0$. Thus, $\sup_n u_{\mu_n}(X_0)\le\mathsf{E}_{\mathsf{Q}_0}X_0<u_\mu(X_0)$. On the other hand, $u_{\mu_n}(X_0)\to u_\mu(X_0)$ since $u_\lambda(X_0)$ is continuous in λ. The obtained contradiction yields the inclusion $\mathcal{D}_\mu\supseteq\bigcap_n\mathcal{D}_{\mu_n}$. The reverse inclusion follows from the previous lemma. □

Lemma 4.3. *Let $\mu=\sum_{n=1}^N a_n\delta_{\lambda_n}$, where $\lambda_1>\dots>\lambda_N\ge0$. Then $\mathcal{D}_\mu=\sum_{n=1}^N a_n\mathcal{D}_{\lambda_n}$.*

`Proof`. Denote $\sum_n a_n\mathcal{D}_{\lambda_n}$ by $\mathcal{D}$. For any $\mathsf{Q}=\sum_n a_n\mathsf{Q}_n\in\mathcal{D}$ and any $X\in L^\infty$, we have

$$\mathsf{E}_\mathsf{Q}X=\sum_{n=1}^N a_n\mathsf{E}_{\mathsf{Q}_n}X\ge\sum_{n=1}^N a_n u_{\lambda_n}(X)=u_\mu(X),$$

so that $\mathcal{D} \subseteq \mathcal{D}_\mu$.

Let us prove the reverse inclusion. Clearly, $\mathcal{D}$ is convex. Let us prove that $\mathcal{D}$ is L^1-closed. Take a sequence $\xi^k = \sum_{n=1}^N a_n Z_n^k \in \mathcal{D}$ that converges in L^1 to a random variable ξ. Applying Komlos' principle of subsequences (see [19] or [24; Lem. 2.10]) to the random vectors $Z^k = (Z_1^k, \dots, Z_N^k)$, we get $\widetilde{Z}^k \in \operatorname{conv}(Z^k, Z^{k+1}, \dots)$ that converge P-a.s. to a random vector $\widetilde{Z}^\infty$. Clearly, $\sum_n a_n \widetilde{Z}_n^k \in \mathcal{D}$ and $\widetilde{Z}_n^k \xrightarrow{L^1} \widetilde{Z}_n^\infty$ for any $n = 1, \dots, N-1$ (note that $\widetilde{Z}_n^k \le \lambda_n^{-1}$). Hence,

$$\sum_{n=1}^{N-1} a_n \widetilde{Z}_n^k \xrightarrow[k\to\infty]{L^1} \sum_{n=1}^{N-1} a_n \widetilde{Z}_n^\infty,$$

and therefore, $\widetilde{Z}_N^k \xrightarrow{L^1} \widetilde{Z}_N^\infty$. As a result, $\xi = \sum_n a_n \widetilde{Z}_n^\infty \in \mathcal{D}$.

Now, assume that there exists $\mathsf{Q}_0 \in \mathcal{D}_\mu \setminus \mathcal{D}$. The Hahn-Banach theorem yields the existence of $X_0 \in L^\infty$ such that $\mathsf{E}_{\mathsf{Q}_0} X_0 < \inf_{\mathsf{Q}\in\mathcal{D}} \mathsf{E}_\mathsf{Q} X_0$. Thus, $u_\mu(X_0) < \inf_{\mathsf{Q}\in\mathcal{D}} \mathsf{E}_\mathsf{Q} X_0$. On the other hand, it is easy to check that $u_\mu(X) = \inf_{\mathsf{Q}\in\mathcal{D}} \mathsf{E}_\mathsf{Q} X$ for any $X \in L^\infty$. The obtained contradiction shows that $\mathcal{D}_\mu \subseteq \mathcal{D}$. □

Theorem 4.4. *We have*

$$\mathcal{D}_\mu = \left\{ \int_{[0,1]} Z_\lambda \mu(d\lambda) : Z(\lambda,\omega) \text{ is jointly measurable and } Z_\lambda \in \mathcal{D}_\lambda \text{ for any } \lambda \in [0,1] \right\}. \tag{4.1}$$

`Proof`. Denote the right-hand side of (4.1) by $\mathcal{D}$. Set

$$\mu_n = \sum_{k=1}^{n-1} \mu\Big(\Big[\frac{k-1}{n}, \frac{k}{n}\Big)\Big)\delta_{\frac{k-1}{n}} + \mu\Big(\Big[\frac{n-1}{n}, 1\Big]\Big)\delta_1, \quad n \in \mathbb{N}.$$

Due to Lemma 4.3, $\mathcal{D} \subseteq \mathcal{D}_{\mu_n}$, and by Lemma 4.2, $\mathcal{D} \subseteq \mathcal{D}_\mu$.

Let us prove the reverse inclusion. Clearly, $\mathcal{D}$ is convex. Arguing in the same way as in the previous proof, we conclude that $\mathcal{D}$ is L^1-closed. Take

$$\mu_n = \mu\Big(\Big[0, \frac{1}{n}\Big]\Big)\delta_{\frac{1}{n}} + \sum_{k=2}^{n} \mu\Big(\Big(\frac{k-1}{n}, \frac{k}{n}\Big]\Big)\delta_{\frac{k}{n}}, \quad n \in \mathbb{N}.$$

Due to Lemma 4.3, $\mathcal{D}_{\mu_n} \subseteq \mathcal{D}$, and therefore, $u_{\mu_n} \ge u$, where $u(X) = \inf_{\mathsf{Q}\in\mathcal{D}} \mathsf{E}_\mathsf{Q} X$. As $u_{\mu_n}(X) \to u_\mu(X)$ for any $X \in L^\infty$, we get $u_\mu \ge u$. Using the same argument as in the proof of Lemma 4.3, we conclude that $\mathcal{D}_\mu \subseteq \mathcal{D}$. □

Now, we pass on to another representation of $\mathcal{D}_\mu$. The key step for establishing it is the following lemma.

Lemma 4.5. *Let $\lambda_1 > \dots > \lambda_N \ge 0$, $a_1, \dots, a_N \in (0,\infty)$. Then a random variable Z can be represented as $\sum_{n=1}^N a_n Z_n$ with $Z_n \in \mathcal{D}_{\lambda_n}$ if and only if $Z \ge 0$, $\mathsf{E}_\mathsf{P} Z = \sum_{n=1}^N a_n$, and*

$$\mathsf{E}_\mathsf{P}\Big(Z - \sum_{k=1}^n a_k \lambda_k^{-1}\Big)^+ \le \sum_{k=n+1}^N a_k, \quad n = 1, \dots, N. \tag{4.2}$$

`Proof`. The "only if" part follows from the inequality

$$\Big(Z-\sum_{k=1}^{n}a_k\lambda_k^{-1}\Big)^+\le\sum_{k=n+1}^{N}a_kZ_k,\quad n=1,\dots,N.$$

Let us prove the "if" part. We proceed by the induction in N. The base of induction is obvious, so we assume that the statement is true for $N-1$ and will prove it for N.

First, suppose that $\lambda_N=0$. Choose $h\in\mathbb{R}_+$ such that, for $Y=Z\wedge h$, we have $\mathsf{E}_\mathsf{P}Y=\sum_{n=1}^{N-1}a_n$. Then

$$\mathsf{E}_\mathsf{P}\Big(Y-\sum_{k=1}^{n}a_k\lambda_k^{-1}\Big)^+\le\sum_{k=n+1}^{N}a_n-a_N=\sum_{k=n+1}^{N-1}a_n,\quad n=1,\dots,N-1.$$

By taking $Z_N=a_N^{-1}(Z-Y)$ and applying the induction assumption to Y, we get the desired statement.

Now, suppose that $\lambda_N>0$. Let us first prove the inequality

$$\mathsf{E}_\mathsf{P}Z\wedge a_N\lambda_N^{-1}\ge a_N. \tag{4.3}$$

Assume the contrary. Then $\mathsf{E}_\mathsf{P}a_N\lambda_N^{-1}I(Z\ge a_N\lambda_N^{-1})<a_N$, and hence, $\mathsf{P}(Z\ge a_N\lambda_N^{-1})<\lambda_N$. It follows from (4.2) that $Z\le\sum_{n=1}^{N}a_n\lambda_n^{-1}$, so that

$$\mathsf{E}_\mathsf{P}(Z-a_N\lambda_N^{-1})^+\le\mathsf{P}(Z\ge a_N\lambda_N^{-1})\sum_{n=1}^{N-1}\lambda_n^{-1}a_n<\lambda_N\sum_{n=1}^{N-1}\lambda_n^{-1}a_n<\sum_{n=1}^{N-1}a_n.$$

But then

$$\mathsf{E}_\mathsf{P}Z=\mathsf{E}_\mathsf{P}(Z\wedge a_N\lambda_N^{-1})+\mathsf{E}_\mathsf{P}(Z-a_N\lambda_N^{-1})^+<\sum_{n=1}^{N}a_n,$$

which is a contradiction.

Consider the function

$$f(h)=\mathsf{E}_\mathsf{P}((Z-a_N\lambda_N^{-1})^+\vee h)\wedge Z,\quad h\in\big[0,\textstyle\sum_{n=1}^{N-1}a_n\lambda_n^{-1}\big].$$

It follows from (4.3) that

$$f(0)=\mathsf{E}_\mathsf{P}(Z-a_N\lambda_N^{-1})^+=\mathsf{E}_\mathsf{P}Z-\mathsf{E}_\mathsf{P}Z\wedge a_N\lambda_N^{-1}\le\sum_{n=1}^{N-1}a_n.$$

It follows from (4.2) that

$$f\Big(\sum_{n=1}^{N-1}a_n\lambda_n^{-1}\Big)=\mathsf{E}_\mathsf{P}Z\wedge\sum_{n=1}^{N-1}a_n\lambda_n^{-1}=\mathsf{E}_\mathsf{P}Z-\mathsf{E}_\mathsf{P}\Big(Z-\sum_{n=1}^{N-1}a_n\lambda_n^{-1}\Big)^+\ge\sum_{n=1}^{N-1}a_n.$$

Thus, there exists $h_0\in\big[0,\sum_{n=1}^{N-1}a_n\lambda_n^{-1}\big]$ such that $f(h_0)=\sum_{n=1}^{N-1}a_n$. Set $Y=((Z-a_N\lambda_N^{-1})^+\vee h_0)\wedge Z$. Clearly, $Z_N:=a_N^{-1}(Z-Y)$ belongs to $\mathcal{D}_{\lambda_N}$. Take $n\in\{1,\dots,N-1\}$ and let us prove that

$$\mathsf{E}_\mathsf{P}\Big(Y-\sum_{k=1}^{n}a_k\lambda_k^{-1}\Big)^+\le\sum_{k=n+1}^{N-1}a_k. \tag{4.4}$$

First, consider the case, where $\sum_{k=1}^{n} a_k\lambda_k^{-1} \geq h_0$. Then, on the set $A := \left\{Y > \sum_{k=1}^{n} a_k\lambda_k^{-1}\right\}$ we have $Z - Y = a_N\lambda_N^{-1}$. If $\mathsf{P}(A) = 0$, then (4.4) is trivially satisfied, so we assume that $\mathsf{P}(A) > 0$. As $Y \leq \sum_{n=1}^{N-1} a_n\lambda_n^{-1}$, we have

$$\mathsf{E}_{\mathsf{P}}\left(Y - \sum_{k=1}^{n} a_k\lambda_k^{-1}\right)^+ \leq \sum_{k=n+1}^{N-1} a_k\lambda_k^{-1}\mathsf{P}(A).$$

Combining this with the equality $\mathsf{E}_{\mathsf{P}}(Z-Y)I_A = a_N\lambda_N^{-1}\mathsf{P}(A)$, we get

$$\frac{\mathsf{E}_{\mathsf{P}}\left(Y - \sum_{k=1}^{n} a_k\lambda_k^{-1}\right)^+}{\mathsf{E}_{\mathsf{P}}(Z-Y)I_A} \leq \frac{\sum_{k=n+1}^{N-1} a_k}{a_N}.$$

Taking into account the inequality

$$\mathsf{E}_{\mathsf{P}}\left[(Z-Y)I_A + \left(Y - \sum_{k=1}^{n} a_k\lambda_k^{-1}\right)^+\right] \leq \mathsf{E}_{\mathsf{P}}\left(Z - \sum_{k=1}^{n} a_k\lambda_k^{-1}\right)^+ \leq \sum_{k=n+1}^{N} a_k,$$

we get (4.4).

Now, suppose that $\sum_{k=1}^{n} a_k\lambda_k^{-1} < h_0$. Consider the set

$$A := \left\{Y > \sum_{k=1}^{n} a_k\lambda_k^{-1}\right\} = \left\{Z > \sum_{k=1}^{n} a_k\lambda_k^{-1}\right\}.$$

As $Z = Y$ on A^c, we get $\mathsf{E}_{\mathsf{P}}(Z-Y)I_A = a_N$. Consequently,

$$\mathsf{E}_{\mathsf{P}}\left(Y - \sum_{k=1}^{n} a_k\lambda_k^{-1}\right)^+ = \mathsf{E}_{\mathsf{P}}\left(Z - \sum_{k=1}^{n} a_k\lambda_k^{-1}\right)^+ - a_N \leq \sum_{k=n+1}^{N-1} a_n.$$

Thus, (4.4) is proved and an application of the induction hypothesis to Y completes the proof. □

Lemma 4.6. *A random variable Z belongs to $\mathcal{D}_\mu$ if and only if $Z \geq 0$, $\mathsf{E}_{\mathsf{P}}Z = 1$, and*

$$\mathsf{E}_{\mathsf{P}}\left(Z - \int_{(x,1]} \lambda^{-1}\mu(d\lambda)\right)^+ \leq \mu([0,x]), \quad x \in [0,1]. \tag{4.5}$$

`Proof`. Let us prove the "only if" part. Take $Z \in \mathcal{D}_\mu$ and $x \in (0,1]$. Choose discrete measures μ_n such that $\mu_n \preccurlyeq \mu$, $\mu_n([0,x]) = \mu([0,x])$, and μ_n converge weakly to μ. By Lemma 4.1, $Z \in \mathcal{D}_{\mu_n}$. By Lemma 4.5,

$$\mathsf{E}_{\mathsf{P}}\left(Z - \int_{(x,1]} \lambda^{-1}\mu_n(d\lambda)\right)^+ \leq \mu_n([0,x]).$$

Passing on to the limit as $n \to \infty$, we prove (4.5) for $x \in (0,1]$. Passing on to the limit as $x \downarrow 0$, we extend (4.5) to $x = 0$.

Now, let us prove the "if" part. Set $\mu_n = \sum_{k=1}^{n} a_k^n \delta_{\lambda_k^n}$, where $\lambda_k^n = \frac{n-k}{n}$, $a_1^n = \mu([\lambda_1^n, 1])$, $a_k^n = \mu([\lambda_k^n, \lambda_{k-1}^n))$, $k = 2, \dots, n$. Then

$$\mathsf{E}_{\mathsf{P}}\left(Z - \sum_{i=1}^{k} a_i^n(\lambda_i^n)^{-1}\right)^+ \leq \mathsf{E}_{\mathsf{P}}\left(Z - \int_{[\lambda_k^n,1]} \lambda^{-1}\mu(d\lambda)\right)^+ \leq \mu([0,\lambda_k^n)) = \mu_n([0,\lambda_k^n)) = \sum_{i=k+1}^{n} a_i^n$$

(the second inequality here follows from (4.5) by passing on to the limit $x_m \uparrow \lambda_k^n$). Due to Lemma 4.5, $Z \in \mathcal{D}_{\mu_n}$ for any n. By Lemma 4.2, $Z \in \mathcal{D}_\mu$. □

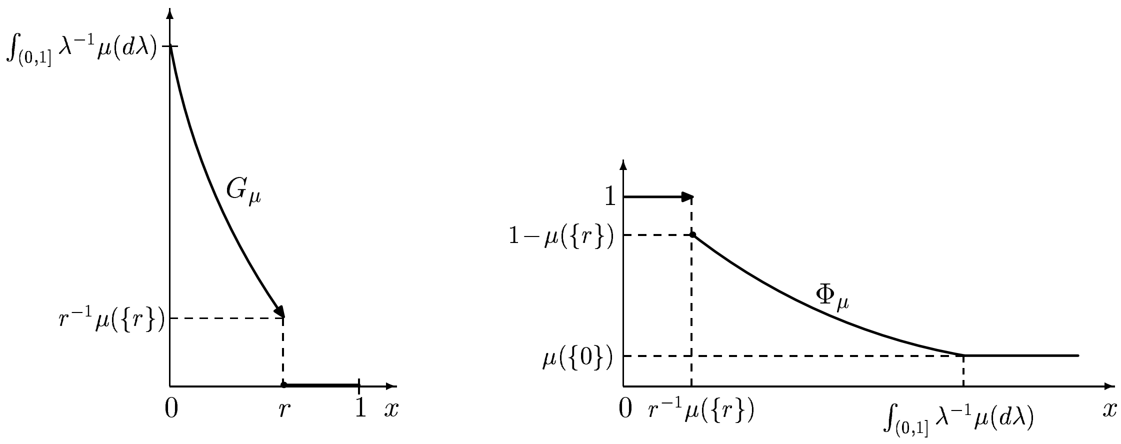
For a probability measure μ on $[0,1]$, we introduce the notation

$$\begin{aligned} F_\mu(x) &= \mu([0,x]), \quad x \in [0,1], \\ G_\mu(x) &= \int_{(x,1]} \lambda^{-1}\mu(d\lambda), \quad x \in [0,1], \\ G_\mu^{-1}(x) &= \inf\{y \in [0,1] : G_\mu(y) \le x\}, \quad x \in \mathbb{R}_+, \\ \Phi_\mu^0(x) &= F_\mu(G_\mu^{-1}(x)), \quad x \in \mathbb{R}_+. \end{aligned}$$

Clearly, Φ_μ^0 is decreasing, so that there exists its right-continuous modification, which we denote by Φ_μ (Φ_μ^0 need not be right-continuous as seen from the example $\mu = \frac{1}{2}\mu_L + \frac{1}{2}\delta_1$, where μ_L is the Lebesgue measure). Set

$$r = \inf\{x \in [0,1] : \mu([0,x]) = 1\}. \tag{4.6}$$

Note that $\Phi_\mu = 1$ on $[0, r^{-1}\mu(\{r\}))$, $\Phi_\mu(r^{-1}\mu(\{r\})) = 1 - \mu(\{r\})$, $\lim_{x\to\infty} \Phi_\mu(x) = \mu(\{0\})$, and $\Phi_\mu = \mu(\{0\})$ on $\big[\int_{(0,1]} \lambda^{-1}\mu(d\lambda), \infty\big)$.



`Figure 1`. The form of G_μ and Φ_μ

Theorem 4.7. *We have*

$$\mathcal{D}_\mu = \{Z \in L^0 : Z \ge 0,\ \mathsf{E_P} Z = 1, \text{ and } \mathsf{E_P}(Z - x)^+ \le \Phi_\mu(x) \text{ for any } x \in [0,1]\}.$$

`Proof`. Clearly, we can replace in this statement Φ_μ by Φ_μ^0 and vice versa, so we will prove it for Φ_μ^0. Let $Z \in \mathcal{D}_\mu$. Then, due to Lemma 4.6,

$$\mathsf{E_P}(Z - x)^+ \le \mathsf{E_P}(Z - G_\mu(G_\mu^{-1}(x)))^+ \le \mu([0, G_\mu^{-1}(x)]) = \Phi^0(x), \quad x \in \mathbb{R}_+.$$

The reverse inclusion follows from the line

$$\mathsf{E_P}(Z - G_\mu(x))^+ \le \Phi_\mu^0(G_\mu(x)) = F_\mu \circ G_\mu^{-1} \circ G_\mu(x) = F_\mu(x) = \mu([0,x]), \quad x \in [0,1]$$

and Lemma 4.6. □

We conclude this section by the third representation of $\mathcal{D}_\mu$. It was obtained by Carlier and Dana [7] (the proof can also be found in [16; Th. 4.73] or [24; Th. 1.53]).

Theorem 4.8. *We have*

$$\begin{aligned}\mathcal{D}_\mu &= \{\mathsf{Q} \in \mathcal{P} : \mathsf{Q}(A) \le \Psi_\mu(\mathsf{P}(A)) \text{ for any } A \in \mathcal{F}\}\\ &= \Big\{Z \in L^0 : Z \ge 0,\ \mathsf{E}_\mathsf{P} Z = 1, \text{ and } \int_{1-x}^1 q_s(Z)ds \le \Psi_\mu(x) \text{ for any } x \in [0,1]\Big\},\end{aligned}$$

where q_s is the s-quantile and Ψ_μ is given by (3.4).

To conclude this section, we compare the representations of Theorems 4.7 and 4.8 by an example. Let $\Omega = [0,1]$ endowed with the Lebesgue measure. Let Z be a decreasing function of ω. Then $\mathsf{E}_\mathsf{P}(Z-x)^+$ is the shaded area in the left graph of Figure 2, while $\int_{1-y}^1 q_s(X)ds$ is the shaded area in the right graph of Figure 2.

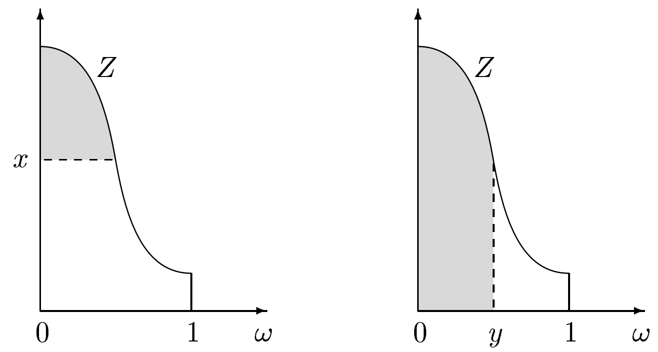


Figure 2

5 Strict Diversification and Optimization

Let us introduce the notation

$$L^1_\mu = \{X \in L^0 : u_\mu(X) > -\infty \text{ and } u_\mu(-X) > -\infty\}.$$

Theorem 5.1. *Suppose that* $\operatorname{supp}\mu = [0,1]$. *For $X, Y \in L^1_\mu$, we have*

$$u_\mu(X+Y) > u_\mu(X) + u_\mu(Y) \tag{5.1}$$

if and only if X and Y are not comonotone.

`Proof`. The "only if" part for bounded X and Y is a consequence of Theorem 2.9 (i). The statement for unbounded X and Y is obtained by passing on to the limit with the help of the representation

$$L^1_\mu = \Big\{X \in L^0 : \lim_{n\to\infty} \sup_{\mathsf{Q}\in\mathcal{D}_\mu} \mathsf{E}_\mathsf{Q}|X|I(|X|>n) = 0\Big\},$$

which was proved in [10; Subsect. 1.2].

Let us prove the "if" part. Suppose that (5.1) is not true. Combining representation (3.2) with the property $u_\lambda(X+Y) \ge u_\lambda(X) + u_\lambda(Y)$, we conclude that $u_\lambda(X+Y) = u_\lambda(X) + u_\lambda(Y)$ for μ-a.e. $\lambda \in [0,1]$. As $\operatorname{supp}\mu = [0,1]$ and the functions u_λ are continuous in λ, we get $u_\lambda(X+Y) = u_\lambda(X) + u_\lambda(Y)$ for any $\lambda \in [0,1]$. In view of

Proposition 2.6 (i), for any $\lambda \in (0,1]$, there exists $Z_\lambda^* \in \mathcal{D}_\lambda$ such that $\mathsf{E}_\mathsf{P} X Z_\lambda^* = u_\lambda(X)$, $\mathsf{E}_\mathsf{P} Y Z_\lambda^* = u_\lambda(Y)$. It is seen from Proposition 2.6 (i) that this is possible only if

$$\begin{aligned}
&\mathsf{P}\big((X,Y) \in (-\infty, q_\lambda(X)) \times (q_\lambda(Y), \infty)\big) = 0, \quad \lambda \in (0,1],\\
&\mathsf{P}\big((X,Y) \in (q_\lambda(X), \infty) \times (-\infty, q_\lambda(Y))\big) = 0, \quad \lambda \in (0,1].
\end{aligned}$$

From this it is easy to deduce that $\mathsf{P}((X,Y) \in f((0,1])) = 1$, where $f(\lambda) = (q_\lambda(X), q_\lambda(Y))$. Thus, X and Y are comonotone. □

`Remark`. Without the condition $\operatorname{supp} \mu = [0,1]$, the theorem does not hold. In particular, it does not hold for Tail V@R. Let us remark that this problem as well as the problem to prove (5.1) for independent X and Y were proposed at the Fourth Kolmogorov Students' Competition on Probability Theory (see [11]).

Property (5.1) can be called the *strict diversification property.* It holds, in particular, if X and Y are independent or if X and Y have a joint density (with respect to the Lebesgue measure). The strict diversification property leads to the uniqueness of a solution of several optimization problems based on coherent risk measures that were considered in [10]. Let us briefly describe two of them.

Let $S_0 \in \mathbb{R}^d$ be a vector of initial prices of several assets and S_1 be a d-dimensional random vector of their terminal (discounted) prices. Let $H \subseteq \mathbb{R}^d$ be a convex set of possible trading strategies, so that the (discounted) income of a strategy $h \in H$ is $\langle h, S_1 - S_0\rangle$. Consider the problem

$$\mathrm{RAROC}(\langle h, S_1 - S_0\rangle) \xrightarrow[h \in H]{} \max, \tag{5.2}$$

where

$$\mathrm{RAROC}(X) = \begin{cases} +\infty & \text{if } \mathsf{E}_\mathsf{P} X > 0 \text{ and } u(X) \ge 0,\\ \dfrac{\mathsf{E}_\mathsf{P} X}{-u(X)} & \text{otherwise} \end{cases} \tag{5.3}$$

with the convention $\frac{0}{0} = 0$, $\frac{\infty}{\infty} = 0$. Here u is a coherent utility function. The solution of this problem was presented in [10; Subsect. 3.2]. Here we give a sufficient condition for the uniqueness.

Corollary 5.2. *Let $u = u_\mu$ with $\operatorname{supp} \mu = [0,1]$. Suppose that each component of S_1 belongs to L_μ^1, S_1 has a density, and $\sup_{h \in H} \mathrm{RAROC}(\langle h, S_1 - S_0\rangle) < \infty$. Then a solution of* (5.2) (*if it exists*) *is unique up to multiplication by a positive constant.*

`Proof`. Suppose that there exist two solutions h_*^1 and h_*^2 that are not collinear. Denote $S_1 - S_0$ by X. After multiplying h_*^1 by a positive constant, we can assume that $\mathsf{E}_\mathsf{P}\langle h_*^1, X\rangle = \mathsf{E}_\mathsf{P}\langle h_*^2, X\rangle$. As h_*^1 and h_*^2 solve (5.2), $u_\mu(\langle h_*^1, X\rangle) = u_\mu(\langle h_*^2, X\rangle)$. Consider $h_* = \frac{h_*^1 + h_*^2}{2}$. Then $\mathsf{E}_\mathsf{P}\langle h_*, X\rangle = \mathsf{E}_\mathsf{P}\langle h_*^i, X\rangle$, while it follows from (5.1) that $u_\mu(\langle h_*, X\rangle) > u_\mu(\langle h_*^i, X\rangle)$. Thus, $\mathrm{RAROC}(\langle h_*, X\rangle) > \mathrm{RAROC}(\langle h_*^i, X\rangle)$, which is a contradiction. □

`Remark`. Without the assumption $\operatorname{supp} \mu = [0,1]$, the statement above is not true (see [10; Ex. 3.2]).

Consider now a single-agent optimization problem. Thus, in addition to the objects introduced above, we have a random variable W, which means the current endowment of some agent. Consider the problem

$$u(W + \langle h, S_1 - S_0\rangle) \xrightarrow[h \in H]{} \max. \tag{5.4}$$

Let us remark that one can also consider the problem $\mathrm{RAROC}(\langle h, S_1 - S_0\rangle) \to \max$, but it can be reduced to (5.4) by the technique of Lagrange multipliers. The solution of (5.4) was presented in [10; Subsect. 3.5].

Corollary 5.3. *Let $u = u_\mu$ with $\operatorname{supp}\mu = [0,1]$. Suppose that each component of S_1 belongs to L^1_μ, S_1 has a density, $W \in L^1_\mu$, and $\sup_{h\in H} u_\mu(W + \langle h, S_1 - S_0\rangle) < \infty$. Then a solution of (5.4) (if it exists) is unique.*

This statement is an immediate consequence of Theorem 5.1.

6 Minimal Extreme Measure and Capital Allocation

The following definition was introduced in [10].

Definition 6.1. Let u be a coherent utility function on L^0 with the determining set $\mathcal{D}$. Let $X \in L^0$. We call a measure $\mathsf{Q} \in \mathcal{D}$ an *extreme measure* for X if $\mathsf{E}_\mathsf{Q} X = u(X)$.
The set of extreme measures for $u = u_\mu$ will be denoted by $\mathcal{X}_\mu(X)$.

It was proved in [10; Subsect. 1.2] that if $\mu(\{0\}) = 0$ and $X \in L^1_\mu$, then $\mathcal{X}_\mu(X) \neq \emptyset$.

Proposition 6.2. *Suppose that $\mu(\{0\}) = 0$ and $X \in L^1_\mu$. Then an element $Z = \int_{(0,1]} Z_\lambda \mu(d\lambda) \in \mathcal{D}_\mu$ (here we use the representation of $\mathcal{D}_\lambda$ provided by Theorem 4.4) belongs to $\mathcal{X}_\mu$ if and only if*

$$Z_\lambda = \begin{cases} \lambda^{-1} & \text{on } \{X < q_\lambda(X)\}, \\ 0 & \text{on } \{X > q_\lambda(X)\} \end{cases}$$

for μ-a.e. λ.

`Proof`. For $Z = \int_{(0,1]} Z_\lambda \mu(d\lambda) \in \mathcal{D}$, we have

$$\begin{aligned} \mathsf{E}_\mathsf{P} XZ &= \lim_{m\to-\infty} \lim_{n\to\infty} \mathsf{E}_\mathsf{P}(m \vee X \wedge n) Z \\ &= \lim_{m\to-\infty} \lim_{n\to\infty} \int_{(0,1]} [\mathsf{E}_\mathsf{P}(m \vee X \wedge n) Z_\lambda] \mu(d\lambda) \\ &= \int_{(0,1]} (\mathsf{E}_\mathsf{P} X Z_\lambda) \mu(d\lambda). \end{aligned} \tag{6.1}$$

The inclusion $X \in L^1_\mu$ implies that the function $\lambda \mapsto \mathsf{E}_\mathsf{P} X Z_\lambda$ is μ-integrable. An application of Proposition 2.6 (i) completes the proof. □

It is seen from the above proposition that if X has a continuous distribution, then $\mathcal{X}_\mu(X)$ consists of a unique element $Z = g(X)$, where

$$g(x) = \int_{[F(x),1]} \lambda^{-1} \mu(d\lambda), \quad x \in \mathbb{R} \tag{6.2}$$

and F denotes the distribution function of X.

If X has atoms, then, clearly, $\mathcal{X}_\mu$ need not be a singleton. However, it turns out that there exists a minimal element of $\mathcal{X}_\mu(X)$ with respect to the convex stochastic order. (For other applications of this order in financial mathematics, see [21], [22], [23]).

Theorem 6.3. *Suppose that $\mu(\{0\})=0$ and $X\in L^1_\mu$. Let*

$$Z^*(X)=\int_{(0,1]} Z^*_\lambda(X)\mu(d\lambda),$$

where $Z^*_\lambda(X)$ is defined by (3.1). Then, for any $Z\in\mathcal{X}_\mu(X)$ and any convex function $f:\mathbb{R}_+\to\mathbb{R}_+$, we have $\mathsf{E}_\mathsf{P} f(Z^*(X))\le \mathsf{E}_\mathsf{P} f(Z)$. Moreover, $Z^*(X)$ is the unique element of $\mathcal{X}_\mu(X)$ with this property.

`Proof`. Take an arbitrary $Z=\int_{(0,1]} Z_\lambda\mu(d\lambda)\in\mathcal{X}_\mu(X)$. It follows from Proposition 6.2 that $Z^*_\lambda(X)=\mathsf{E}_\mathsf{P}(Z_\lambda\mid X)$ for μ-a.e. λ. By Fubini's theorem, $Z^*(X)=\mathsf{E}_\mathsf{P}(Z\mid X)$. An application of Jensen's inequality yields the first statement.

Now, suppose that there exists another minimal (in the convex order) element Z' of $\mathcal{X}_\mu(X)$. Then $\widetilde{Z}:=\frac{Z^*(X)+Z'}{2}$ belongs to $\mathcal{X}_\mu(X)$ and, for a strictly convex function f with a linear growth, we get $\mathsf{E}_\mathsf{P} f(\widetilde{Z})<\mathsf{E}_\mathsf{P} f(Z^*(X))=\mathsf{E}_\mathsf{P} f(Z')$, which is a contradiction. $\square$

Definition 6.4. We will call the measure $\mathsf{Q}^*_\mu(X)=Z^*(X)\mathsf{P}$ the *minimal extreme measure* for X.

The minimal extreme measure admits a representation similar to (6.2). Let F denote the distribution function of X. Then, for any $\lambda\in(0,1]$, $Z^*_\lambda(X)=g_\lambda(X)$, where

$$g_\lambda(x)=\frac{1-\lambda^{-1}F(x-)}{F(x)-F(x-)}I(F(x-)<\lambda<F(x))+\lambda^{-1}I(\lambda\ge F(x)),\quad x\in\mathbb{R}.$$

Hence, $Z^*(X)=g(X)$, where

$$g(x)=\int_{(F(x-),F(x))}\frac{1-\lambda^{-1}F(x-)}{F(x)-F(x-)}\mu(d\lambda)+\int_{[F(x),1]}\lambda^{-1}\mu(d\lambda),\quad x\in\mathbb{R}.$$

The following statement will be used in financial applications below.

Theorem 6.5. *Suppose that $\mu(\{0\})=0$ and $X,Y\in L^1_\mu$. Let (ξ_n) be a sequence of random variables such that $\xi_n\in L^1_\mu$, each ξ_n is independent of (X,Y), and $\xi_n\xrightarrow{\mathsf{P}}0$. Then*

$$\mathsf{E}_{\mathsf{Q}^*_\mu(X+\xi_n)}Y\underset{n\to\infty}{\longrightarrow}\mathsf{E}_{\mathsf{Q}^*_\mu(X)}Y.$$

`Proof`. Denote $q_\lambda=q_\lambda(X)$, $q^n_\lambda=q_\lambda(X+\xi_n)$. Then, for $\lambda\in(0,1]$,

$$Z^*_\lambda(X)=\begin{cases}\lambda^{-1} & \text{on } \{X<q_\lambda\},\\ c_\lambda & \text{on } \{X=q_\lambda\},\\ 0 & \text{on } \{X>q_\lambda\},\end{cases}\qquad Z^*_\lambda(X+\xi_n)=\begin{cases}\lambda^{-1} & \text{on } \{X+\xi_n<q^n_\lambda\},\\ c^n_\lambda & \text{on } \{X+\xi_n=q^n_\lambda\},\\ 0 & \text{on } \{X+\xi_n>q^n_\lambda\}.\end{cases}$$

Fix $\lambda\in(0,1]$. By Fubini's theorem, $\mathsf{E}_\mathsf{P}(Z^*_\lambda(X+\xi_n)\mid X,Y)=f^n_\lambda(X)$, where

$$f^n_\lambda(x)=\lambda^{-1}F_n(q^n_\lambda-x)+c^n_\lambda\Delta F_n(q^n_\lambda-x),\quad x\in\mathbb{R}$$

and $F_n(x)=\mathsf{P}(\xi_n<x)$, $\Delta F_n(x)=\mathsf{P}(\xi_n=x)$. Obviously, $q^n_\lambda\to q_\lambda$, and therefore, $f^n_\lambda\to\lambda^{-1}$ on $(-\infty,q_\lambda)$, $f^n_\lambda\to 0$ on (q_λ,∞). Employing the normalization condition $\mathsf{E}_\mathsf{P} f^n_\lambda(X)=1$, we conclude that $f^n_\lambda(q_\lambda)\to c_\lambda$. Thus, $f^n_\lambda(X)\xrightarrow{\text{a.s.}}Z^*_\lambda(X)$. As $0\le f^n_\lambda\le\lambda^{-1}$, we get

$$\mathsf{E}_\mathsf{P} YZ^*_\lambda(X+\xi_n)=\mathsf{E}_\mathsf{P} Yf^n_\lambda(X)\underset{n\to\infty}{\longrightarrow}\mathsf{E}_\mathsf{P} YZ^*_\lambda(X),\quad\lambda\in(0,1].$$

Note that $u_\lambda(Y) \le \mathsf{E}_\mathsf{P} Y Z_\lambda^*(X+\xi_n) \le -u_\lambda(-Y)$. Furthermore, it follows from the inclusion $Y \in L^1_\mu$ and representation (3.2) that the functions $\lambda \mapsto u_\lambda(Y)$ and $\lambda \mapsto u_\lambda(-Y)$ are μ-integrable. Applying now (6.1), we complete the proof. $\Box$

`Remark`. Without the assumption that ξ_n is independent of (X,Y), the theorem does not hold. As an example, consider $X=0$, $\xi_n = Y/n$. Then $\mathsf{Q}_\mu^*(X) = \mathsf{P}$, while $\mathsf{Q}_\mu^*(\xi_n) = \mathsf{Q}_\mu^*(Y)$.

Let us now present a financial application of the notion of the minimal extreme measure. It is related to the *capital allocation problem.* Delbaen [14; Sect. 9] proposed the following formulation of this problem. Let $X^1,\dots,X^d$ be random variables meaning the (discounted) incomes produced by several components of a firm. Let ρ be a coherent risk measure. A *capital allocation between* $X^1,\dots,X^d$ is a vector $x^1,\dots,x^d$ such that

$$\rho\Bigl(\sum_{i=1}^d X^i\Bigr) = \sum_{i=1}^d x^i, \tag{6.3}$$

$$\forall h^1,\dots,h^d \in \mathbb{R}_+,\quad \rho\Bigl(\sum_{i=1}^d h^i X^i\Bigr) \ge \sum_{i=1}^d h^i x^i. \tag{6.4}$$

From the financial point of view, x^i means the contribution of the i-th component to the total risk of the firm, or, equivalently, the capital that should be allocated to this component. In order to illustrate the meaning of (6.4), consider the example $h^i = I(i \in J)$, where J is a subset of $\{1,\dots,d\}$. Then (6.4) means that the capital allocated to a part of the firm does not exceed the risk carried by that part.

It was proved in [10; Subsect. 1.4] under the assumption $u(X^i) > -\infty$, $u(-X^i) > -\infty$ that the set of capital allocations has the form

$$\Bigl\{-\mathsf{E}_\mathsf{Q}(X^1,\dots,X^d) : \mathsf{Q} \in \mathcal{X}\Bigl(\sum_{i=1}^d X^i\Bigr)\Bigr\}, \tag{6.5}$$

where $\mathcal{X}$ denotes the set of extreme measures corresponding to u.

Suppose now that $u = u_\mu$ with $\mu(\{0\}) = 0$. It is seen from Proposition 6.1 that if $\sum_i X^i$ has a continuous distribution, then a capital allocation is unique. But in general, this is not the case. For example, if $X^2 = -X^1$, then the set of capital allocations is the interval $[a,b]$ in $\mathbb{R}^2$, where $a = (-u_\mu(X^1), u_\mu(X^1))$, $b = (u_\mu(-X^1), -u_\mu(-X^1))$.

However, if $u = u_\mu$ with $\mu(\{0\}) = 0$, then there exists a particular element of (6.5), namely $x_0 = -\mathsf{E}_{\mathsf{Q}_\mu^*(\sum X^i)}(X^1,\dots,X^d)$ (for the example considered above, $x_0 = (-\mathsf{E}_\mathsf{P} X^1, \mathsf{E}_\mathsf{P} X^1)$). We call x_0 the *central solution* of the capital allocation problem. Its role is as follows. Let us disturb X^i, i.e. we pass from X^i to $\widetilde{X}_n^i = X^i + \xi_n^i$, where each ξ_n^i is independent of $(X^1,\dots,X^d)$, $\xi_n^i \in L^1_\mu$, and $\xi_n^i \xrightarrow{\mathsf{P}} 0$. If $\sum_i \xi_n^i$ has a continuous distribution, then $\mathcal{X}\bigl(\sum_i X^i\bigr)$ is a singleton, so that the capital allocation x_n between $\widetilde{X}_n^1,\dots,\widetilde{X}_n^d$ is unique. By Theorem 6.5, $x_n \to x_0$.

7 Pricing in an Option-Based Model

Let S_0 be the initial price of some asset and S_1 be a positive random variable meaning its terminal (discounted) price. Let $\mathbb{K} \subseteq \mathbb{R}_+$ be the set of strike prices of traded European call options on this asset with maturity 1 and let $\varphi(K)$, $K\in\mathbb{K}$ be the price at time 0 of an option that pays $(S_1-K)^+$ at time 1. The set

$$A=\left\{\sum_{n=1}^N h_n[(S_1-K_n)^+-\varphi(K_n)]:N\in\mathbb{N},\ K_n\in\mathbb{K},\ h_n\in\mathbb{R}\right\}$$

means the set of incomes that can be obtained in the model under consideration (we assume that $0\in\mathbb{K}$, which corresponds to the possibility of trading the underlying asset).

According to [10], we say that the model satisfies the *No Good Deals* (*NGD*) condition if there exists no $X\in A$ with $u(X)>0$, where u is a fixed coherent utility function.

Now, let $F\in L^0$ be the payoff of some contingent claim. According to [10], we say that a real number z is an *NGD price* of F if there exist no $X\in A$, $h\in\mathbb{R}$ with $u(X+h(F-z))>0$. The set of NGD prices will be denoted by $I_{NGD}(F)$.

It was proved in [10; Subsect. 2.1] (under some additional conditions that are automatically satisfied for $u=u_\mu$ provided that $\mu(\{0\})=0$) that the NGD condition is satisfied if and only if $\mathcal{D}\cap\mathcal{R}\neq\emptyset$, where $\mathcal{D}$ is the determining set of u and $\mathcal{R}$ is the set of *risk-neutral measures*, which in this model has the form

$$\mathcal{R}=\{\mathsf{Q}\in\mathcal{P}:\mathsf{E_Q}(S_1-K)^+=\varphi(K)\text{ for any }K\in\mathbb{K}\}$$

(the notation $\mathcal{P}$ was introduced in Section 2). Furthermore, for $F\in L^0$ such that $u(F)>-\infty$ and $u(-F)>-\infty$,

$$I_{NGD}(F)=\{\mathsf{E_Q}F:\mathsf{Q}\in\mathcal{D}\cap\mathcal{R}\}. \tag{7.1}$$

Below we give more concrete versions of these results for $u=u_\mu$. Let us introduce the notation

$$\begin{aligned}\mathfrak{F}_\mu=\Big\{\psi:\psi\text{ is a convex function }\mathbb{R}_+\to\mathbb{R}_+,\ \psi'_+(0)\geq-1,\ \lim_{x\to\infty}\psi(x)=0,\ \psi|_{\mathbb{K}}=\varphi|_{\mathbb{K}},\\ \psi''\sim\mathsf{P}_0,\text{ and }\int_{\mathbb{R}_+}\Big(\frac{d\psi''}{d\mathsf{P}_0}(y)-x\Big)^+\mathsf{P}_0(dy)\leq\Phi_\mu(x)\text{ for any }x\in\mathbb{R}_+\Big\}.\end{aligned}$$

Here ψ'_+ denotes the right-hand derivative, ψ'' is the second derivative taken in the sense of distributions (i.e. $\psi''((a,b])=\psi'_+(b)-\psi'_+(a)$) with the convention $\psi''(\{0\})=\psi'_+(0)+1$, $\mathsf{P}_0=\mathrm{Law}_{\mathsf{P}}\,S_1$, and Φ_μ is the function introduced in Section 4.

Theorem 7.1. *Let $u=u_\mu$ with $\mu(\{0\})=0$.*
(i) *The NGD is satisfied if and only if $\mathfrak{F}_\mu\neq\emptyset$.*
(ii) *For $F=f(S_1)\in L^1_\mu$, we have*

$$I_{NGD}(F)=\left\{\int_{\mathbb{R}_+}f(x)\psi''(dx):\psi\in\mathfrak{F}_\mu\right\}.$$

`Proof.` **(i)** Let us prove the "only if" part. By the result mentioned above, there exists $\mathsf{Q}\in\mathcal{D}_\mu\cap\mathcal{R}$. The function $\psi(x):=\mathsf{E_Q}(S_1-x)^+$ is convex, $\lim_{x\to\infty}\psi(x)=0$, and $\psi|_{\mathbb{K}}=\varphi|_{\mathbb{K}}$. Denote $Z=\frac{d\mathsf{Q}}{d\mathsf{P}}$ and set $g(x)=\mathsf{E_P}(Z\mid S_1=x)$. Then

$$\psi(x)=\mathsf{E_P}(S_1-x)^+g(S_1)=\int_{\mathbb{R}_+}(y-x)^+g(y)\mathsf{P}_0(dy),\quad x\in\mathbb{R}.$$

This representation shows that $\psi'_+(0) \geq -1$ and $\psi'' = g\mathsf{P}_0$. By Theorem 4.7,

$$\int_{\mathbb{R}_+} (g(y)-x)^+ \mathsf{P}_0(dy) = \mathsf{E}_\mathsf{P}(g(S_1)-x)^+ \leq \mathsf{E}_\mathsf{P}(Z-x)^+ \leq \Phi_\mu(x), \quad x \in \mathbb{R},$$

so that $\psi \in \mathfrak{F}_\mu$.

Let us prove the "if" part. Take $\psi \in \mathfrak{F}_\mu$ and set $g = \frac{d\psi''}{d\mathsf{P}_0}$, $\mathsf{Q} = g(S_1)\mathsf{P}$. Then $\mathsf{Q} \in \mathcal{P}$. The inequality

$$\mathsf{E}_\mathsf{Q}(g(S_1)-x)^+ = \int_{\mathbb{R}_+} (g(y)-x)^+ \mathsf{P}_0(dy) \leq \Phi_\mu(x), \quad x \in \mathbb{R}_+,$$

combined with Theorem 4.7, shows that $\mathsf{Q} \in \mathcal{D}_\mu$. Furthermore,

$$\mathsf{E}_\mathsf{Q}(S_1-K)^+ = \int_{\mathbb{R}_1} (y-K)^+ g(y)\mathsf{P}_0(dy) = \int_{\mathbb{R}_+} (y-K)^+ \psi''(dy) = \psi(K) = \varphi(K), \quad K \in \mathbb{K},$$

so that $\mathsf{Q} \in \mathcal{R}$.

(ii) Take $z \in I_{NGD}(F)$. By (7.1), $z = \mathsf{E}_\mathsf{Q} f(S_1)$ with some $\mathsf{Q} \in \mathcal{D}_\mu \cap \mathcal{R}$. The proof of (i) shows that

$$z = \mathsf{E}_\mathsf{P} f(S_1)g(S_1) = \int_{\mathbb{R}_+} f(x)g(x)\mathsf{P}_0(dx) = \int_{\mathbb{R}_+} f(x)\psi''(dx),$$

where $g(x) = \mathsf{E}_\mathsf{P}\big(\frac{d\mathsf{Q}}{d\mathsf{P}} \mid S_1 = x\big)$ and $\psi = \mathsf{E}_\mathsf{Q}(S_1 - \cdot)^+ \in \mathfrak{F}_\mu$.

Conversely, take $z = \int_{\mathbb{R}_+} f(x)\psi''(dx)$ with $\psi \in \mathfrak{F}_\mu$. Then $z = \mathsf{E}_\mathsf{Q} f(S_1)$, where $\mathsf{Q} = g(S_1)\mathsf{P}$, $g = \frac{d\psi''}{d\mathsf{P}_0}$. The proof of (i) shows that $\mathsf{Q} \in \mathcal{D}_\mu \cap \mathcal{R}$, and by (7.1), $z \in I_{NGD}(F)$. □

8 Optimization in a Complete Model

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space. We consider a complete model, in which an agent can obtain by trading any income from the class

$$A = \{X \in L^1(\mathsf{Q}) : \mathsf{E}_\mathsf{Q} X = 0\},$$

where Q is a fixed probability measure, which is absolutely continuous with respect to P. We will consider the problem

$$\mathrm{RAROC}(X) \xrightarrow[X \in A]{} \max, \tag{8.1}$$

where RAROC is given by (5.3) and $u = u_\mu$ with some fixed μ. We will assume that $\mu \neq \delta_1$, since otherwise $u_\mu(X) = \mathsf{E}_\mathsf{P} X$, and the above problem is meaningless.

Let Φ_μ be the function defined in Section 4 and set

$$\varphi(x) = \mathsf{E}_\mathsf{P}(Z^0 - x)^+, \quad x \in \mathbb{R}_+,$$

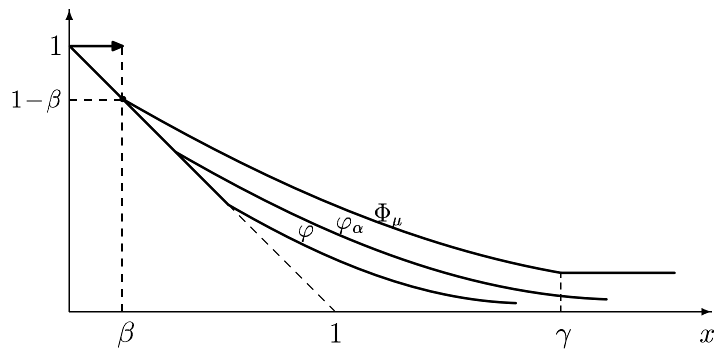
where $Z^0 = \frac{d\mathsf{Q}}{d\mathsf{P}}$. We will assume that there exists no $X \in A$ with $u_\mu(X) > 0$ (indeed, for such X we would have $\mathrm{RAROC}(X) = \infty$). In view of the results of [10; Subsect. 2.2], this is equivalent to the inclusion $\mathsf{Q} \in \mathcal{D}_\mu$. This, in turn, is equivalent to the inequality $\varphi \leq \Phi_\mu$ (see Theorem 4.7). For $\alpha \geq 1$, we set

$$\varphi_\alpha(x) = \alpha\varphi\Big(\frac{x-1}{\alpha} + 1\Big), \quad x \in \mathbb{R}_+,$$

i.e. the graph of φ_α is the α-stretching of the graph of φ with respect to the point $(1,0)$ (see Figure 3). Set $\alpha_* = \sup\{\alpha : \varphi_\alpha \le \Phi_\mu\}$, $\beta = \mu(\{1\})$, $\gamma = \int_{(0,1]} \lambda^{-1}\mu(d\lambda)$, and $\overline{Z} = \alpha_*\bigl(Z^0 - \frac{\alpha_*-1}{\alpha_*}\bigr)$, so that $\varphi_{\alpha_*}(x) = \mathsf{E_P}(\overline{Z} - x)^+$. Note that the left-hand and the right-hand derivatives of φ_{α_*} are given by

$$(\varphi_{\alpha_*})'_-(x) = -\mathsf{P}(\overline{Z} \ge x), \quad x > 0, \tag{8.2}$$

$$(\varphi_{\alpha_*})'_+(x) = -\mathsf{P}(\overline{Z} > x), \quad x \ge 0. \tag{8.3}$$



`Figure 3.` The graphs of Φ_μ, φ, and φ_α. Here we consider the situation, where r defined by (4.6) equals 1.

Theorem 8.1. *Suppose that $\mu \ne \delta_1$, $\mathsf{Q} \ne \mathsf{P}$, and $\varphi \le \Phi_\mu$.*

(i) *We have* $\sup_{X\in A} \mathrm{RAROC}(X) = (\alpha_* - 1)^{-1}$.

(ii) *If $\alpha_* > 1$ and $(\varphi_{\alpha_*})'_+(\beta) > -1$, then $\mathsf{P}(\overline{Z} = \beta) > 0$ and any random variable of the form $X = bI_B + cI_{B^c}$, where $B \subseteq \{\overline{Z} = \beta\}$ and $b > 0 > c$ are such that $\mathsf{E_Q}X = 0$, is optimal for* (8.1).

(iii) *If $\alpha_* > 1$, $\gamma < \infty$, and either $\varphi_{\alpha_*}(\gamma) = \Phi_\mu(\gamma) > 0$ or $\Phi_\mu(\gamma) = 0$ and $(\varphi_{\alpha_*})'_-(\gamma) < 0$, then $\mathsf{P}(\overline{Z} \ge \gamma) > 0$ and any random variable of the form $X = bI_B + cI_{B^c}$, where $B \subseteq \{\overline{Z} \ge \gamma\}$ and $b < 0 < c$ are such that $\mathsf{E_Q}X = 0$, is optimal for* (8.1).

(iv) *If $\alpha_* > 1$ and there exists $x_0 \in (\beta, \gamma)$ such that $\varphi_{\alpha_*}(x_0) = \Phi_\mu(x_0)$, then $\mathsf{P}(\overline{Z} > x_0) > 0$ and any random variable of the form $X = bI_B + cI_{B^c}$, where $B = \{\overline{Z} > x_0\}$ and $b < 0 < c$ are such that $\mathsf{E_Q}X = 0$, is optimal for* (8.1). *If moreover* $\operatorname{supp}\mu = [0,1]$ *and x_0 is the unique point of (β,γ), at which $\varphi_{\alpha_*} = \Phi_\mu$, then an optimal element of A is unique up to multiplication by a positive constant.*

(v) *If $\alpha_* > 1$, but neither of conditions* (ii)–(iv) *is satisfied, then the maximum in* (8.1) *is not attained.*

`Remarks.` `(i)` It is easy to check that if μ has no gap near 1, i.e. $\mu((1-\varepsilon, 1)) > 0$ for any $\varepsilon > 0$, then $(\Phi_\mu)'_+(\beta) = -1$. Similarly, if μ has no gap near 0, i.e. $\mu((0,\varepsilon)) > 0$ for any $\varepsilon > 0$, then $(\Phi_\mu)'_-(\gamma) = 0$. Thus, the situation of (ii) (resp., (iii)) can be realized only if μ has a gap near 1 (resp., near 0). Another verification of these statements follows from the arguments in the proof of (v) below.

`(ii)` In many natural complete models (for instance, in the Black–Scholes model), we have $\operatorname{essinf}_\omega Z^0(\omega) = 0$. Then $\varphi(\varepsilon) > 1 - \varepsilon$ for any $\varepsilon > 0$, and hence, $\alpha_* = 0$. By Theorem 8.1 (i), $\sup_{X\in A} \mathrm{RAROC}(X) = \infty$. A sequence of elements $X_n \in A$ with $\mathrm{RAROC}(X_n) \to \infty$ is provided by $X_n = I(Z \le n^{-1}) - \mathsf{Q}(Z \le n^{-1})$.

`Proof of Theorem 8.1.` **(i)** Take $R \in (0, \infty)$. It follows from the result in [10; Subsect. 2.2] that $\sup_{X \in A} \mathrm{RAROC}(X) \le R$ if and only if

$$\frac{1+R}{R}\left(Z^0 - \frac{1}{1+R}\right) \in \mathcal{D}_\mu.$$

In view of Theorem 4.7, this is equivalent to the conditions $Z^0 \ge \frac{1}{1+R}$ and

$$\forall x \ge 0, \ \ \mathsf{E_P}\left(\frac{1+R}{R}\left(Z^0 - \frac{1}{1+R}\right) - x\right)^+ \le \Phi_\mu(x). \tag{8.4}$$

Note that

$$\mathsf{E_P}\left(\frac{1+R}{R}\left(Z^0 - \frac{1}{1+R}\right) - x\right)^+ = \frac{1+R}{R}\,\mathsf{E_P}\left(Z^0 - \frac{R}{1+R}(x-1) - 1\right)^+ = \varphi_{\alpha(R)}(x), \ \ x \in \mathbb{R}_+,$$

where $\alpha(R) = \frac{1+R}{R}$. Thus, (8.4) is satisfied if and only if $\varphi_{\alpha(R)} \le \Phi_\mu$.

Set $h = \operatorname{essinf}_\omega Z^0(\omega)$. Note that $\varphi(h) = 1 - h$ and $\varphi(h+\varepsilon) > 1 - h - \varepsilon$ for any $\varepsilon > 0$. As $\Phi_\mu(0) = 1$, we conclude that the condition $\varphi_{\alpha(R)} \le \Phi_\mu$ automatically implies that $(1-h)\alpha(R) \le 1$, which, in turn, is equivalent to $Z^0 \ge \frac{1}{1+R}$. As a result,

$$\sup_{X \in A} \mathrm{RAROC}(X) \le R \iff \varphi_{\alpha(R)} \le \Phi_\mu \iff \alpha(R) \le \alpha_* \iff R \ge (\alpha_* - 1)^{-1}.$$

(ii) The inequality

$$\mathsf{E_P}(\overline{Z} - x)^+ = \varphi_{\alpha_*}(x) \le \Phi_\mu(x), \quad x \in \mathbb{R}_+, \tag{8.5}$$

combined with Theorem 4.7, shows that $\overline{Z} \in \mathcal{D}_\mu$. Consequently, $\overline{Z} \ge \beta$, and it follows from (8.3) that $\mathsf{P}(\overline{Z} = \beta) > 0$. Due to Theorem 4.4, we can write $\overline{Z} = \int_{[0,1]} \overline{Z}_\lambda \mu(d\lambda)$ with $\overline{Z}_\lambda \in \mathcal{D}_\lambda$. As $\overline{Z}_1 = 1$, we deduce that, for μ-a.e. $\lambda \in [0,1)$, $\overline{Z}_\lambda = 0$ P-a.e. on $\{\overline{Z} = \beta\}$. Due to the structure of X, for μ-a.e. $\lambda \in [0,1]$, we have $\mathsf{E_P} X \overline{Z}_\lambda = u_\lambda(X)$. Thus, $\mathsf{E_P} X \overline{Z} = u_\mu(X)$. Applying now the equality

$$0 = \mathsf{E_P} X Z^0 = \frac{\alpha_* - 1}{\alpha_*}\mathsf{E_P} X + \frac{1}{\alpha_*}\mathsf{E_P} X \overline{Z} = \frac{\alpha_* - 1}{\alpha_*}\mathsf{E_P} X + \frac{1}{\alpha_*} u_\mu(X),$$

we deduce that $\mathrm{RAROC}(X) = (\alpha_* - 1)^{-1}$, so that X is optimal.

(iii) Consider first the case $\varphi_{\alpha_*}(\gamma) = \Phi_\mu(\gamma) > 0$. Due to (8.5), $\overline{Z} \in \mathcal{D}_\mu$, so that we can write $\overline{Z} = \int_{[0,1]} \overline{Z}_\lambda \mu(d\lambda) = \xi + \eta$, where $\xi = \int_{(0,1]} \overline{Z}_\lambda \mu(d\lambda)$ and $\eta = \mu(\{0\})\overline{Z}_0$. As $\overline{Z}_\lambda \le \lambda^{-1}$, we get $\xi \le \gamma$, so that

$$\varphi_{\alpha_*}(\gamma) = \mathsf{E_P}(\overline{Z} - \gamma)^+ \le \mathsf{E_P}\eta = \mu(\{0\}) = \Phi_\mu(\gamma).$$

The fact that this inequality should be an equality means that $\mathsf{P}(\xi = \gamma) > 0$ and $\eta = 0$ P-a.e. outside $\{\xi = \gamma\}$. This implies that, for μ-a.e. $\lambda \in (0,1]$, $\overline{Z}_\lambda = \lambda^{-1}$ P-a.e. on $\{\overline{Z} > \gamma\}$ and $\overline{Z}_0 = 0$ P-a.e. outside $\{\overline{Z} > \gamma\}$. The proof is now completed in the same way as in (ii).

Now, consider the case $\Phi_\mu(\gamma) = 0$ and $(\varphi_{\alpha_*})'_-(\gamma) < 0$. As $\Phi_\mu(\gamma) = \mu(\{0\})$, we get $\mu(\{0\}) = 0$. Thus, $\overline{Z} = \int_{(0,1]} \overline{Z}_\lambda \mu(d\lambda)$. As $\overline{Z}_\lambda \le \lambda^{-1}$, we get $\overline{Z} \le \gamma$. It follows from (8.2) that $\mathsf{P}(\overline{Z} = \gamma) > 0$. This implies that, for μ-a.e. $\lambda \in (0,1]$, $\overline{Z}_\lambda = \lambda^{-1}$ P-a.e. on $\{\overline{Z} = \gamma\}$. The proof is now completed in the same way as in (ii).

(iv) Set

$$\begin{aligned}\lambda_0 &= \inf\Big\{x \in [0,1] : \int_{(x,1]} \lambda^{-1}\mu(d\lambda) \le x_0\Big\},\\ l &= \int_{(\lambda_0,1]} \lambda^{-1}\mu(d\lambda),\\ r &= \int_{[\lambda_0,1]} \lambda^{-1}\mu(d\lambda).\end{aligned}$$

As Φ_μ is constant on $[l,r)$ and φ_{α_*} is strictly decreasing on $[l,r)$, we have either $x_0 = l$ or $x_0 = r$. Consider the first case (the other one is analyzed similarly). Due to (8.5), $\overline{Z} \in \mathcal{D}_\mu$, so that we can write $\overline{Z} = \int_{[0,1]} \lambda^{-1}\mu(d\lambda)$. As $\int_{(\lambda_0,1]} \overline{Z}_\lambda \mu(d\lambda) \le l = x_0$, we have

$$\mathsf{E}_\mathsf{P}(\overline{Z} - x_0)^+ \le \mathsf{E}_\mathsf{P} \int_{[0,\lambda_0]} \overline{Z}_\lambda \mu(d\lambda) = \mu([0,\lambda_0]) = \Phi_\mu(l) = \Phi_\mu(x_0).$$

The fact that this inequality should be an equality shows that, for μ-a.e. $\lambda \in (\lambda_0,1]$, $\overline{Z}_\lambda = \lambda^{-1}$ P-a.e. on $\{\overline{Z} > x_0\}$ and, for μ-a.e. $\lambda \in [0,\lambda_0]$, $\overline{Z}_\lambda = 0$ P-a.e. outside $\{\overline{Z} > x_0\}$. The proof is now completed in the same way as in (ii).

Let us now prove the uniqueness. Let X be optimal for (8.1). Then X is not degenerate since otherwise it should be equal to 0. Thus, we can find $c \in \mathbb{R}$ such that $\lambda_0 = \mathsf{P}(X \le c)$ belongs to $(0,1)$. The analysis of the proof of (ii) shows that $u_\mu(X) = \mathsf{E}_\mathsf{P} X\overline{Z}$. Consequently, for μ-a.e. $\lambda \in [0,1]$, $u_\lambda(X) = \mathsf{E}_\mathsf{P} X\overline{Z}_\lambda$, where $\overline{Z}_\lambda$ are taken from the representation $\overline{Z} = \int_{[0,1]} \overline{Z}_\lambda \mu(d\lambda)$. This means that, for μ-a.e. $\lambda \in (\lambda_0,1]$, $\overline{Z}_\lambda = \lambda^{-1}$ P-a.e. on $\{X \le c\}$ and, for μ-a.e. $\lambda \in [0,\lambda_0]$, $\overline{Z}_\lambda = 0$ P-a.e. outside $\{X \le c\}$. Consequently, for $x_0 = \int_{(\lambda_0,1]} \lambda^{-1}\mu(d\lambda)$, we have

$$\mathsf{E}_\mathsf{P}(\overline{Z} - x_0)^+ = \mu([0,\lambda_0]) = \Phi_\mu(x_0).$$

Moreover, as $\lambda_0 \in (0,1)$ and $\operatorname{supp}\mu = [0,1]$, we have $x_0 \in (\beta,\gamma)$. Since such x_0 is unique, we conclude that X takes on only two values. Thus, any optimal X has the form $bI_B + cI_{B^c}$ with some $B \in \mathcal{F}$ and some constants $b < c$. It is clear from the reasoning given above that $\mathsf{P}(B)$ is determined uniquely. Using the same arguments as in the proof of Corollary 5.2, we deduce that different optimal elements should be comonotone. Consequently, B is determined uniquely, so that an optimal strategy is unique up to multiplication by a positive constant.

(v) Assume the contrary, i.e. the existence of an optimal element X. As X is not degenerate, we can find $c \in \mathbb{R}$ such that $\lambda_0 = \mathsf{P}(X \le c)$ belongs to $(0,1)$. Arguing in the same way as above, we prove that $\varphi_{\alpha_*}(x_0) = \Phi_\mu(x_0)$, where $x_0 = \int_{(\lambda_0,1]} \lambda^{-1}\mu(d\lambda)$. We will now consider three cases.

Case 1. Assume that $x_0 = \beta$. This means that $\mu((\lambda_0,1)) = 0$. The arguments given above show that, for μ-a.e. $\lambda \in [0,\lambda_0]$, $\overline{Z} = 0$ P-a.e. outside $\{X \le c\}$. Consequently, $\overline{Z} = \beta$ P-a.e. on $\{X > c\}$, so that

$$(\varphi_{\alpha_*})'_+(\beta) = -\mathsf{P}(\overline{Z} > \beta) > -1.$$

Thus, in this case conditions of (ii) are satisfied.

Case 2. Assume that $x_0 = \gamma$. This means that $\mu((0,\lambda_0)) = 0$. If $\mu(\{0\}) > 0$, then $\varphi_{\alpha_*}(\gamma) = \Phi_\mu(\gamma) = \mu(\{0\}) > 0$, so that conditions of (iii) are satisfied.

Now, assume that $\mu(\{0\}) = 0$. Then $\Phi_\mu(\gamma) = 0$. The arguments given above show that, for μ-a.e. $\lambda \in [\lambda_0, 1]$, $\overline{Z}_\lambda = \lambda^{-1}$ P-a.e. on $\{X \le c\}$. As $\mu([0, \lambda_0)) = 0$, we get $\overline{Z} = \int_{(0,1]} \lambda^{-1}\mu(d\lambda) = \gamma$ P-a.e. on $\{X \le c\}$, so that

$$(\varphi_{\alpha_*})'_-(\gamma) = -\mathsf{P}(\overline{Z} = \gamma) < 0.$$

Thus, in this case conditions of (iii) are satisfied.

Case 3. Assume that $x_0 \in (\beta, \gamma)$. Then conditions of (iv) are satisfied. □

Corollary 8.2. *Suppose that* $\operatorname{supp}\mu = [0, 1]$, $\mathsf{Q} \ne \mathsf{P}$, $\varphi \le \Phi_\mu$, *and* $\alpha_* > 1$.

There exists an optimal element of A if and only if there exists $x_0 \in (\beta, \gamma)$ such that $\varphi_{\alpha_*}(x_0) = \Phi_\mu(x_0)$.

An optimal element is unique up to multiplication by a positive constant if and only if such x_0 is unique.

`Proof`. The proof of point (v) shows that, under the condition $\operatorname{supp}\mu = [0, 1]$, the situations of (ii), (iii) are not realized. Now, the statement follows from Theorem 8.1. □

The financial interpretation of the obtained results is as follows. In most natural situations, the optimal strategy consists in buying the binary option with the payoff

$$I\big(\overline{Z} \le x_0\big) = I\big(Z^0 \le \alpha_*^{-1}(x_0 - 1) + 1\big) = I\bigg(\frac{d\mathsf{Q}}{d\mathsf{P}} \le c_*\bigg).$$

The geometric recipe for finding c_* is given in Figure 4.

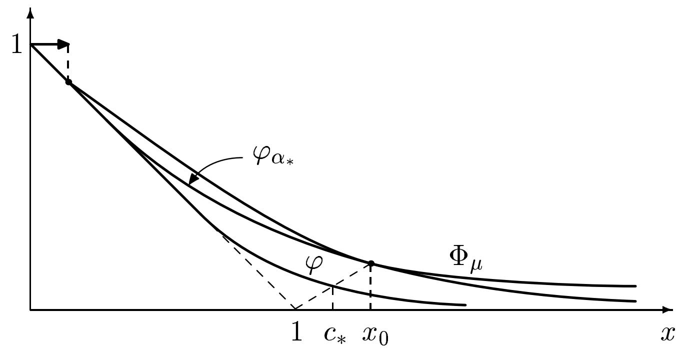


Figure 4

References

[1] Current and emerging research opportunities in probability. Workshop report, available at: `http://www.math.cornell.edu/~durrett/probrep/probrep.html`.

[2] *P. Artzner, F. Delbaen, J.-M. Eber, D. Heath.* Thinking coherently. Risk, **10** (1997), No. 11, p. 68–71.

[3] *P. Artzner, F. Delbaen, J.-M. Eber, D. Heath.* Coherent measures of risk. Mathematical Finance, **9** (1999), No. 3, p. 203–228.

[4] *R. Banz, M. Miller.* Prices of state-contingent claims: some estimates and applications. Journal of Business, **51** (1978), No. 4, p. 653–672.

[5] *A. Bernardo, O. Ledoit.* Gain, loss, and asset pricing. Journal of political economy, **108** (2000), No. 1, p. 144–172.

[6] *D.T. Breeden, R.H. Litzenberger.* Prices of state-contingent claims implicit in option prices. Journal of Business, **51** (1978), No. 4, p. 621–651.

[7] *G. Carlier, R.A. Dana.* Core of convex distortions of a probability. Journal of Economic Theory, **113** (2003), No. 2, p. 199–222.

[8] *P. Carr, H. Geman, D. Madan.* Pricing and hedging in incomplete markets. Journal of Financial Economics, **62** (2001), p. 131–167.

[9] *A.S. Cherny.* General arbitrage pricing model: probability approach. Preprint, available at: `http://mech.math.msu.su/~cherny`.

[10] *A.S. Cherny.* Pricing, optimality, and equlibrium based on coherent risk measures. Preprint, available at: `http://mech.math.msu.su/~cherny`.

[11] *A.S. Cherny.* The Kolmogorov Students' Competitions on Probability Theory. Preprint, available at: `http://mech.math.msu.su/~cherny`.

[12] *J.H. Cochrane, J. Saá-Requejo.* Beyond arbitrage: good-deal asset price bounds in incomplete markets. Journal of Political Economy, **108** (2000), No. 1, p. 79–119.

[13] *F. Delbaen.* Coherent risk measures on general probability spaces. In: Advances in Finance and Stochastics. Essays in honor of Dieter Sondermann. K. Sandmann, P. Schönbucher (Eds.). Springer, 2002, p. 1–37.

[14] *F. Delbaen.* Coherent monetary utility functions. Preprint, available at `http://www.math.ethz.ch/~delbaen` under the name "Pisa lecture notes".

[15] *D. Denneberg.* Distorted probabilities and insurance premium. Methods of Operations Research, **52** (1990), p. 21–42.

[16] *H. Föllmer, A. Schied.* Stochastic finance. An introduction in discrete time. 2nd Ed., Walter de Gruyter, 2004.

[17] *J. Jackwerth.* Option-implied risk-neutral distributions and implied binomial trees: a literature review. Journal of Derivatives, **7** (1999), No. 2, p. 66–82.

[18] *S. Jaschke, U. Küchler.* Coherent risk measures and good deal bounds. Finance and Stochastics, **5** (2001), p. 181–200.

[19] *J. Komlos.* A generalization of a problem of Steinhaus. Acta Math. Acad. Sci. Hung., **18** (1967), p. 217–229.

[20] *S. Kusuoka.* On law invariant coherent risk measures. Advances in Mathematical Economics, **3** (2001), p. 83–95.

[21] *D. Madan, M. Yor.* Making Markov martingales meet marginals: with explicit constructions.// Bernoulli, **8** (2002), No. 4, p. 509–536.

[22] *M. Rotschild, J. Stiglitz.* Increasing risk: I. A definition. Journal of Economic Theory, **2** (1970), p. 225–243.

[23] *M. Rotschild, J. Stiglitz.* Increasing risk: II. Its economic consequences. Journal of Economic Theory, **3** (1971), p. 66–84.

[24] *A. Schied.* Risk measures and robust optimization problems. Lecture notes of a minicourse held at the 8th symposium on probability and stochastic processes. Preprint.

[25] *M. Shaked, J. Shanthikumar.* Stochastic orders and their applications. Academic Press, 1994.

[26] *J. Staum.* Fundamental theorems of asset pricing for good deal bounds. Mathematical Finance, **14** (2004), No. 2, p. 141–161.

[27] *G. Szegö* (Ed.). Risk measures for the 21st century. Wiley, 2004.