Sensitivity analysis of utility based prices and risk-tolerance wealth processes

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Model of a financial market

There are $d + 1$ **traded** or **liquid** assets:

1. a **savings account**. We assume that the interest rate is 0:

$$B = \text{const}$$

2. d **stocks**. The price process S of the stocks is a semimartingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le T}, \mathbb{P})$, where T is a maturity.

Assumption *(No Arbitrage)*

$$\mathcal{Q} \neq \emptyset$$

where $\mathcal{Q}$ is the family of equivalent local martingale measures for S.

Economic agent or investor

x: initial capital

U: utility function for consumption at T such that

- $U : (0, \infty) \to \mathbf{R}$
- U is strictly increasing
- U is strictly concave
- the Inada conditions:

$$U'(0) = \infty \quad U'(\infty) = 0$$

Pricing problem

Consider a family of N **non-traded** or **illiquid** European contingent claims with

- maturity T

- payment functions $f = (f_i)_{1 \leq i \leq N}$

Question What is the (marginal) price $p(x)$ of the contingent claims f?

Intuitive definition: $p(x)$ is the **threshold** such that given the chance to buy or sell at p^{trade} the investor will

- buy at $p^{trade} < p(x)$

- sell at $p^{trade} > p(x)$

- do nothing at $p^{trade} = p(x)$

Denote

$$u(x,q) = \sup_{X_0=x} \mathbb{E}[U(X_T + \langle q, f\rangle)]$$

the maximal **expected utility** from the portfolio (x,q), where

x: total wealth invested into liquid securities

$q = (q^i)_{1\le i\le N}$: a number of non traded options

Definition A **(marginal) utility based price** is a vector $p(x)$ such that

$$u(x) := u(x,0) \ge u(x',q')$$

for any portfolio (x',q') such that

$$x = x' + \langle q', p(x)\rangle.$$

Computation of $p(x) = p(x, 0)$

The idea belongs to Mark Davis. Define the conjugate function

$$V(y) = \max_{x>0} \left[U(x) - xy\right], \quad y > 0.$$

and consider the following **dual** optimization problem:

$$v(y) = \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}\left[V\left(y(\frac{d\mathbb{Q}}{d\mathbb{P}})\right)\right], \quad y > 0$$

The measure $\mathbb{Q}(y)$, where the lower bound is attained, is called the **minimal martingale measure** for y.

Mark Davis argued that if $x = -v'(y)$ then

$$p(x) = \mathbb{E}_{\mathbb{Q}(y)}[f].$$

Theorem (Hugonnier,K.,Schachermayer) *Let $y > 0$, $x = -v'(y)$ and X be a non-negative wealth process. The following conditions are equivalent:*

1. *for* **any** *contingent claim f such that*

   $$|f| \leq K(1 + X_T) \text{ for some } K > 0$$

   a utility based price $p(x)$ is uniquely defined.

2. *the minimal martingale measure $\mathbb{Q}(y)$ exists and X is a uniformly integrable martingale under $\mathbb{Q}(y)$.*

Moreover, in this case

$$p(x) = \mathbb{E}_{\mathbb{Q}(y)}[f].$$

“Trading” problem

Question What quantity $q = q(p^{trade})$ the investor should trade (buy or sell) at the price p^{trade}?

Qualitatively, there are two different cases:

1. If f is **replicable**, then

$$q(p^{trade}) = \begin{cases} \textbf{any} & p^{trade} = p(x) \\ \infty, & p^{trade} \neq p(x) \end{cases}$$

2. else if f is **not replicable** and p^{trade} is an arbitrage-free price, then

$$q(p^{trade}) = \begin{cases} 0 & p^{trade} = p(x) \\ \textbf{finite} & p^{trade} \neq p(x) \end{cases}$$

Goal: study the dependence of prices on quantities.

Reservation price (different values for buying or selling)

Marginal utility based price for the claims f given the portfolio (x, q) is a vector $p(x, q)$ such that

$$u(x, q) \geq u(x', q')$$

for any pair (x', q') such that

$$x + \langle q, p(x, q) \rangle = x' + \langle q', p(x, q) \rangle.$$

In other words, given the portfolio (x, q) the investor **will not trade** the options at $p(x, q)$.

If $u = u(x, q)$ is differentiable at (x, q) then

$$p(x, q) = \frac{u_q}{u_x}(x, q).$$

Using the marginal utility based prices $p(x, q)$ we can compute the optimal quantity

$$q = q(p^{trade})$$

from the "equilibrium" condition:

$$p^{trade} = p(x - qp^{trade}, q)$$

Main difficulty: $p(x, q)$ is hard to compute except for the case $q = 0$.

Sensitivity analysis of utility based prices

We study **linear** approximation for $p(x, q)$:

$$p(x + \Delta x, q) \approx p(x) + p'(x)\Delta x + D(x)q,$$

where

$$p^i(x) = p^i(x, 0), \quad i = 1, \ldots, N$$

$p'(x)$ is the derivative of $p(x)$ and

$$D^{ij}(x) = \frac{\partial p^i}{\partial q^j}(x, 0), \quad i, j = 1, \ldots, N.$$

The vector $p'(x)$ and the matrix $D(x)$ measure the **sensitivity** of $p(x, q)$ with respect to x and q at $(x, 0)$

Question (Quantitative) How to compute $p'(x)$ and $D(x)$?

Closely related publications:

J. Kallsen (02) : formula for $D(x)$ for general semimartingale model but in a different framework of local utility maximization.

V. Henderson (02) : formula for $D(x)$ in the case of a Black-Scholes type model with basis risk and for power utility functions.

Question (Qualitative) When the following (desirable) properties hold true for **any** family of contingent claims f?

1. The marginal utility based price $p(x) = p(x, 0)$ **does not depend** (locally) on x, that is,
$$p'(x) = 0$$

2. The sensitivity matrix $D(x)$ has **full rank** for non replicable contingent claims:
$D(x)q = 0 \quad \Leftrightarrow \langle q, f \rangle$ is replicable.

3. The sensitivity matrix $D(x)$ is **symmetric**, that is
$$D^{ij}(x) = D^{ji}(x), \quad i, j = 1, \ldots, N.$$

4. The sensitivity matrix $D(x)$ is **strictly negatively defined** for non-replicable options: $\langle q, D(x)q\rangle < 0 \quad \Leftrightarrow \langle q, f\rangle$ is not replicable

5. **Stability** of the linear approximation: for any p^{trade} the linear approximation to the "equilibrium" equation:

$$p^{trade} = p(x - qp^{trade}, q)$$

that is,

$$p^{trade} \approx p(x) - p'(x)qp^{trade} + D(x)q$$

has the *"correct"* solution.

Remark Using the representation

$$p(x, q) = \frac{u_q}{u_x}(x, q).$$

one can see that all these properties are, in fact, **equivalent**!

Risk-tolerance wealth process

Fix $x > 0$. Recall that $-U'(x)/U''(x)$ is called the **risk-tolerance** coefficient of U at x.

Denote by $\widehat{X}(x)$ the optimal solution of

$$u(x) := u(x,0) = \sup_{X_0=x} \mathbb{E}[U(X_T)].$$

Definition (K., Sirbu) A maximal wealth process $R(x)$ is called the **risk-tolerance wealth process** if

$$R_T(x) = -\frac{U'(\widehat{X}_T(x))}{U''(\widehat{X}_T(x))}.$$

Some properties of $R(x)$ (if it exists):

1. Initial value:

$$R_0(x) = -\frac{u'(x)}{u''(x)}.$$

2. Derivative of optimal wealth strategy:

$$\frac{R(x)}{R_0(x)} = X'(x) := \lim_{\Delta x \to 0} \frac{\widehat{X}(x + \Delta x) - \widehat{X}(x)}{\Delta x}.$$

Theorem (K., Sirbu) *The following assertions are equivalent:*

1. *The risk-tolerance wealth process* $R(x)$ *exists for all* $x > 0$.

2. *The minimal martingale measures* $\mathbb{Q}(y)$ *for different* $y > 0$ *coincide:*

$$\mathbb{Q}(y) = \widehat{\mathbb{Q}}, \quad y > 0.$$

3. *Any (equivalently, each) of the "qualitative" properties 1–5 holds true.*

Extremal cases:

1. Power utility functions:

$$U(x,\alpha) = \frac{x^{1-\alpha} - 1}{1-\alpha}, \quad (\alpha > 0).$$

   For $\alpha = 1$ we have **Bernoulli utility**:

$$U(x;0) = \log x.$$

2. Model is “essentially complete”, that is, it has the **same minimal martingale measure** for any utility function.

Computation of $D(x)$

We choose

$$R(x)/R_0(x) = X'(x)$$

as a **numeraire** and denote

$f^R = fR_0(x)/R(x)$: discounted contingent claims

$X^R = XR_0(x)/R(x)$: discounted wealth processes

$\mathbb{Q}^R$: the martingale measure for $R(x)/R_0(x)$-discounted wealth processes:

$$\frac{d\mathbb{Q}^R}{d\widehat{\mathbb{Q}}} = \frac{R_T(x)}{R_0(x)}$$

Consider the $\mathbb{Q}^{R}$-martingale

$$P_t^{R} = \mathbb{E}_{\mathbb{Q}^{R}}\left[f^{R}|\mathcal{F}_t\right].$$

and let

$$P^{R} = M + N, \quad N_0 = 0,$$

be its Kunita-Watanabe decomposition, where

1. M is $R(x/R_0(x)$-discounted wealth process. Interpretation: **hedging process**.

2. N is a martingale under $\mathbb{Q}^{R}$ which is orthogonal to all $R(x)/R_0(x)$-discounted wealth processes. Interpretation: **risk process**.

Denote by $a(x)$ the relative risk-aversion coefficient of

$$u(x) = \max_{X_0 = x} \mathbb{E}[U(X_T)],$$

that is

$$\alpha(x) = -\frac{x u''(x)}{u'(x)}.$$

Theorem (K., Sirbu) *Assume that the risk-tolerance wealth process $R(x)$ exists. Then*

$$\begin{aligned} p'(x) &= 0 \\ D(x) &= -\frac{a(x)}{x} \mathbb{E}_{\mathbb{Q}^R}\left[N_T N_T'\right] \end{aligned}$$

Question How to compute $D(x)$ in *practice*?

Inputs:

1. $\mathbb{Q}$. *Already implemented!*

2. $R(x)/R_0(x)$. Recall that

$$\frac{R(x)}{R_0(x)} = \lim_{\Delta x \to 0} \frac{\widehat{X}(x+\Delta x) - \widehat{X}(x)}{\Delta x}.$$

Decide what to do with one penny!

3. Relative risk-aversion coefficient $\alpha(x)$. *Deduce from mean-variance preferences*. In any case, this is just a number!

Black and Scholes model with basis risk

Traded asset:

$$dS_t = S_t\,(\mu dt + \sigma dW_t)$$

Non traded asset:

$$d\widetilde{S} = (\widetilde{\mu} dt + \widetilde{\sigma} d\widetilde{W}_t),$$

Denote by

$$\rho = \frac{d\widetilde{W} dW}{dt}$$

the **correlation** coefficient between S and $\widetilde{S}$.
In practice, we want to chose S so that

$$\rho \approx 1.$$

Consider contingent claims

$$\boldsymbol{f} = \boldsymbol{f}(\widetilde{\boldsymbol{S}})$$

whose payoffs are determined by $\widetilde{\boldsymbol{S}}$.

To compute $\boldsymbol{D}(\boldsymbol{x})$ assume (as an example) the following choices:

1. $\mathbb{Q}$ is a martingale measure for $\widetilde{\boldsymbol{S}}$.

2. $\boldsymbol{R}(\boldsymbol{x})/\boldsymbol{R}_0(\boldsymbol{x}) = 1$

Then

$$\boldsymbol{D}_{ij}(\boldsymbol{x}) = -\frac{\boldsymbol{\alpha}(\boldsymbol{x})}{\boldsymbol{x}}(1-\boldsymbol{\rho}^2)\text{Covar}_{\mathbb{Q}}(\boldsymbol{f}_i, \boldsymbol{f}_j).$$

Smoothness of the indirect utility

Consider the problem of expected utility maximization:

$$u(x) = \sup_{X_0=x} \mathbb{E}[U(X_T)]$$

Question When $u''(x)$ exists and is strictly positive?

We need additional conditions on the utility function U and the price processes S.

Assumption *There are strictly positive constants* c_1 *and* c_2 *such that*

$$c_1 < -\frac{xU''(x)}{U'(x)} < c_2, \quad x > 0.$$

Definition A semimartingale $\boldsymbol{R}$ is called **sigma-bounded** if

$$\boldsymbol{R} = \int \boldsymbol{H} d\boldsymbol{S}$$

and $\boldsymbol{S}$ is (locally) bounded.

Assumption *For any numeraire* $\boldsymbol{X}$

$$\boldsymbol{S}^{\boldsymbol{X}} = \left(\frac{\mathbf{1}}{\boldsymbol{X}}, \frac{\boldsymbol{S}}{\boldsymbol{X}}\right)$$

is **sigma-bounded**.

Examples:

1. $\boldsymbol{S}$ is continuous or, more generally, depends on a finite number of Poisson processes

2. complete financial model

Remark These two examples will not satisfy if sigma-boundedness is replaced by local boundedness.

Theorem (K.,Sirbu) *Let both assumptions above hold true. Then* u'' *exists and*

$$c_1 < -\frac{xu''(x)}{u'(x)} < c_2, \quad x > 0.$$

where c_1 *and* c_2 *are the same constants as in the assumption on* U.

Remark Both assumptions are **essential** for the assertion of the theorem to hold true.

Other results:

1. The existence of the derivative processes $X'(x)$ and $Y'(y)$ of the solutions to the primal and dual problems.

2. The computation of $p'(x)$ and $D(x)$ in general case (when $R(x)$ does not exist).

Summary

- For non replicable (in practice, all) contingent claims the fair prices depend on the trading volume.

- The following conditions are equivalent:

  - Approximate utility based prices have nice qualitative properties

  - Risk-tolerance wealth processes exist.

  - Minimal martingale measures do not depend on initial capital.

- We need to solve the mean-variance hedging problem, where the risk-tolerance wealth process plays the role of the numeraire.