Memory Reduction in Iterated Function Systems

Closing off (kind of) an avenue of measuring fractal complexity

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1 Background
- Fractals
- Applications
- Iterated Function Systems

2 IFS with Memory
- 1-IFS
- Transition Graphs/Matrices
- Classification
- 2-IFS and beyond

3 Memory Reduction
- Theory
- Demonstration
- Results
- Conclusions and Future Work

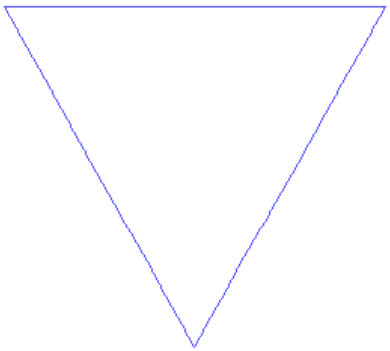
Definitions

A fractal is . . .

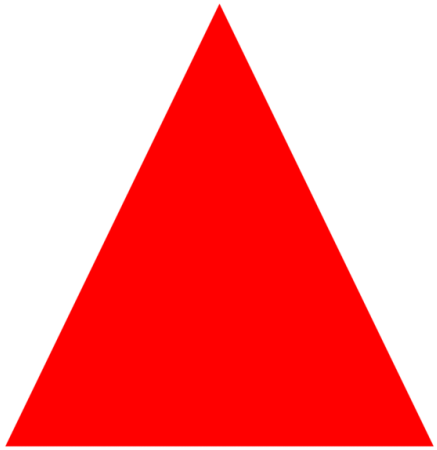
- . . . a set exhibiting self-similarity.
- . . . a set that "looks irregular; but more importantly, after it is magnified it still looks irregular." [1]
- . . . "by definition a set for which the Hausdorff-Besicovitch dimension strictly exceeps the topological dimension." [2]

Canonical examples

Figure: Simple, iteratively generated fractals



(a) Van Koch curve



(b) Sierpinski triangle (deterministic)

(c) Sierpinski triangle (probabilistic)

Dimensional analysis

Let X be a metric space. If $S \subseteq X$ and $d \in [0, \infty)$, the **d-dimensional Hausdorff measure of S** is

$$C_H^d(S) = \inf \left\{ \sum_i r_i^d \mid \exists \text{ cover of } S \text{ by balls with radii } r_i > 0 \right\}$$

and the **Hausdorff dimension** of S is

$$\dim_H(S) = \inf\{d \geq 0 \mid C_H^d(S) = 0$$

The **Lebesgue covering dimension** of a topological space X is the minimum n such that every finite open cover $\mathcal{A}$ of X admits a finite open cover which refines $\mathcal{A}$ in which no point is included in more than $n + 1$ elements.

Dimensional analysis

Example

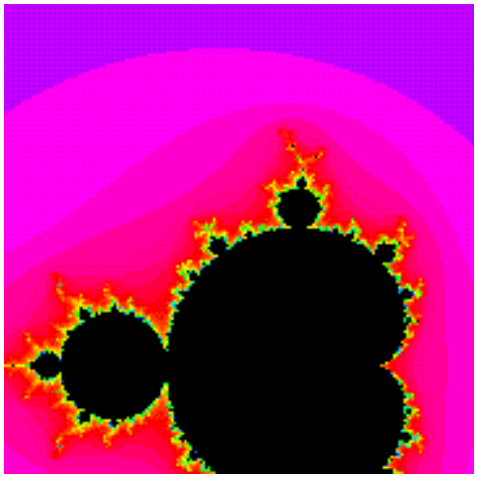
Cantor set: Hausdorff dim. $\frac{\ln 2}{\ln 3}$; topological dim. 0

Sierpinski $\triangle$: Hausdorff dim. $\frac{\ln 3}{\ln 2}$; topological dim. 1

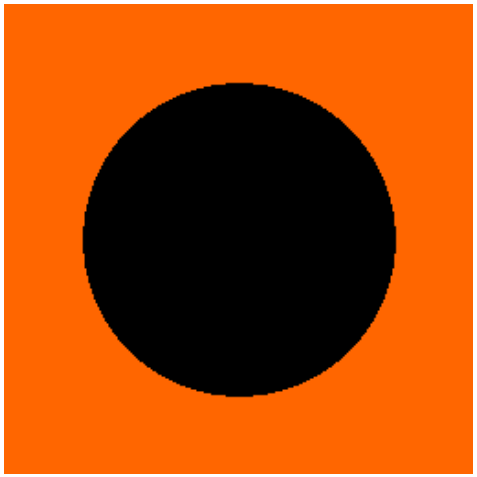
Van Koch curve: Hausdorff dim. $\frac{\ln 4}{\ln 3}$; topological dim. 1

Canonical examples

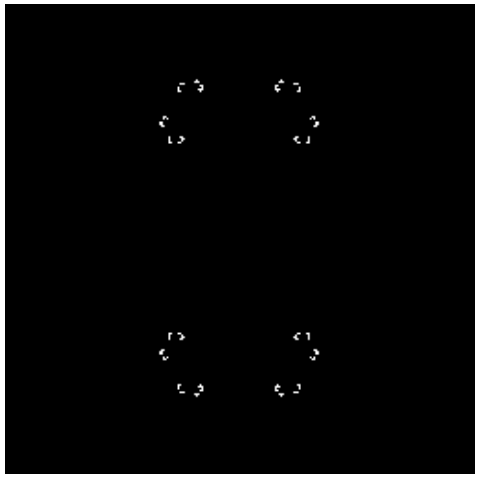
Figure: More complex, iteratively generated fractals



(a) Mandelbrot set



(b) Mandelbrot set growth

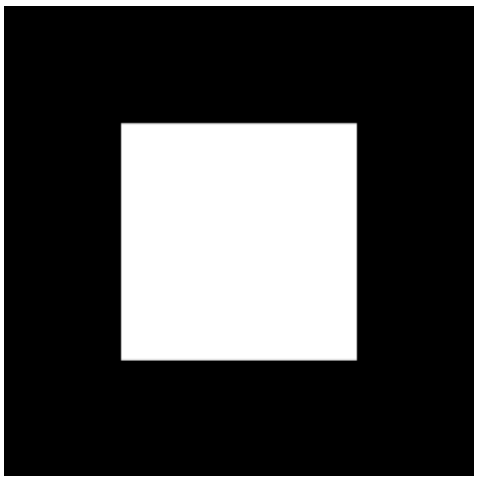


(c) Julia set

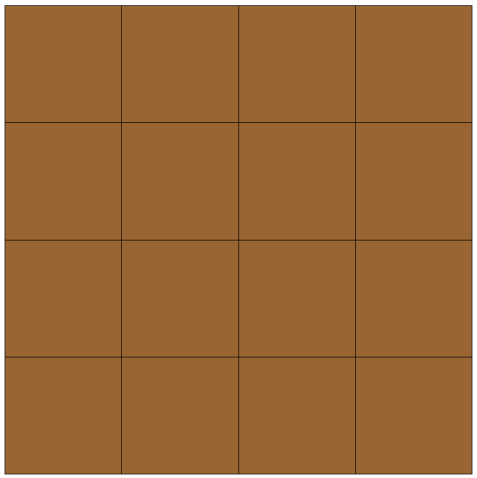
Canonical examples

Figure: Fractals generated by iterated function systems

(a) Shrinking box border

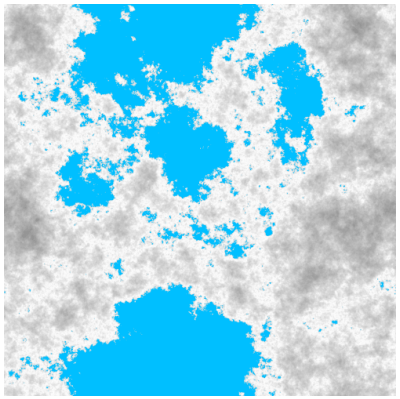


(b) Pointy leaf boxes



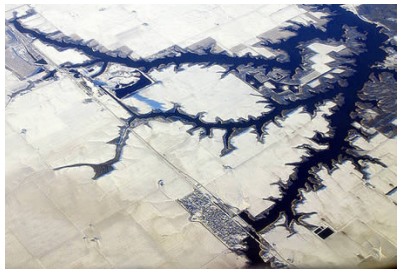
(c) Sierpinskitty trifelinegle

Nature



(a) Clouds [3]

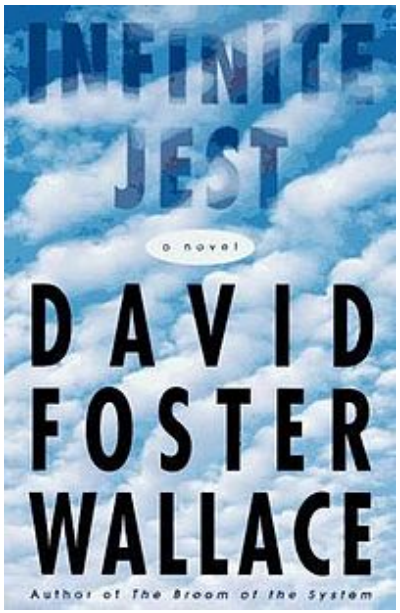
(b) Plants



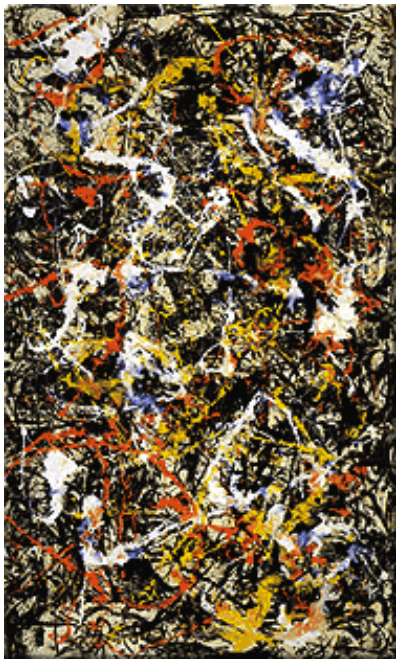
(c) Coastlines

(d) Snowflakes [4]

Art



(a) Literature



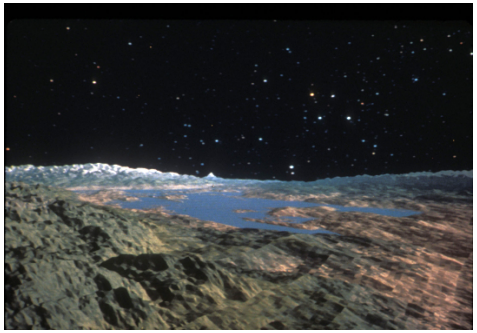
(b) Painting [5]

(c) Music [6]

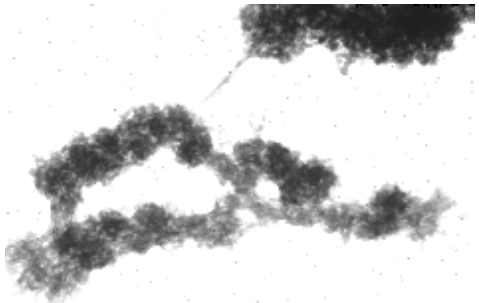
"Wind and Metal" [7]

Science and Computing

(a) Anatomy [9]



(b) Graphics [17]



(c) DNA [18]

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Let (X, ρ) be a metric space and $T : X \to X$ a function.

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Theorem (Banach Fixed Point)

If (X, ρ) is a complete metric space and $T : X \to X$ a contraction map, then T has exactly one fixed point; i.e.

$$\exists!\, \hat{x} \in X.\ T(\hat{x}) = \hat{x}$$

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Question: Does H have any "fixed points"?

Formal definition

Partial Answer:

Theorem (Hutchinson, 1981)

For $X = \mathbb{R}^d$ with the standard metric, every IFS admits a unique compact set $A \subset \mathbb{R}^d$ satisfying $H(A) = A$.

A is the **attractor** of the IFS and is a fractal.

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Theorem (Hutchinson, 1981)

For $X = \mathbb{R}^d$ with the standard metric, every IFS admits a unique compact set $A \subset \mathbb{R}^d$ satisfying $H(A) = A$.

Proof.

Show that H is a contraction map on $K(X)$, the set of compact subsets of X. Apply Banach Fixed Point. □

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Constructive approach:

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Proof.

Corollary to Banach Fixed Point: this limit converges to A for any choice of S_0. □

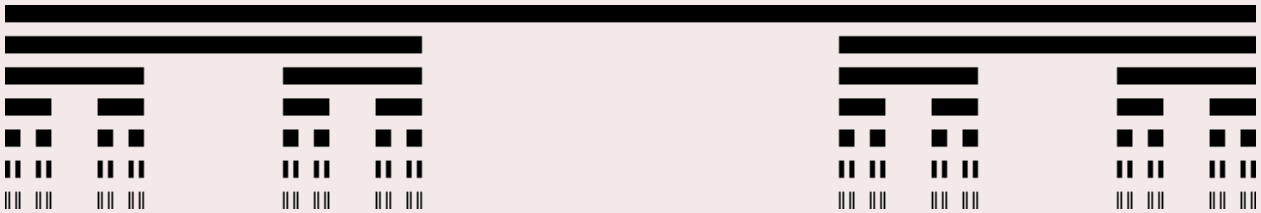
Examples

Standard contraction maps in $\mathbb{R}^d$ are scalings (with factor $r < 1$), rotations, reflections, translations.

Examples

Example

Cantor Set, $\mathcal{C}$:



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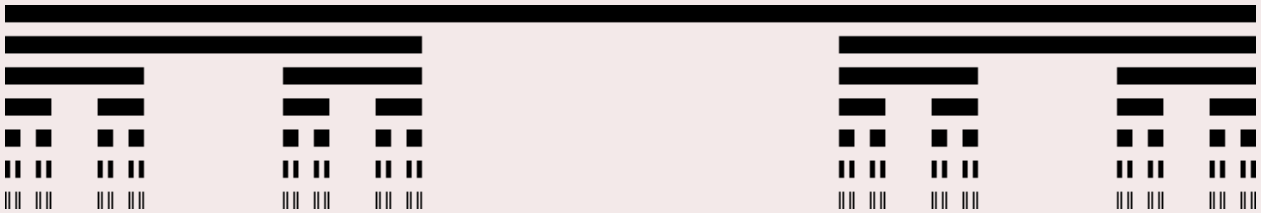

$$T_1(x) = \frac{x}{3} \qquad T_2(x) = \frac{x}{3} + \frac{2}{3}$$

$$\mathcal{C} = T_1(\mathcal{C}) \cup T_2(\mathcal{C})$$

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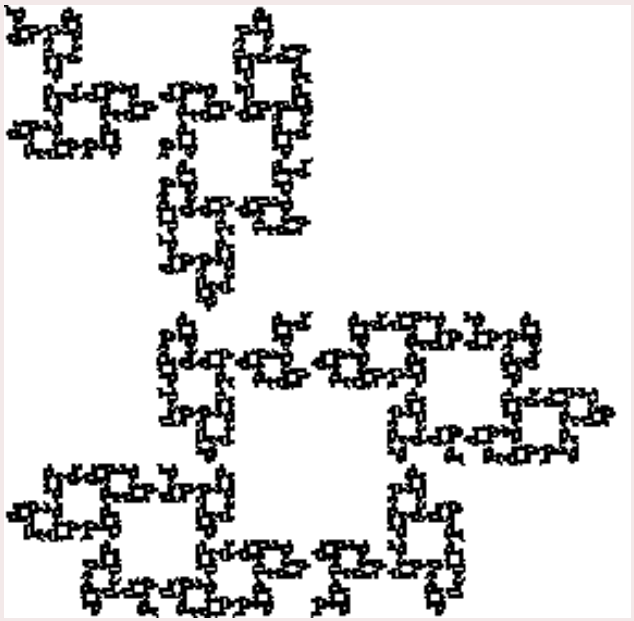
Cantor-style dust:



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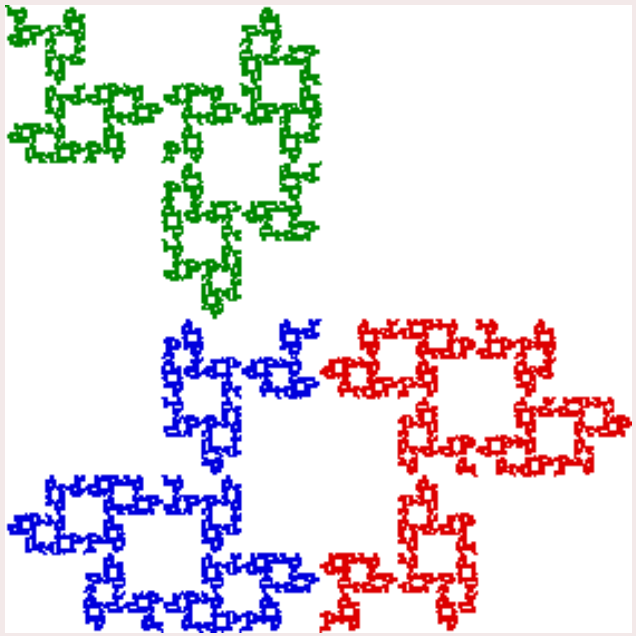

Standard context

$X = [0,1]^2$ with standard metric and $\mathcal{T} = \{T_1, T_2, T_3, T_4\}$, where

$$T_1(x,y) = \left(\frac{x}{2}, \frac{y}{2}\right) + \left(0, 0\right)$$
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3	4
1	2

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Standard context

Being **in state** i means applying T_i
Addresses indicate the *reverse* order of the transformations required to land in that box.

Example

- Map composition: $(T_2 \circ T_1 \circ T_4)(X)$
- Address: 214
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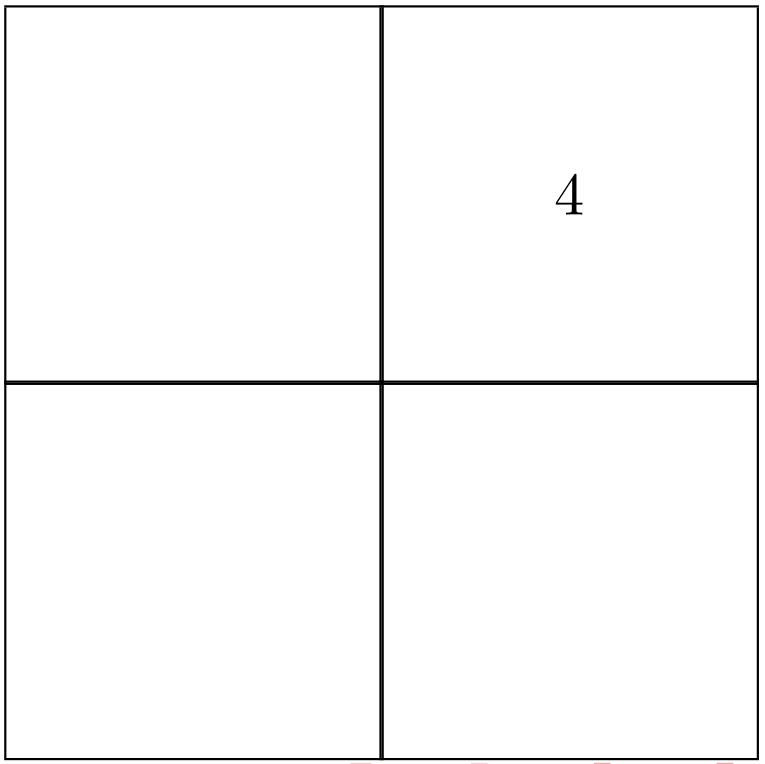

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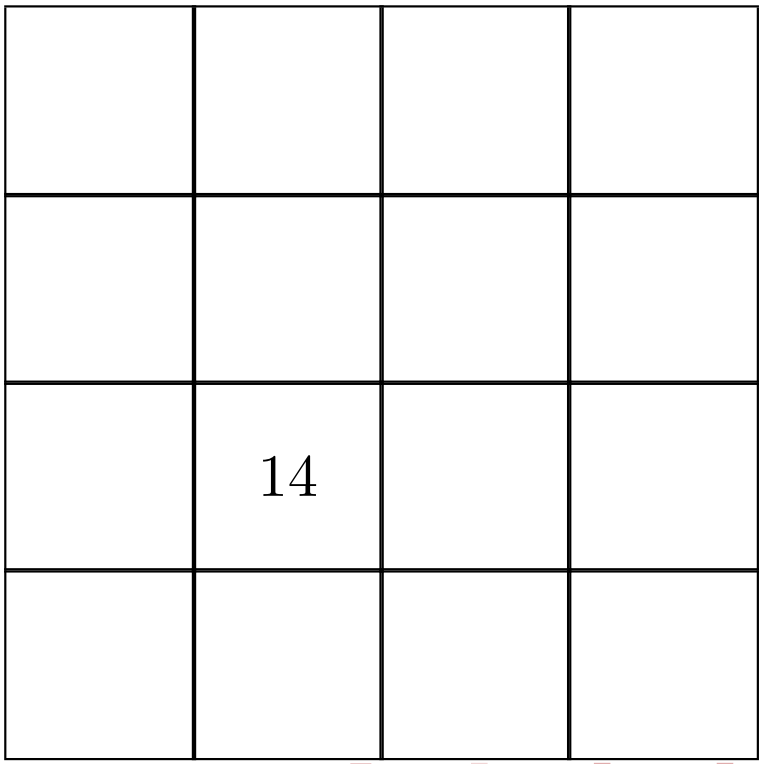

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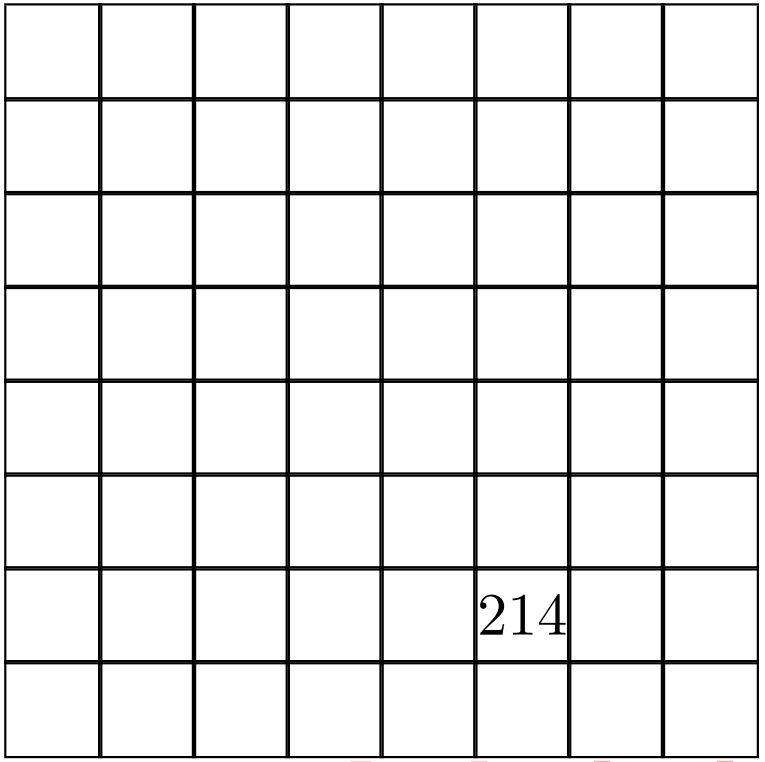

Forbidden pairs: 1-IFS

Main idea: Restrict the constructive approach by disallowing certain pairs of transformations from occurring consecutively.

The system has "1 level of memory" because it looks at the currently-applied transformation in the construction to determine which transformations can be applied next.

Where you are (but not how you got there) affects where you are allowed to go.

Forbidden pairs: 1-IFS

Questions:

- What types of attractors does this yield?
- How does forbidding multiple pairs affect the attractors?
- Can we look at an attractor and determine which pairs were forbidden?
- Which attractors can be realized as a *standard* IFS (with "0 levels of memory") by redefining the set of contraction maps?
- Which attractors can be realized as a standard 0-IFS but require *infinitely* many contraction maps?
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Forbidden pairs: 1-IFS

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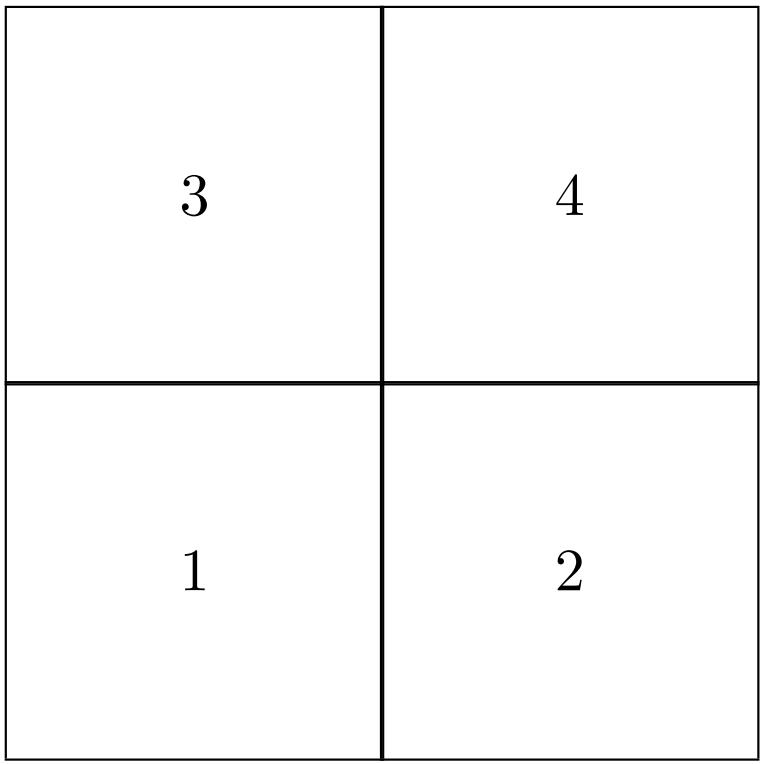

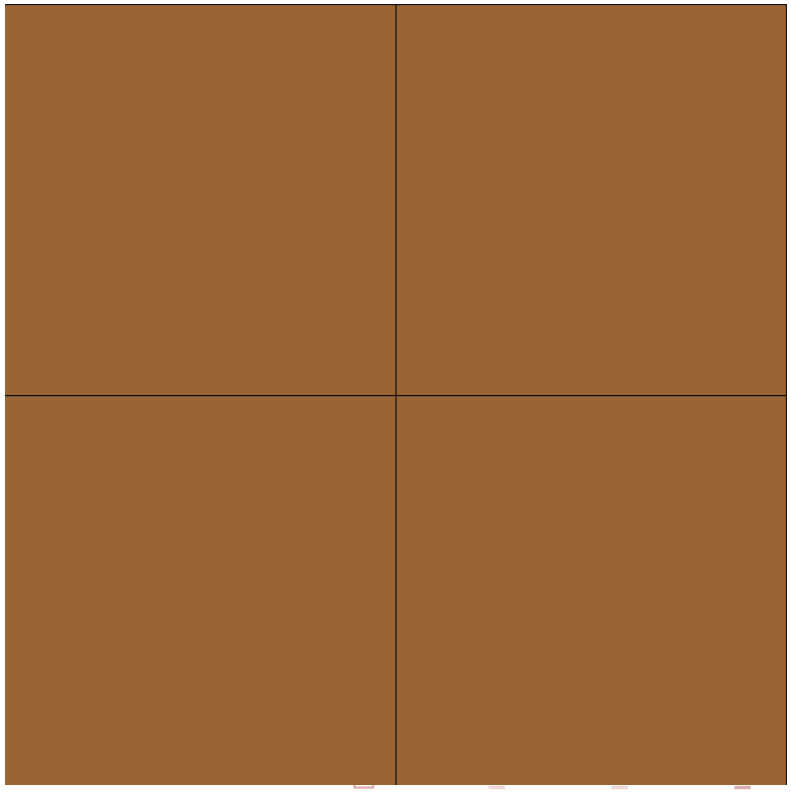

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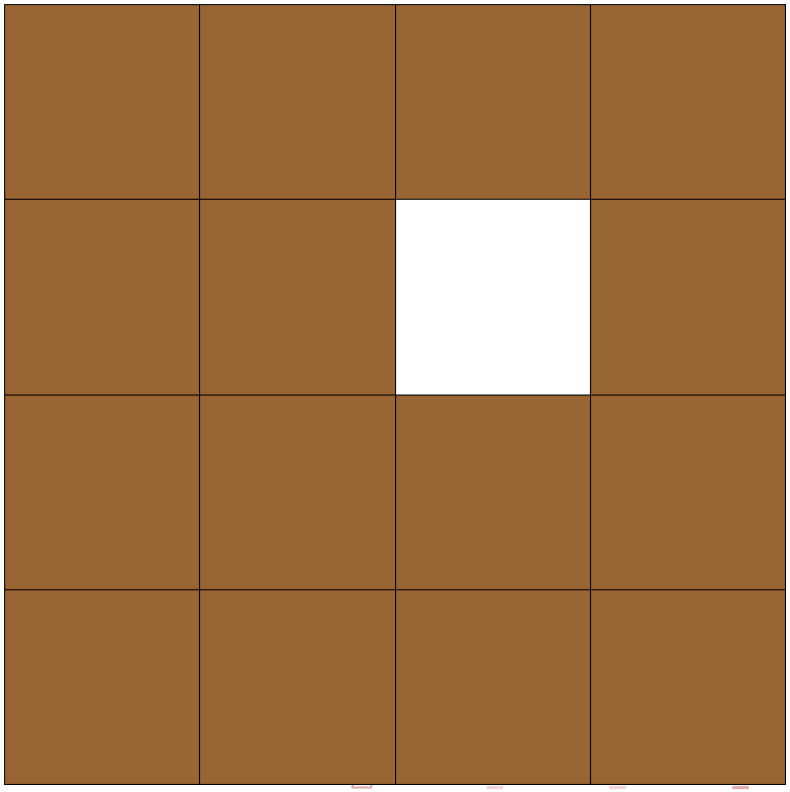

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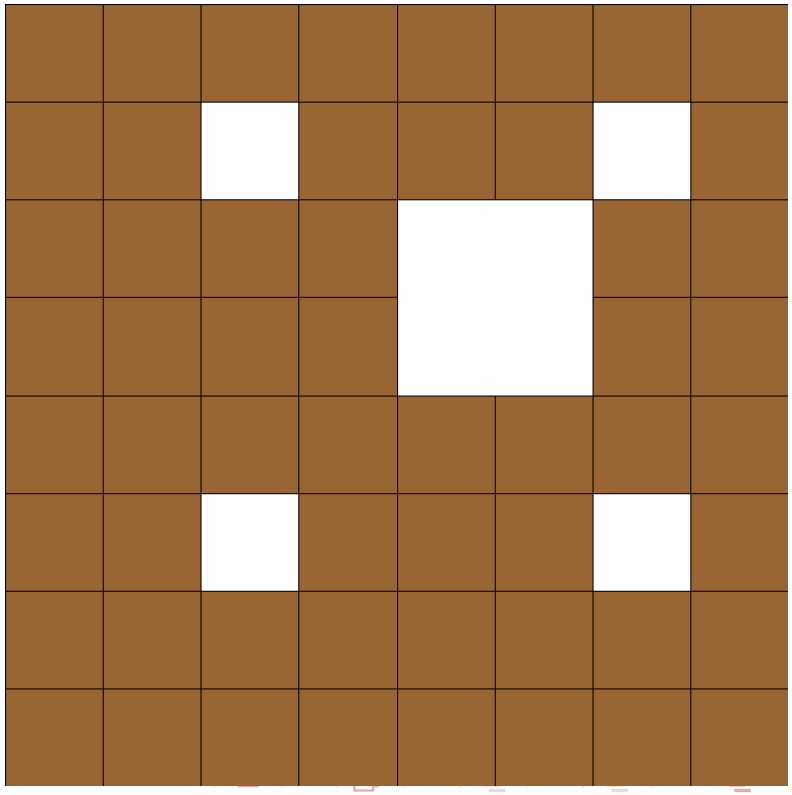

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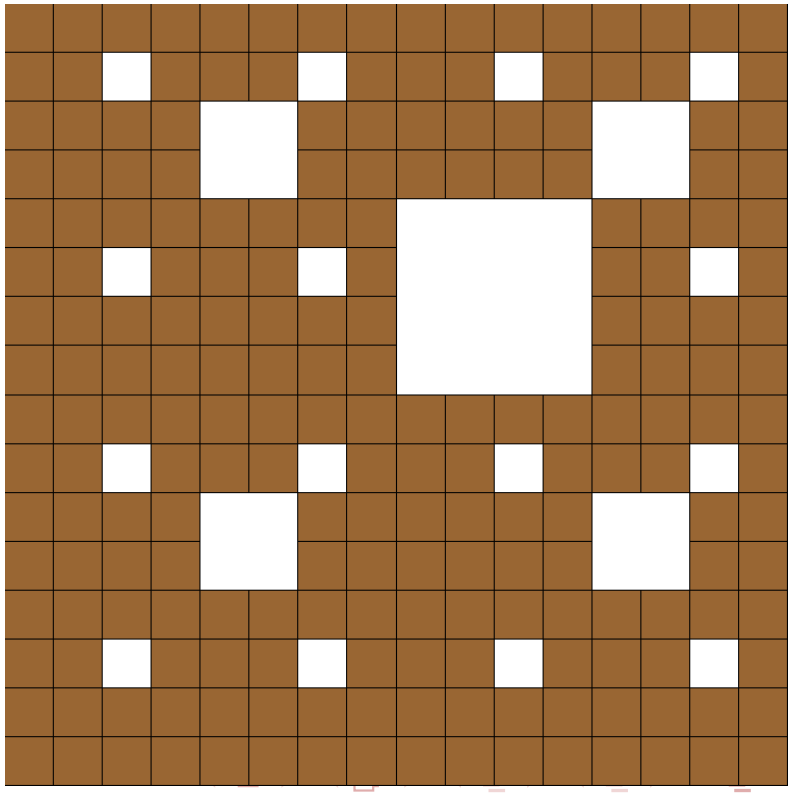

Forbidden pairs: 1-IFS

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Any address with 41 as a substring is empty

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331	332		342	431	432		442
313	314	323	324			423	424
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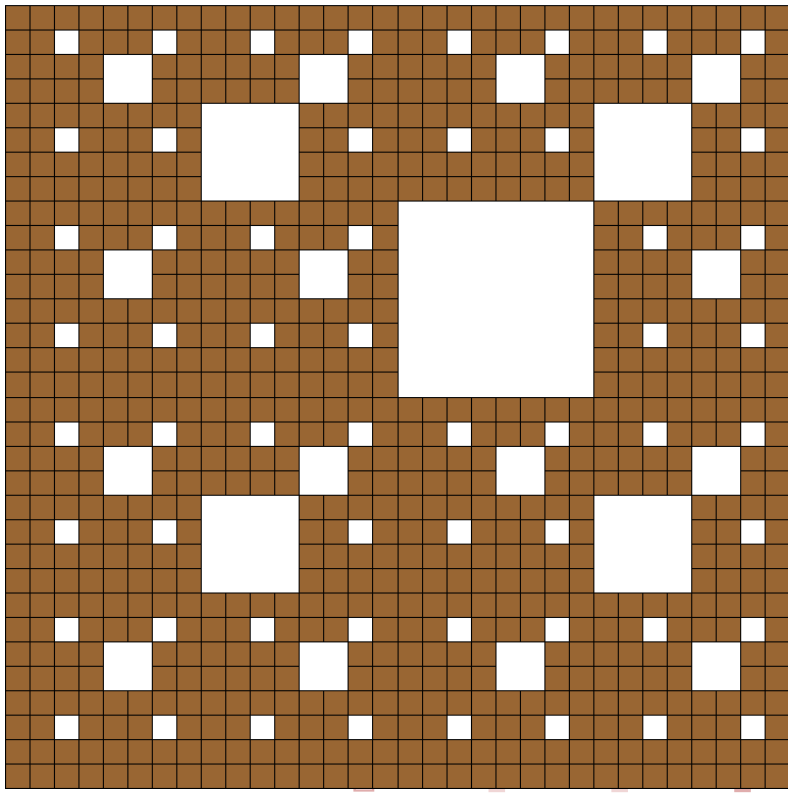

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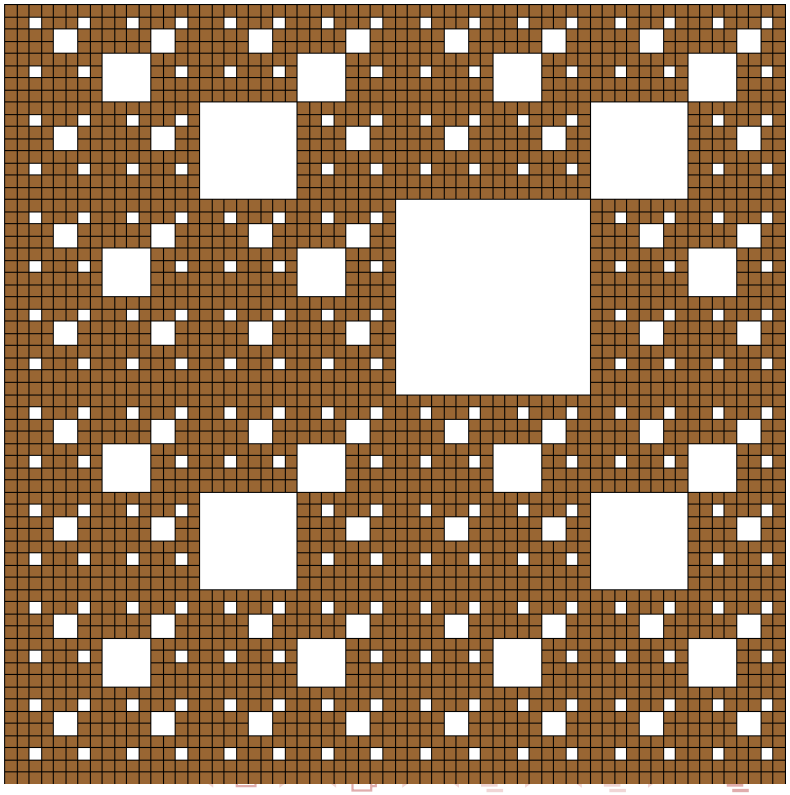

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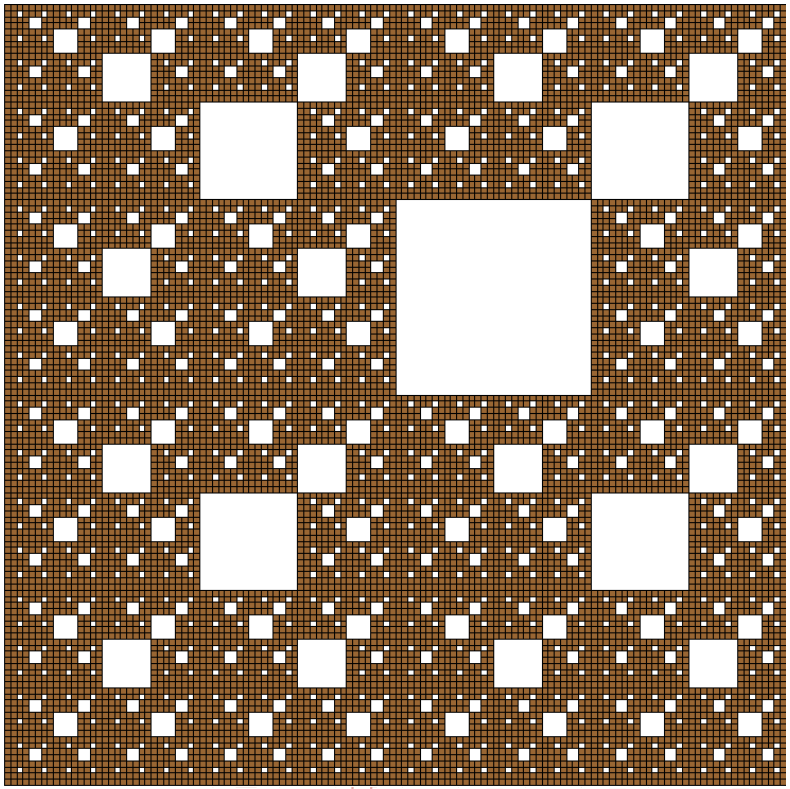

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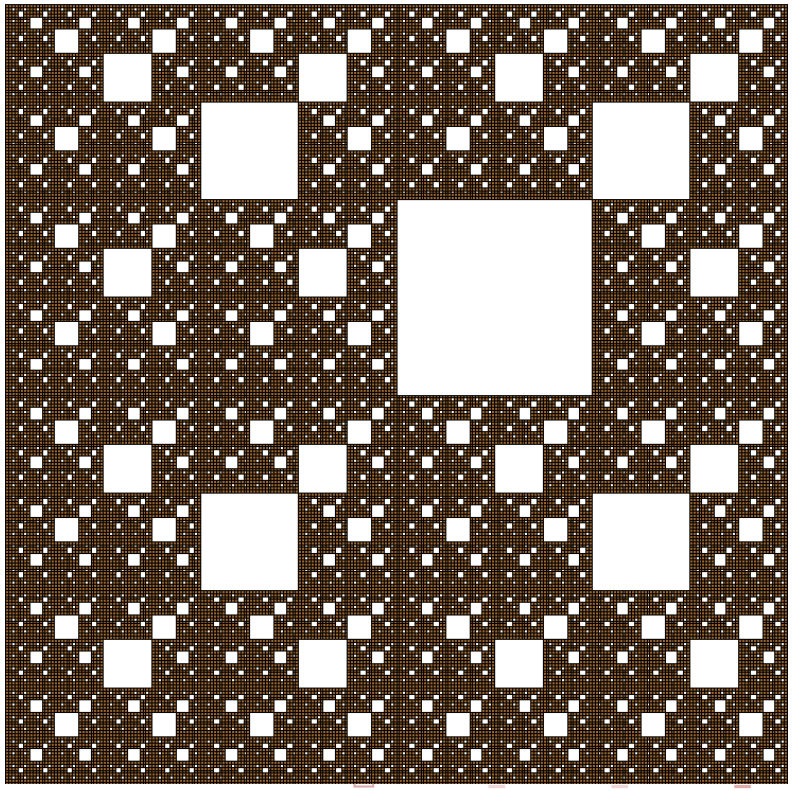

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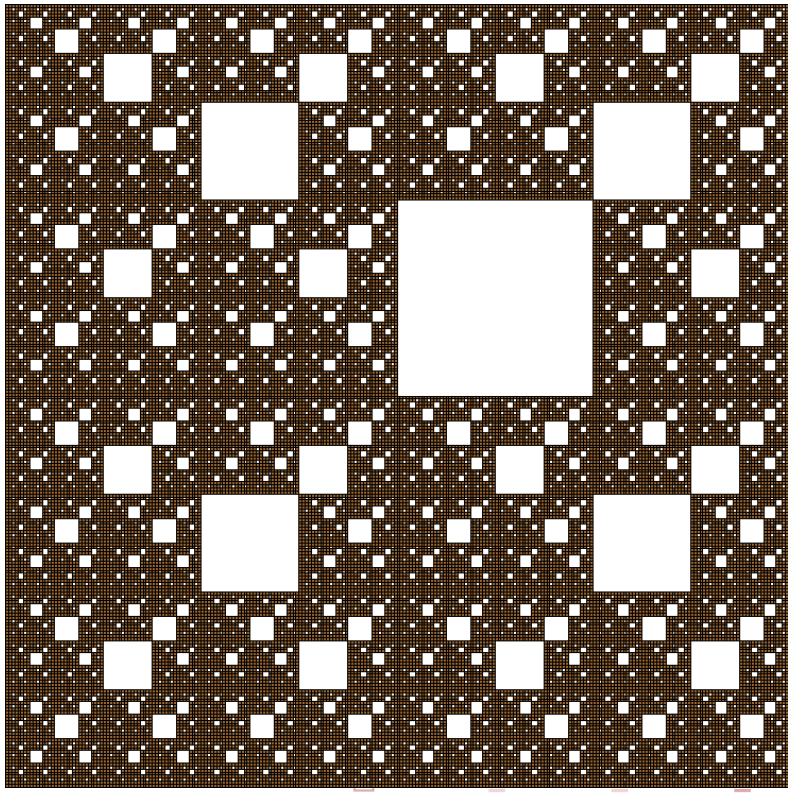


Forbidden pairs: 1-IFS

Example

Forbid $1 \to 4$, $4 \to 1$, $2 \to 3$, $3 \to 2$

Forbidden pairs: 1-IFS

Example

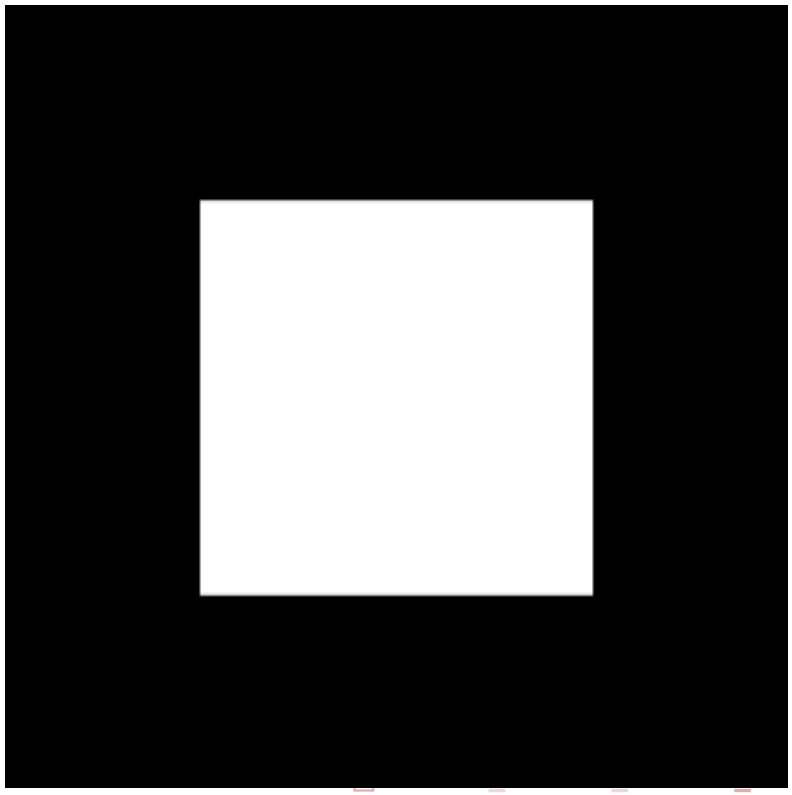
Addresses with 14, 41, 23, 32 are empty

Forbidden pairs: 1-IFS

Example

Addresses with 14, 41, 23, 32 are empty

33	34	43	44
31			42
13			24
11	12	21	22



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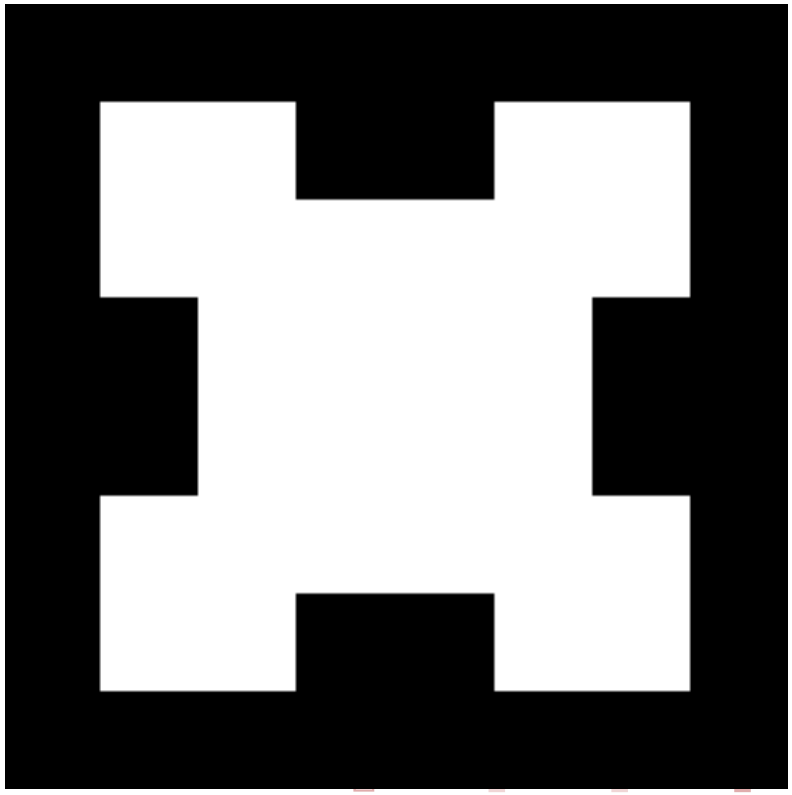

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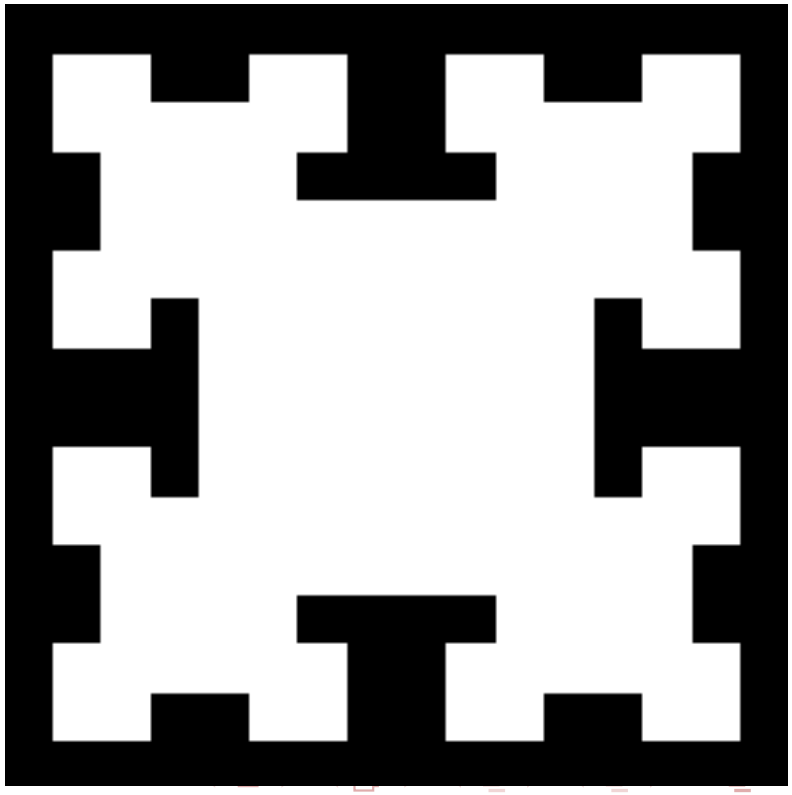

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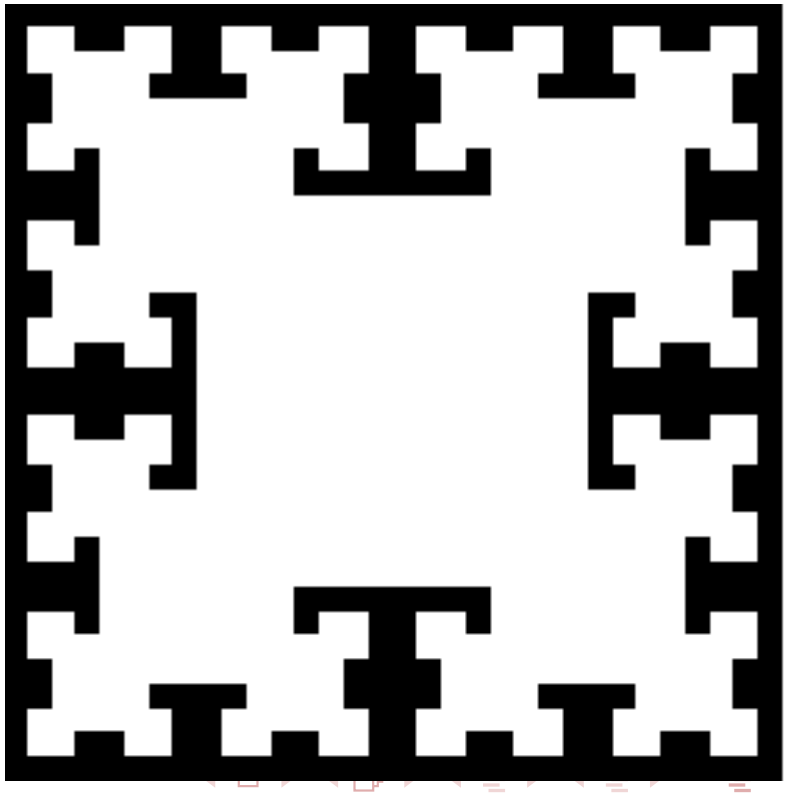

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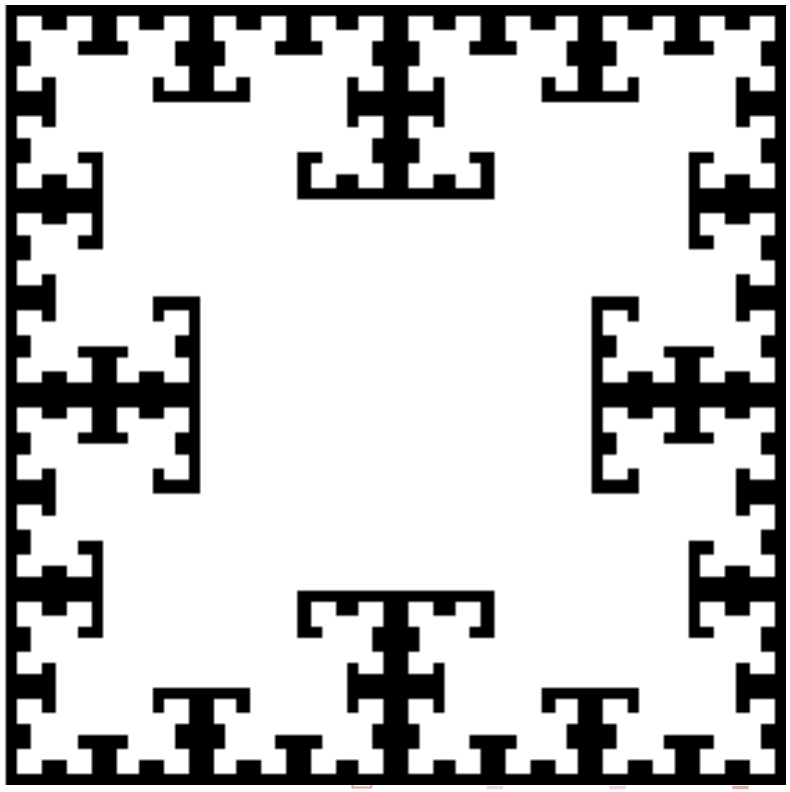

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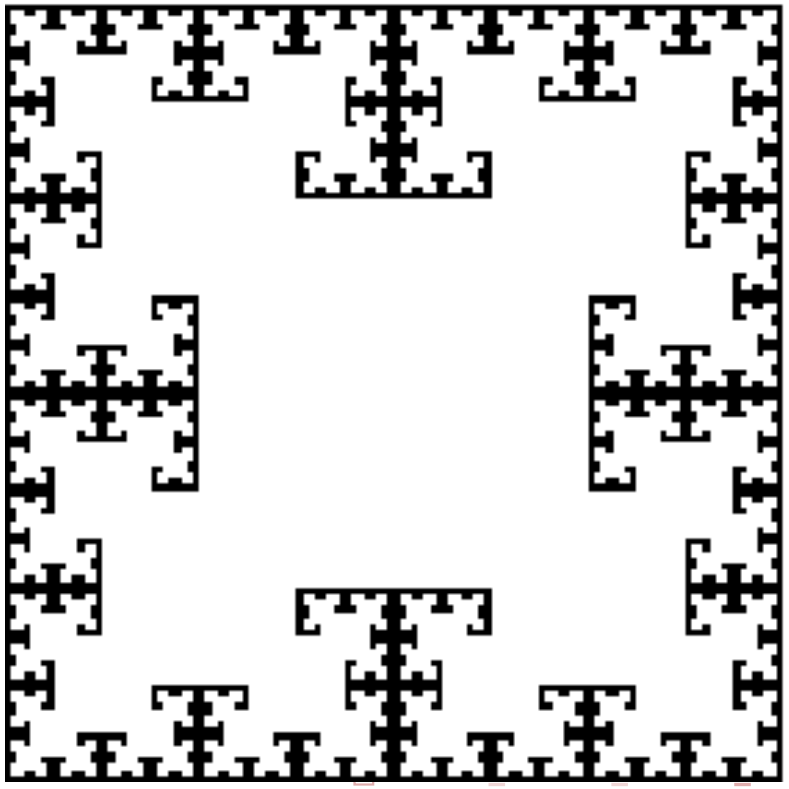

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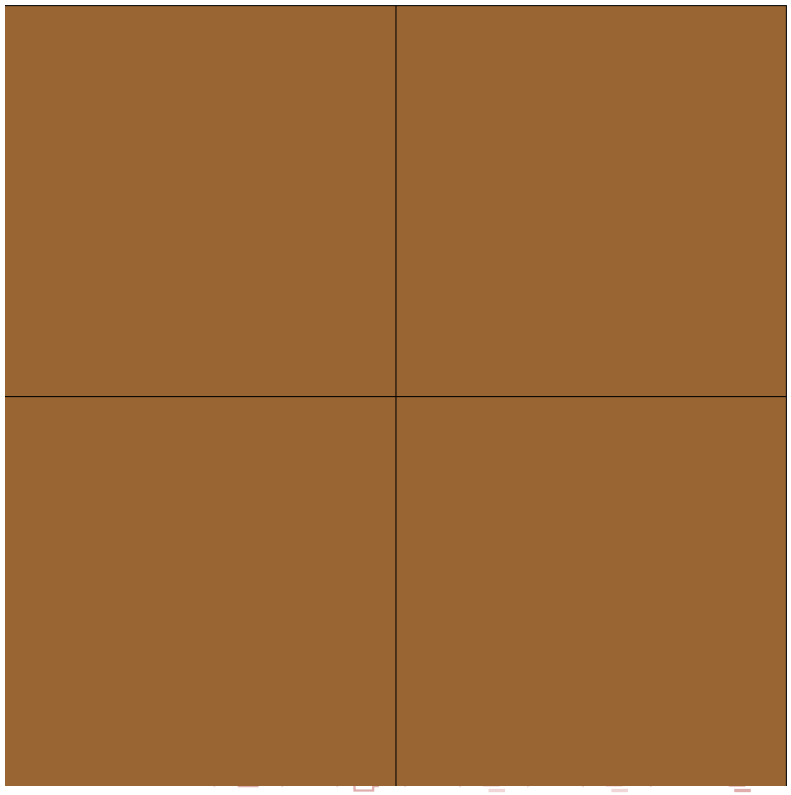

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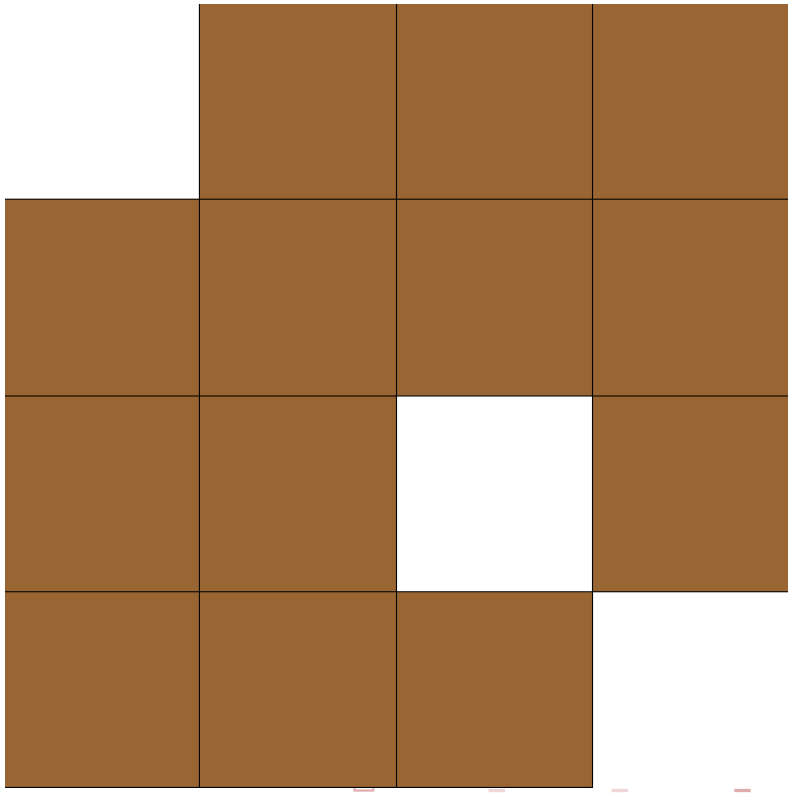

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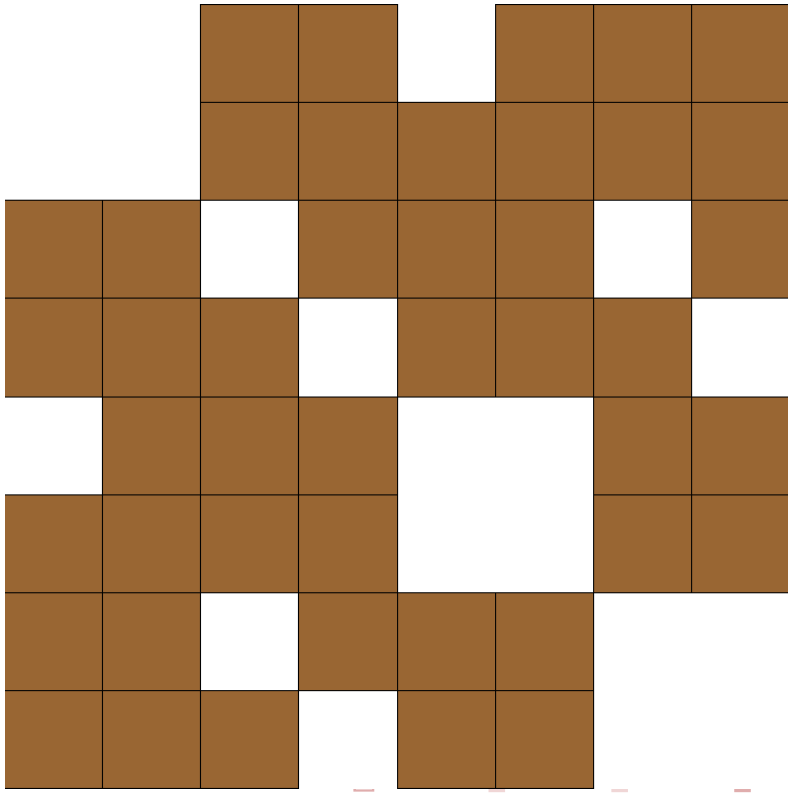

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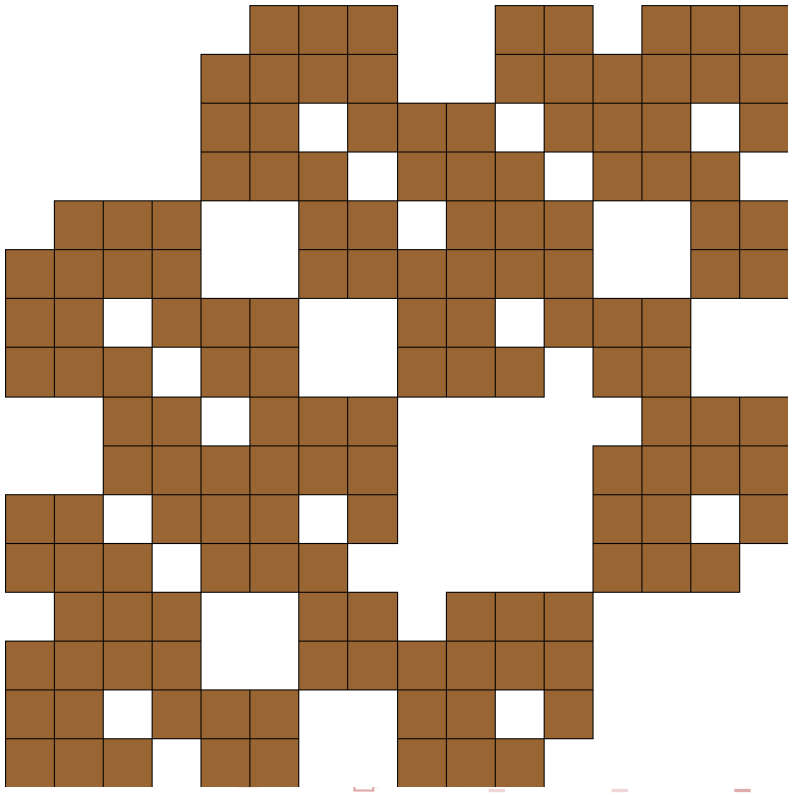

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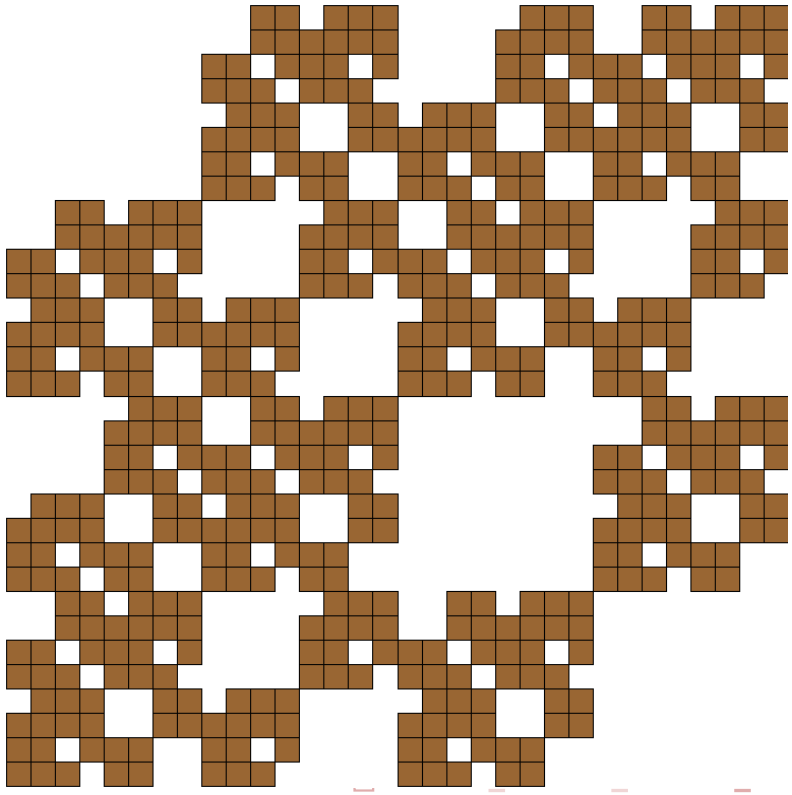

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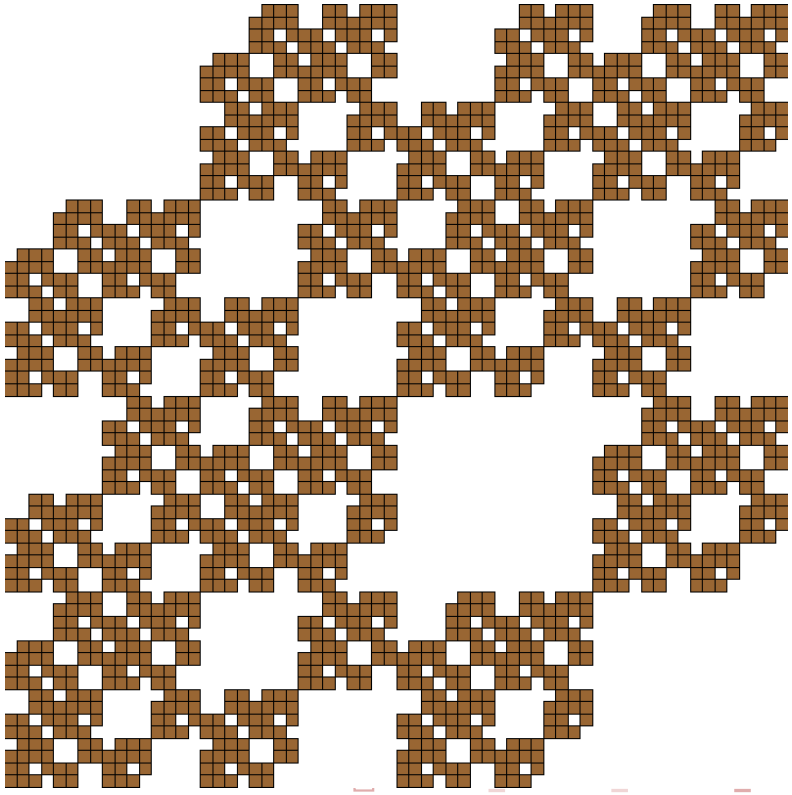

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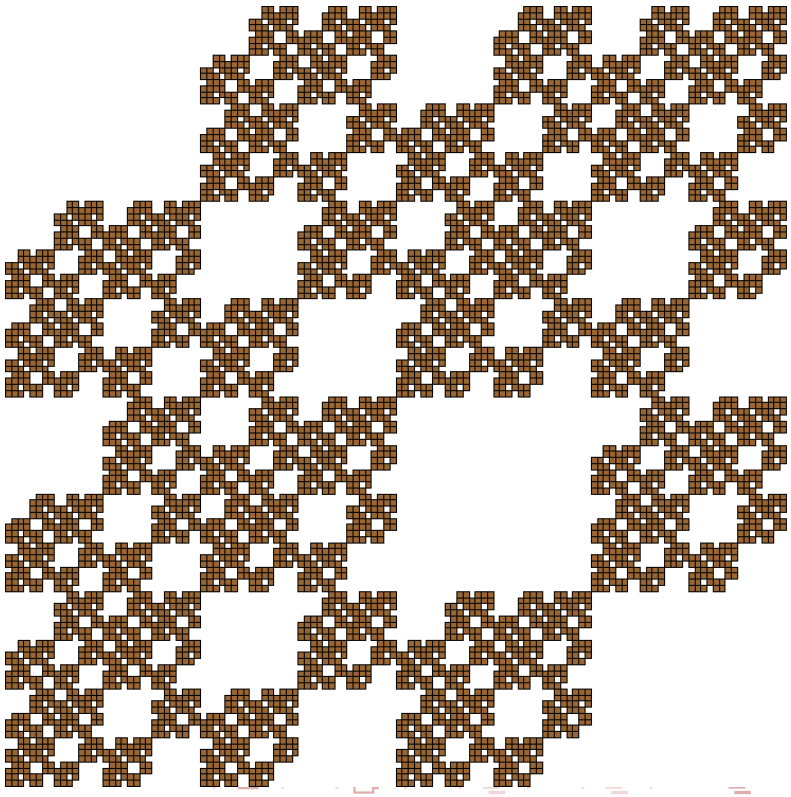

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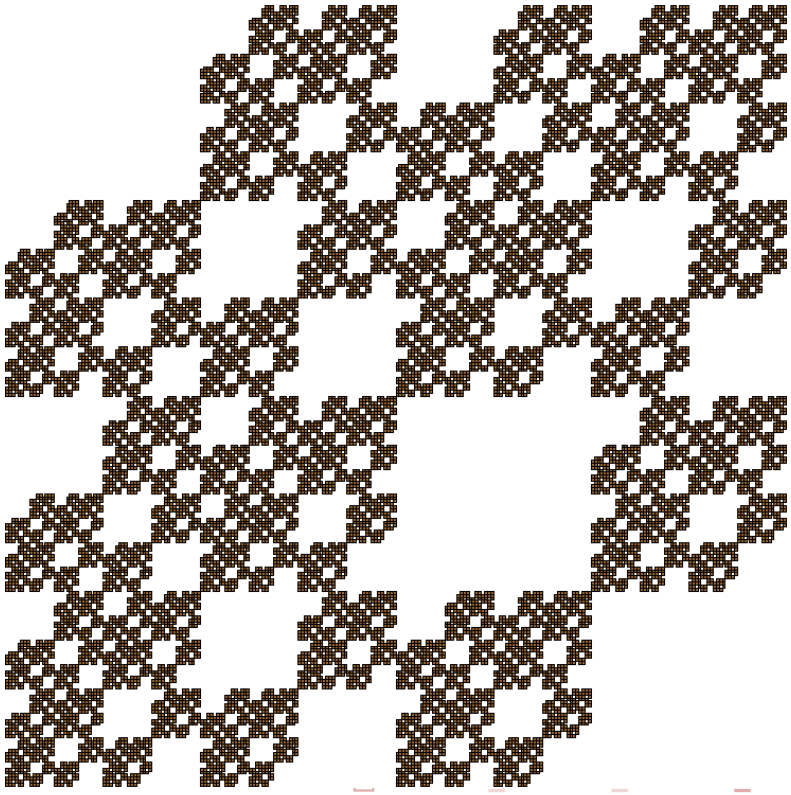

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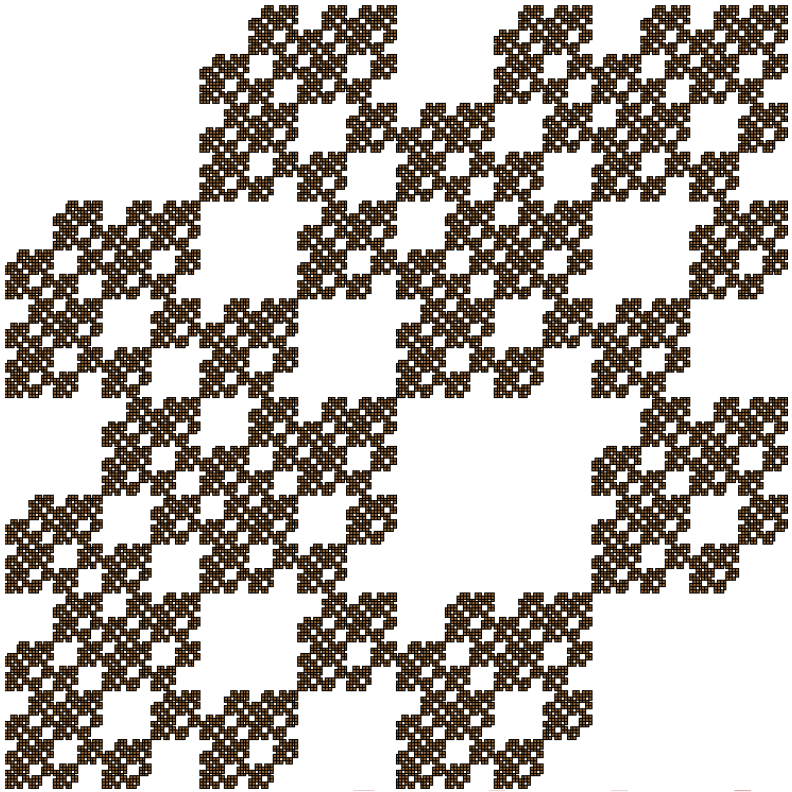

Representing allowed transitions

Vertex set is $\mathcal{T}$.
Directed edge from T_i to T_j if $i \to j$ is allowed.

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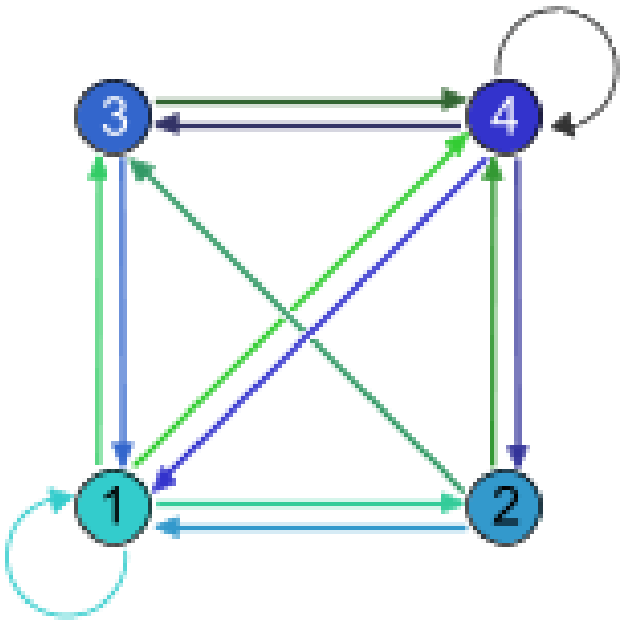

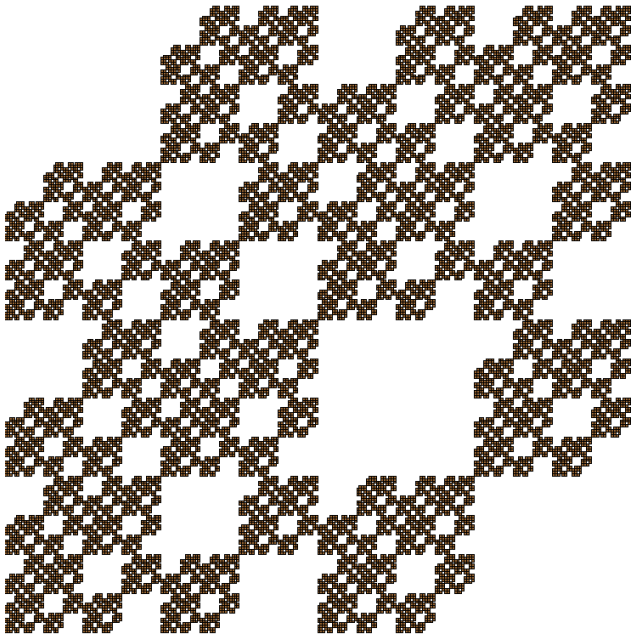

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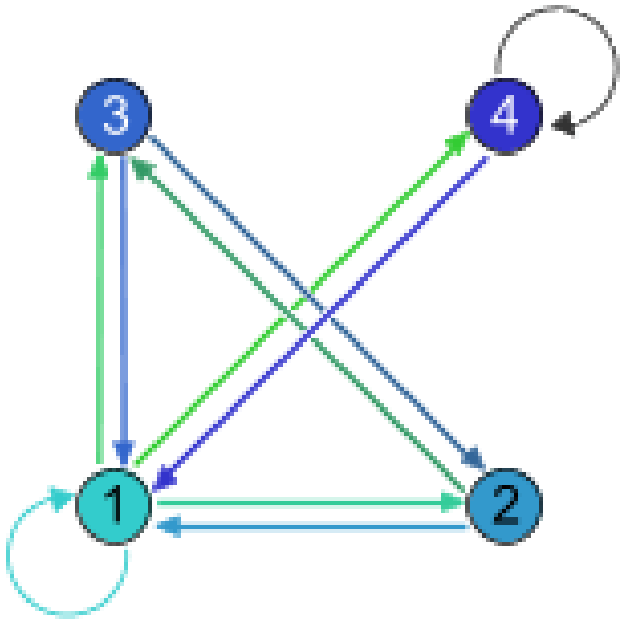

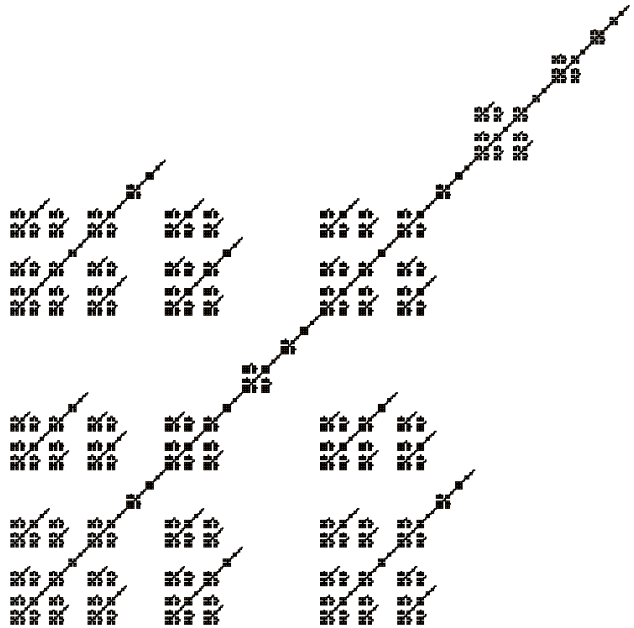

Transition matrices

Represent directed edges by a 0-1 matrix.
Rows/columns indexed by *states*.
$M_{ij} = 1 \iff j \to i$ is allowed.

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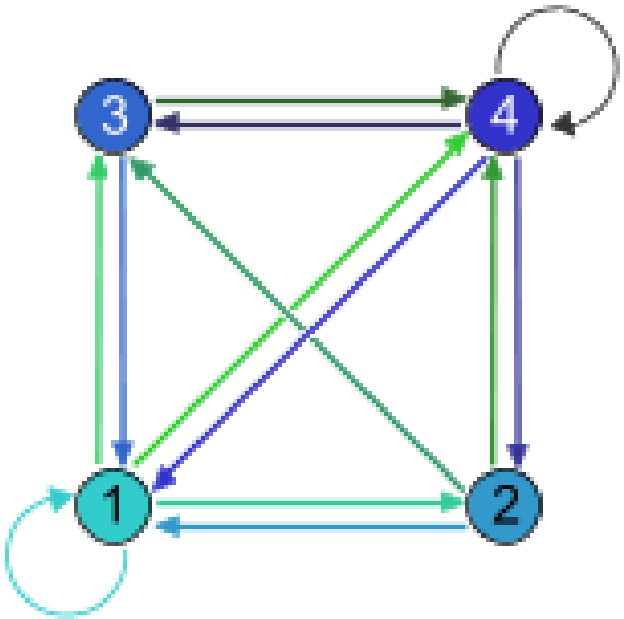

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

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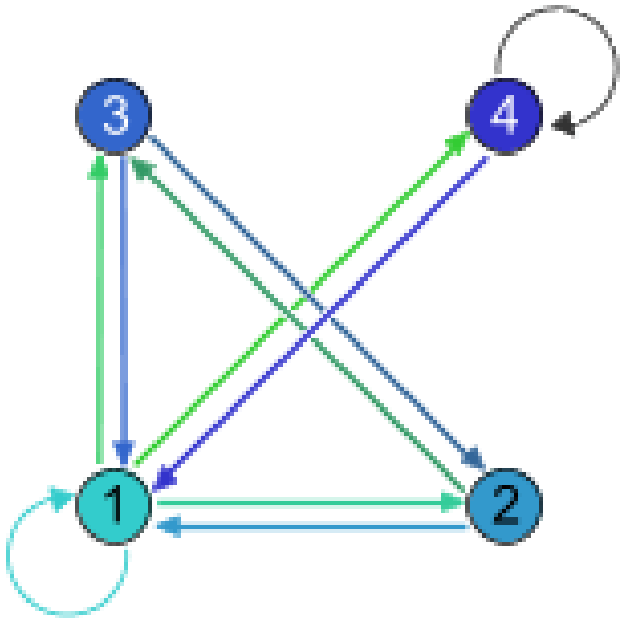

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

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Can use transition matrix M to compute Hausdorff dimension of the attractor A.
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Theorem

The Hausdorff dimension of A is the unique d for which the spectral radius of

$$M(d) = \left[m_{ij} r_j^d \right]_{ij}$$

is exactly 1.

Recall: the **spectral radius** of a matrix M is

$$\rho(M) = \max\{|\lambda_i| \mid \lambda_i \text{ is an eigenvalue of } M\}$$

Terminology

Definition

The attractor of a 1-IFS is . . .

- **IFS-able** *if it can be realized by a 0-IFS.*
- **∞-IFS-able** *if it can be realized by a 0-IFS with countably-many transformations.*
- **non-IFS-able** *if it cannot be realized by any 0-IFS, regardless of how many transformations are used.*

Terminology

Definition

A transformation T_i is called a **full state** *if it can immediately follow any other transformation;*

i.e. $1 \to i$, $2 \to i$, $3 \to i$, $4 \to i$ *are all allowed.*

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Definition

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i.e. $1 \to i$, $2 \to i$, $3 \to i$, $4 \to i$ *are all allowed.*

Also known as a **Rome**, *because all roads in the transition graph lead to it. (Not to be confused with a* **roam**.*)*

Corresponds to a row of 1s in transition matrix.

Main result: 1-IFS to 0-IFS

1. There exists a Rome.
2. Every transformation has a path to it starting at a Rome.
3. There are infinite sequences of non-Romes.

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The attractor of a 1-IFS is ...

- *IFS-able* $\iff$ *(1) and (2) hold.*
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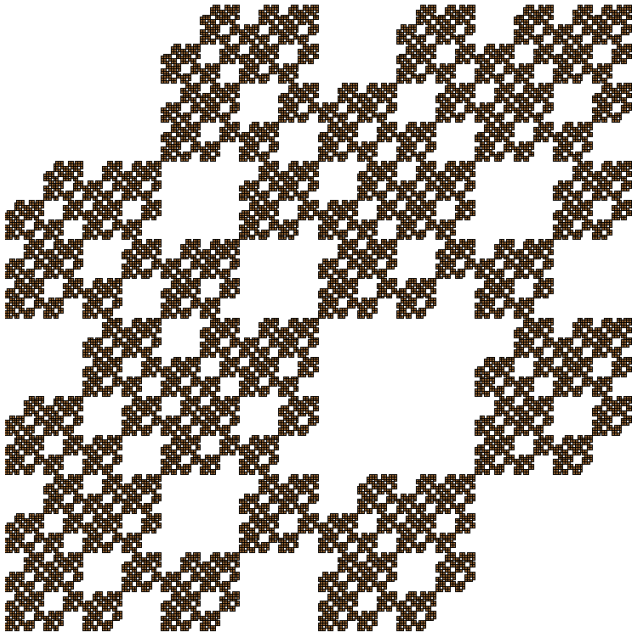
Proof.

Manipulating strings and addresses ... [13] □

Classifying examples

Main idea: look for scaled copies of the attractor within itself.



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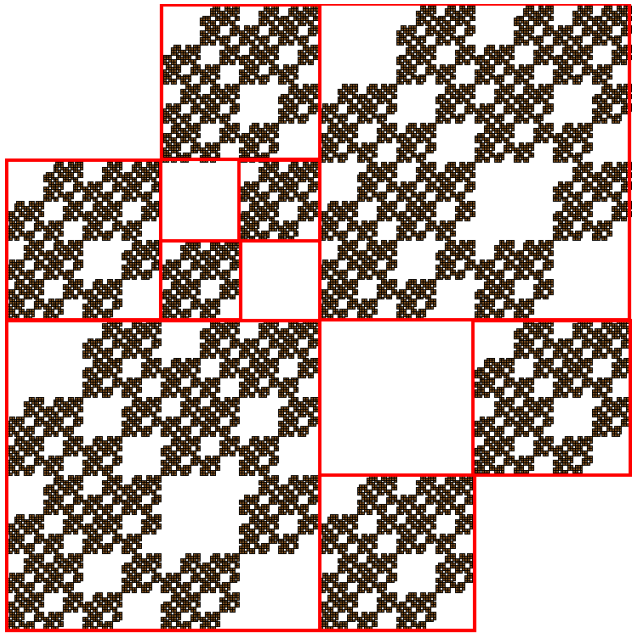

IFS-able with 8 transformations

Classifying examples

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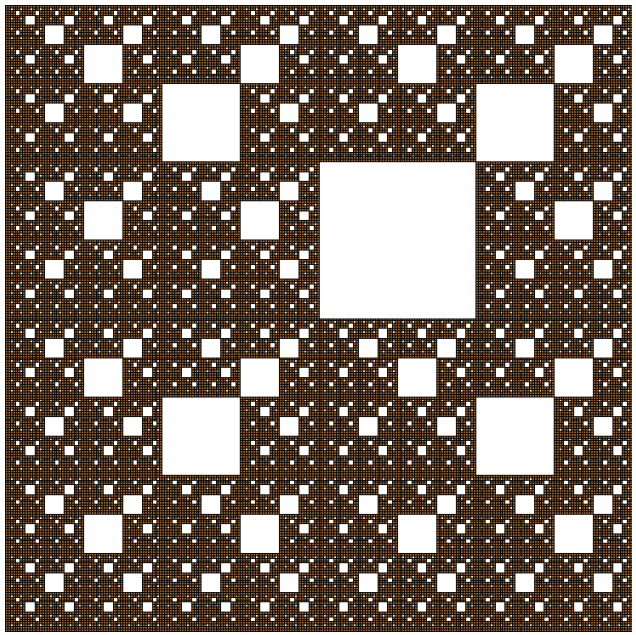

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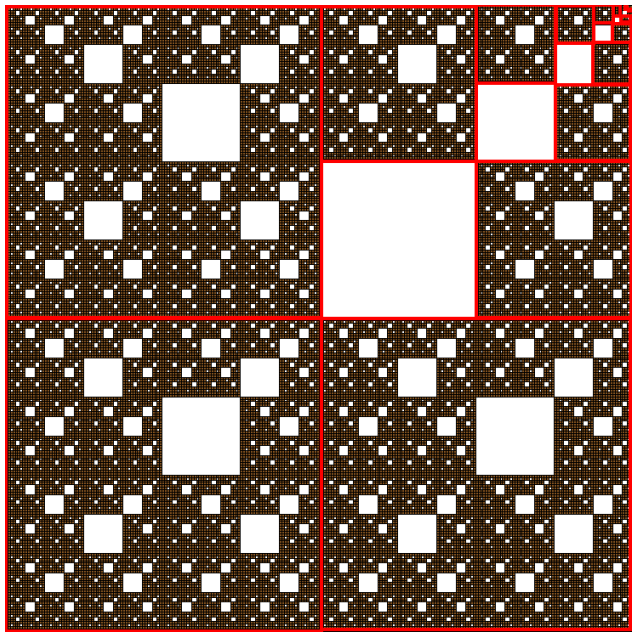

∞-IFS-able

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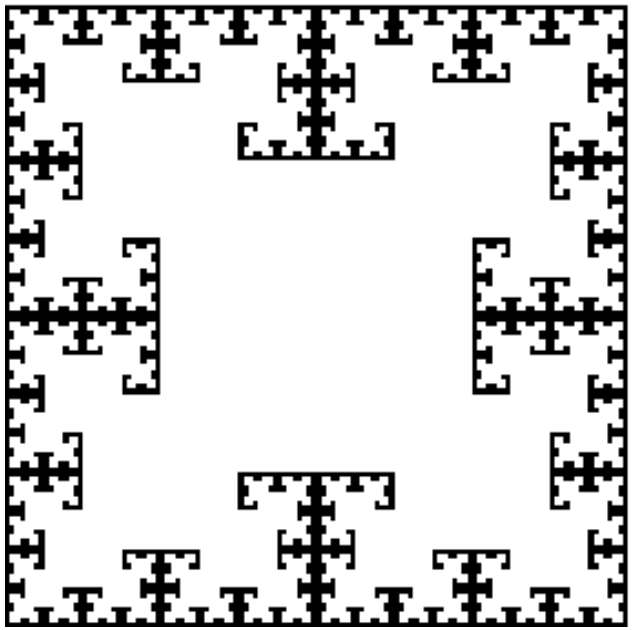
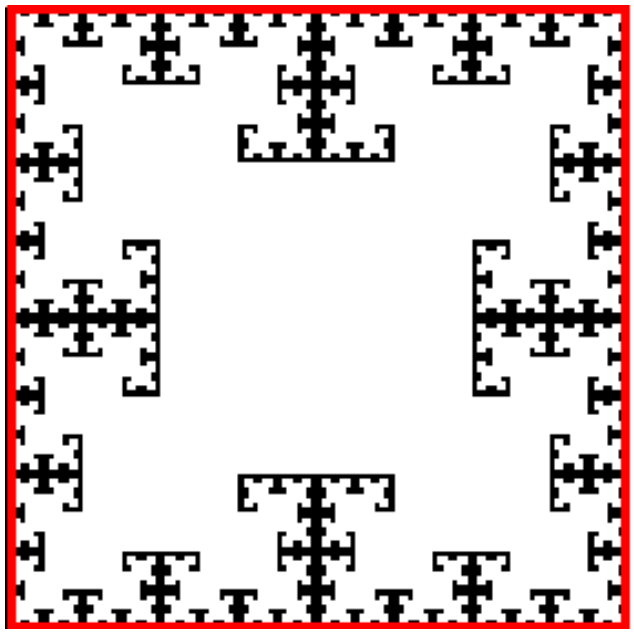


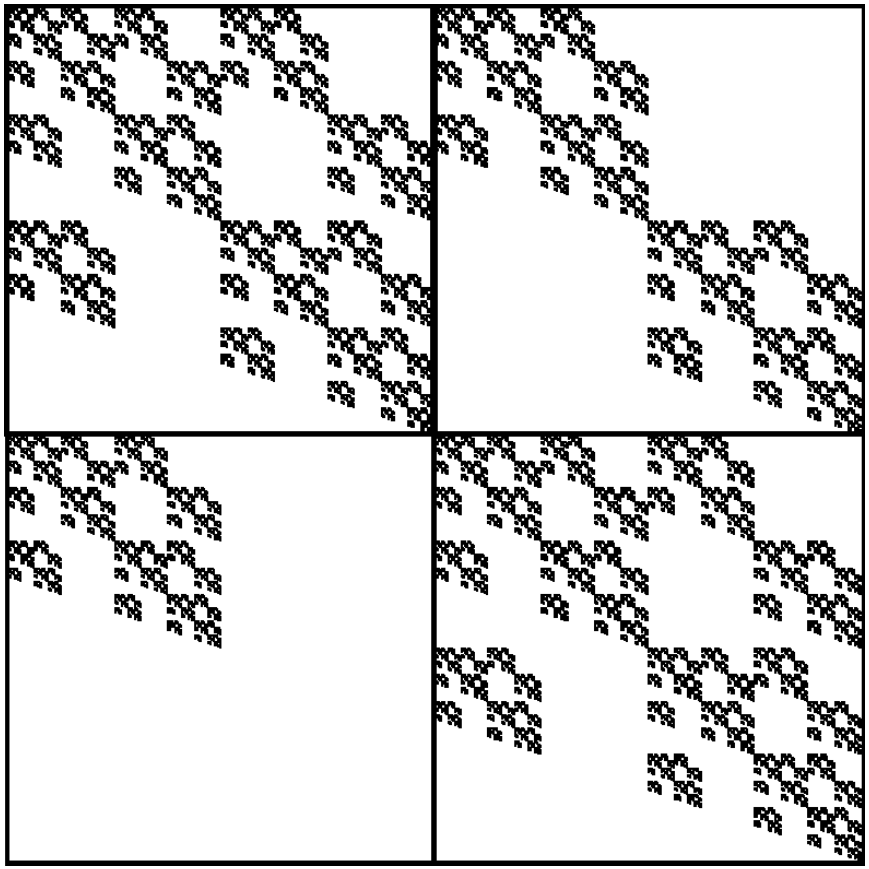
Classifying examples

Main idea: look for scaled copies of the attractor within itself.

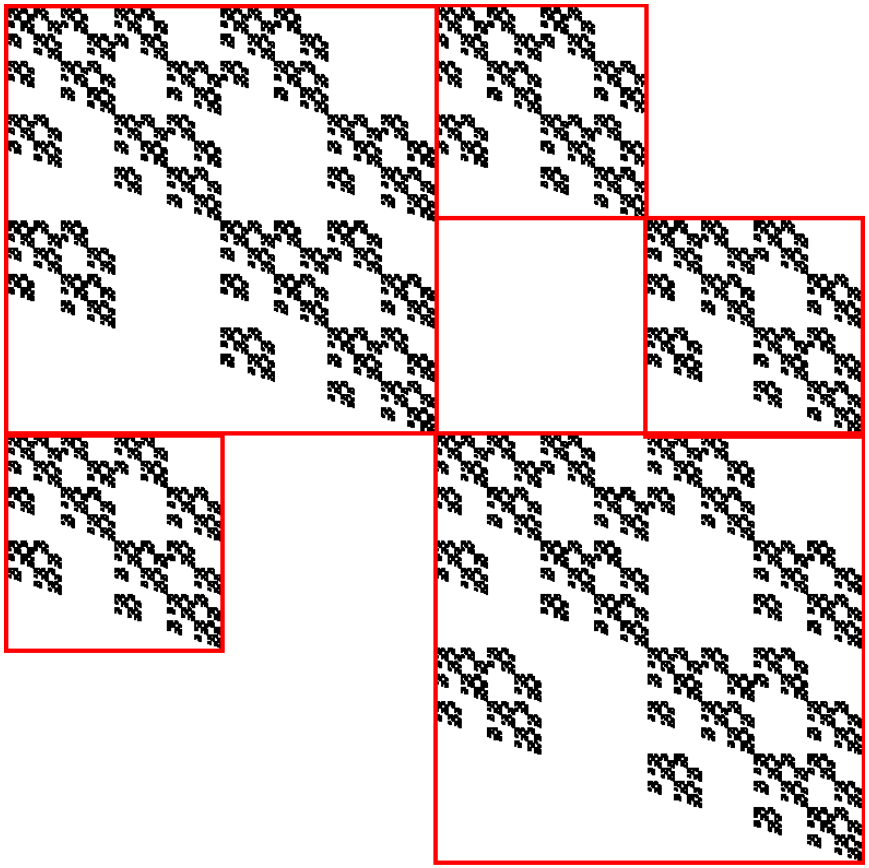


non-IFS-able

Your turn!

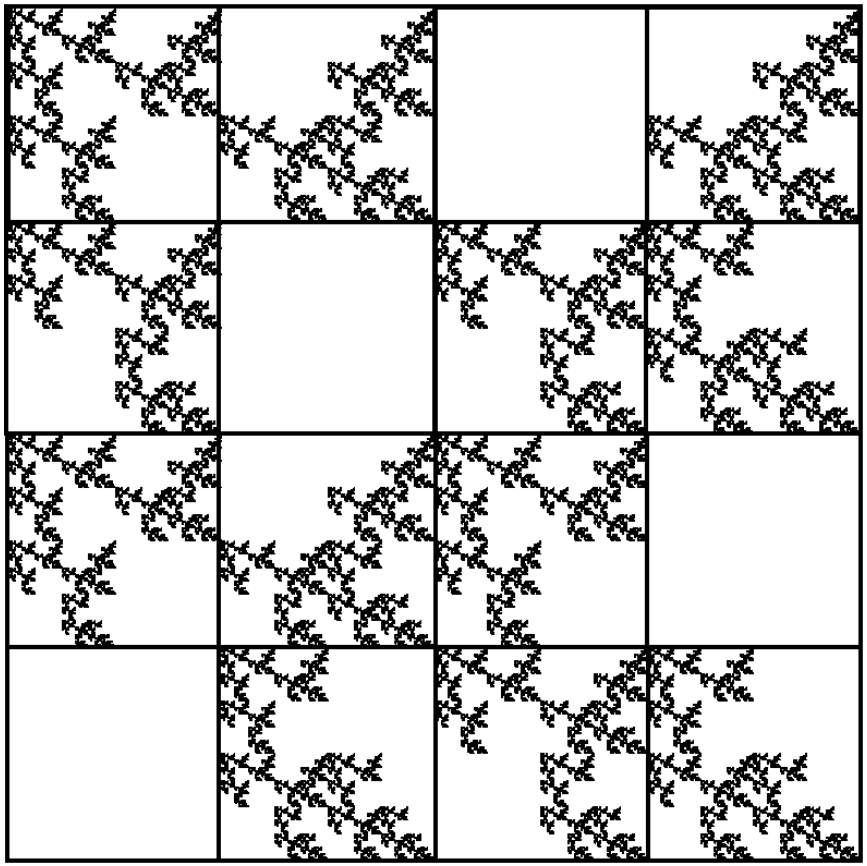


Your turn!

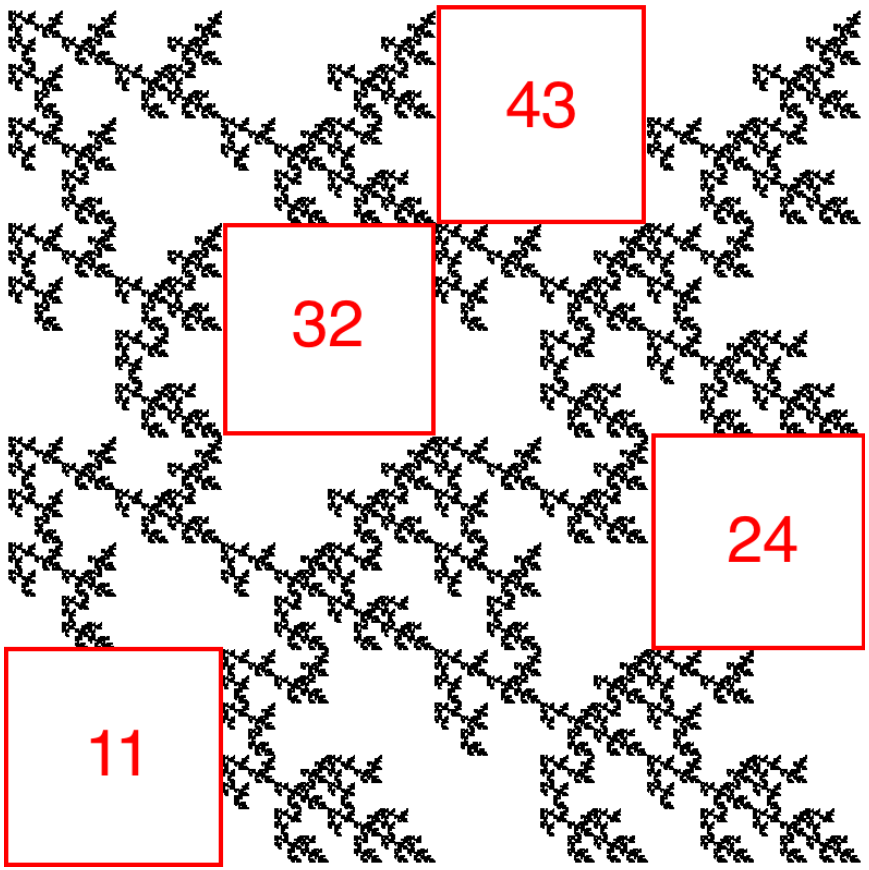


IFS-able with 5 transformations

Your turn!

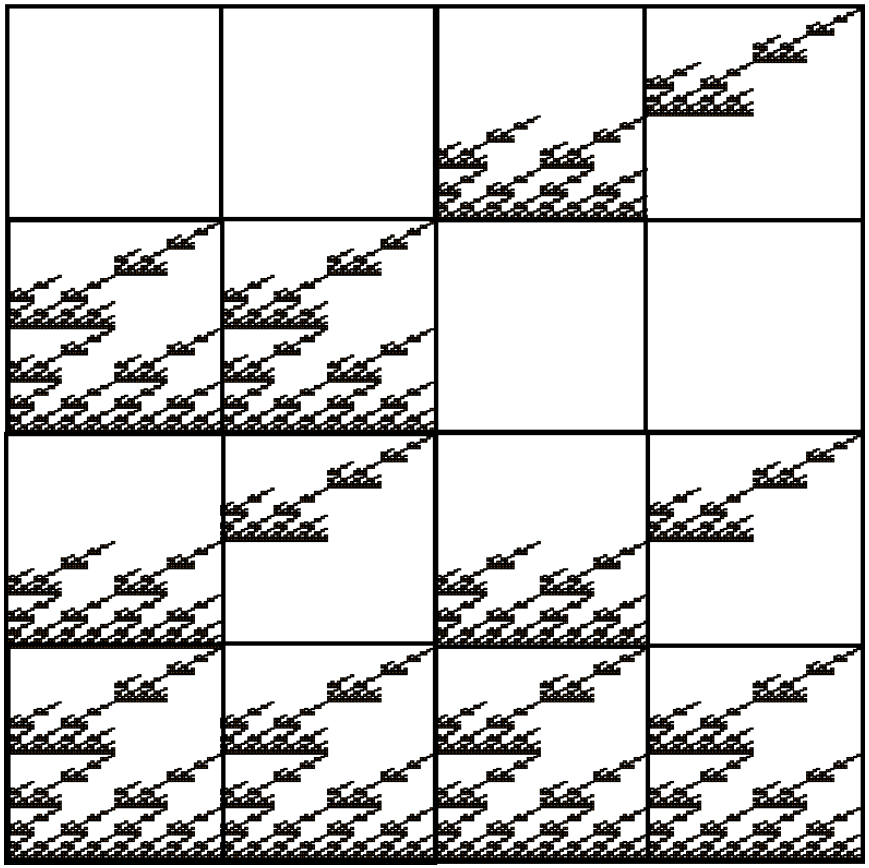


Your turn!

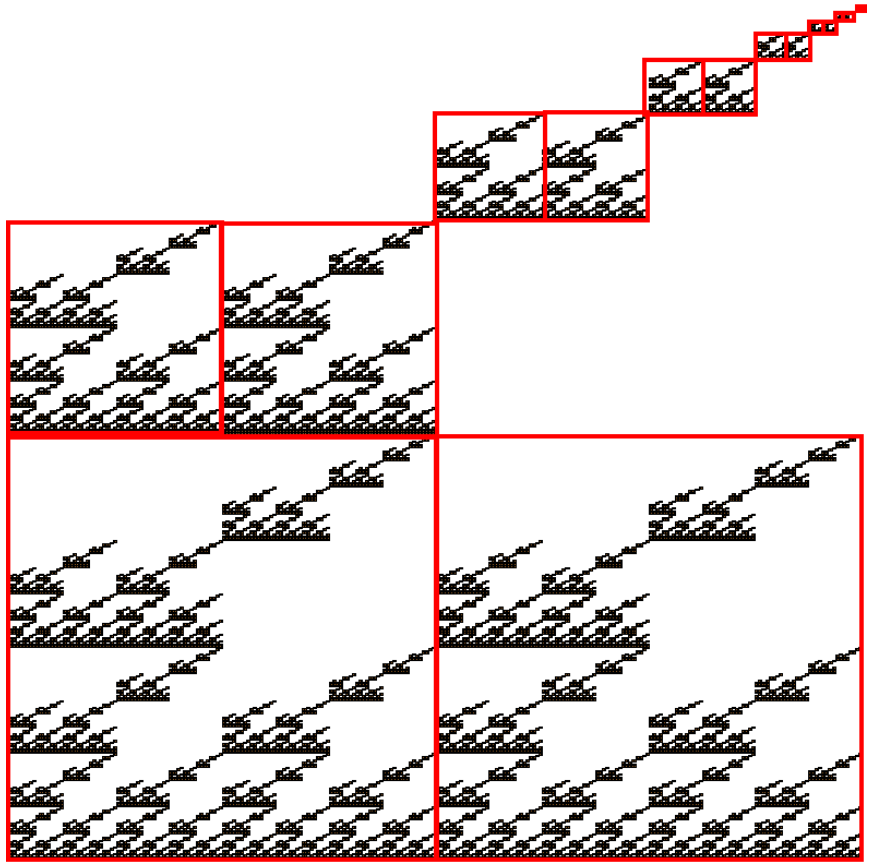


non-IFS-able (no Romes)

Your turn!



Your turn!



∞-IFS-able

Forbidden pairs and triples: 2-IFS

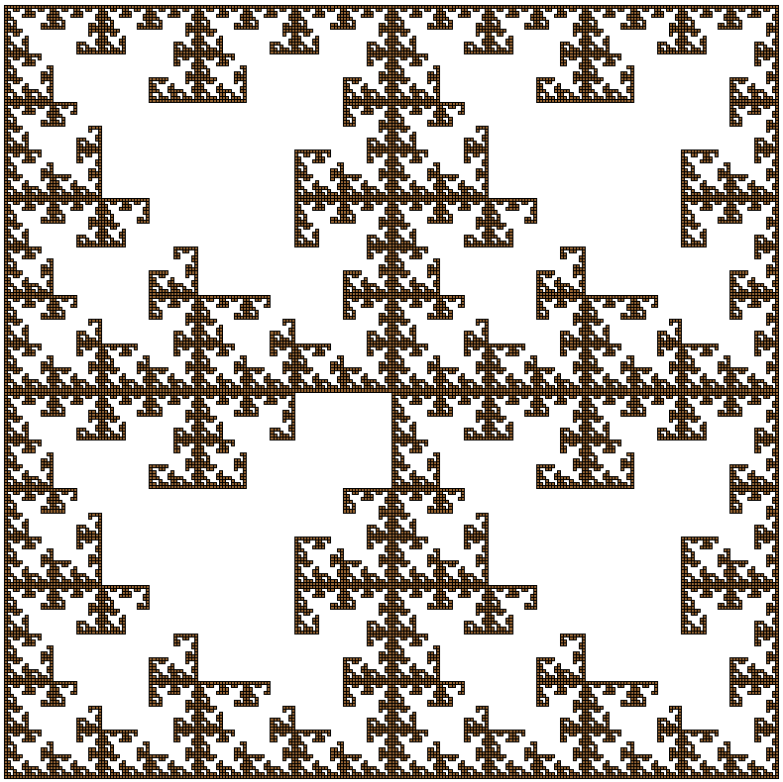
Define IFS by set $\mathcal{F}$ of forbidden strings.

2-IFS means $\mathcal{F}$ contains triples and pairs.

In general, n-IFS means $\mathcal{F}$ contains strings with length at most $n+1$, and contains at least one with exactly that length.

Forbidden pairs and triples: 2-IFS

Working example: $\mathcal{F} = \{14, 23, 32, 441\}$



Subshifts of finite type

Let A be a finite alphabet. Let X be the set of bi-infinite strings from A, called the **full shift**:

$$X = A^{\mathbb{Z}} = \{(\dots x_{-1}.x_0x_1\dots) \mid x_i \in A\}$$

Subshifts of finite type

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Given set of words $\mathcal{F}$ from A, the **shift space determined by** $\mathcal{F}$ is the set of strings from X that contain no element of $\mathcal{F}$ as a substring, written as $X_{\mathcal{F}}$.

If $\mathcal{F}$ is finite, $X_{\mathcal{F}}$ is a **subshift of finite type**.

If the longest string in $\mathcal{F}$ has length $N+1$, say $X_{\mathcal{F}}$ is an **N-step shift**.

Graph vertex shifts

Let G be a directed graph. The **vertex shift** of G has alphabet $A = V$ (vertices of G) and is

$$\hat{X}_G = \{\ldots v_1.v_0v_1\cdots \mid \forall i.\ (v_i, v_{i+1}) \in E\}$$

the set of all infinite paths in G.

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the set of all infinite paths in G.

Lemma

Graph vertex shifts are 1-step shifts of finite type.

Goal: Exploit graph shifts to reduce n-IFS to 1-IFS.

Higher block shifts

Given $A, \mathcal{F}$, and $X_{\mathcal{F}}$ (an N-step shift).

Let $B_N(X_{\mathcal{F}})$ be the set of allowed strings of length N in $X_{\mathcal{F}}$.

Higher block shifts

Given $A, \mathcal{F}$, and $X_{\mathcal{F}}$ (an N-step shift).

Let $B_N(X_{\mathcal{F}})$ be the set of allowed strings of length N in $X_{\mathcal{F}}$.

Let $\beta_N : X_{\mathcal{F}} \to (B_N(X_{\mathcal{F}}))^{\mathbb{Z}}$ be defined by

$$(\beta_N(\underline{x}))_i = x_i x_{i+1} \dots x_{i+N-1}$$

where $\underline{x} = \dots x_{-1}.x_0 x_1 \dots$.

Higher block shifts

Example

Let $A = \{1, 2, 3, 4\}$, $\mathcal{F} = \{14, 23, 32, 441\}$ (i.e. $N = 2$).

Consider $\underline{x} = \ldots 12443 \ldots$

Higher block shifts

Example

Let $A = \{1, 2, 3, 4\}$, $\mathcal{F} = \{14, 23, 32, 441\}$ (i.e. $N = 2$).

Consider $\underline{x} = \ldots 12443 \ldots$

$$\beta_2(\underline{x}) = \ldots 12$$

Higher block shifts

Example

Let $A = \{1, 2, 3, 4\}$, $\mathcal{F} = \{14, 23, 32, 441\}$ (i.e. $N = 2$).

Consider $\underline{x} = \ldots 12443 \ldots$

$$\beta_2(\underline{x}) = \ldots 1224$$

Higher block shifts

Example

Let $A = \{1, 2, 3, 4\}$, $\mathcal{F} = \{14, 23, 32, 441\}$ (i.e. $N = 2$).

Consider $\underline{x} = \ldots 12443 \ldots$

$$\beta_2(\underline{x}) = \ldots 122444$$

Higher block shifts

Example

Let $A = \{1, 2, 3, 4\}$, $\mathcal{F} = \{14, 23, 32, 441\}$ (i.e. $N = 2$).

Consider $\underline{x} = \ldots 12443 \ldots$

$$\beta_2(\underline{x}) = \ldots 12244443 \ldots$$

Higher block shifts

Definition

The N**-th higher block shift** *is the image*

$$X_{\mathcal{F}}^{[N]} = \beta_N(X_{\mathcal{F}})$$

Higher block shifts

Definition

The N**-th higher block shift** *is the image*

$$X_{\mathcal{F}}^{[N]} = \beta_N(X_{\mathcal{F}})$$

Theorem

If $X_{\mathcal{F}}$ *is an* N*-step shift, then there exists a directed graph* G *such that*

$$X_{\mathcal{F}}^{[N]} = \hat{X}_G$$

i.e. N*-th higher block shift can be realized as a 1-step shift!*

Higher block shifts as graph shifts

Proof is in [14].
Vertices of G are $B_N(X_{\mathcal{F}})$, the allowed N-length strings.
Edge from $a_1 \dots a_N$ to $b_1 \dots b_N$ iff

1. $a_2 \dots a_N = b_1 \dots b_{N-1}$
2. $a_1 a_2 \dots a_N b_N$ is an allowed string in $X_{\mathcal{F}}$

Higher block shifts as graph shifts

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Condition (1) ensures correct overlap.
Condition (2) ensures overlap is allowed.

Directed graph encodes which N-length strings of $X_{\mathcal{F}}$ can follow one another; i.e. it encodes which sequences of transformations are allowed by considering longer strings as the most basic elements.

Reducing 2-IFS to 1-IFS: $\mathcal{F} = \{14, 23, 32, 441\}$

$X_{\mathcal{F}}$ is all strings from $I = \{1, 2, 3, 4\}$ without a substring in $\mathcal{F}$.

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$X_{\mathcal{F}}$ is all strings from $I = \{1, 2, 3, 4\}$ without a substring in $\mathcal{F}$.

$X_{\mathcal{F}}^{[2]}$ has alphabet J consisting of 13 allowed pairs:

$$J = \{11, 12, 13, 21, 22, 24, 31, 33, 34, 41, 42, 43, 44\}$$

Construct directed graph with J as vertex set.

Encode edges via transition matrix.

Ensure conditions (1) and (2) from Theorem in [14] are satisfied, i.e. *correct* and *allowed* overlaps.

Reducing 2-IFS to 1-IFS: $\mathcal{F} = \{14, 23, 32, 441\}$

$J = \{11, 12, 13, 21, 22, 24, 31, 33, 34, 41, 42, 43, 44\}$
Column is source of edge, row is target.

$M_{ij,km} = 1 \iff i = m$ and kij is allowed.

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Overlap conditions yield many 0 entries.
(Only 41/169 nonzero entries in this example.)
Helpful in computational applications, such as computing Hausdorff dimension, powers of transition matrix, etc.

$M_{ij,km} = 1 \iff m = i$ and kij is allowed. Recall $\mathcal{F} = \{14, 23, 32, 441\}$

$$M = \begin{array}{c} \\ 11 \\ 12 \\ 13 \\ 21 \\ 22 \\ 24 \\ 31 \\ 33 \\ 34 \\ 41 \\ 42 \\ 43 \\ 44 \end{array} \begin{array}{c} \begin{array}{cccccccccccc} 11 & 12 & 13 & 21 & 22 & 24 & 31 & 33 & 34 & 41 & 42 & 43 & 44 \end{array} \\ \left[\begin{array}{cccccccccccc} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right] \end{array}$$

$M_{ij,km} = 1 \iff m = i$ and kij is allowed. Recall $\mathcal{F} = \{14, 23, 32, 441\}$

$M =$

	11	12	13	21	22	24	31	33	34	41	42	43	44
11	1	1	1	0	0	0	0	0	0	0	0	0	0
12	0	0	0	1	1	1	0	0	0	0	0	0	0
13	0	0	0	0	0	0	1	1	1	0	0	0	0
21	1	1	1	0	0	0	0	0	0	0	0	0	0
22	0	0	0	1	1	1	0	0	0	0	0	0	0
24	0	0	0	0	0	0	0	0	0	1	1	1	1
31	1	1	1	0	0	0	0	0	0	0	0	0	0
33	0	0	0	0	0	0	1	1	1	0	0	0	0
34	0	0	0	0	0	0	0	0	0	1	1	1	1
41	1	1	1	0	0	0	0	0	0	0	0	0	0
42	0	0	0	1	1	1	0	0	0	0	0	0	0
43	0	0	0	0	0	0	1	1	1	0	0	0	0
44	0	0	0	0	0	0	0	0	0	0	1	1	1

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Reducing 2-IFS to 1-IFS: $\mathcal{F} = \{14, 23, 32, 441\}$

Recall $J = \{11, 12, 13, 21, 22, 24, 31, 33, 34, 41, 42, 43, 44\}$.

Define $\mathcal{F}'$ to be the forbidden pairs from alphabet J:

$$\mathcal{F}' = \{ij, km \in J \mid M_{ij,km} = 0\}$$

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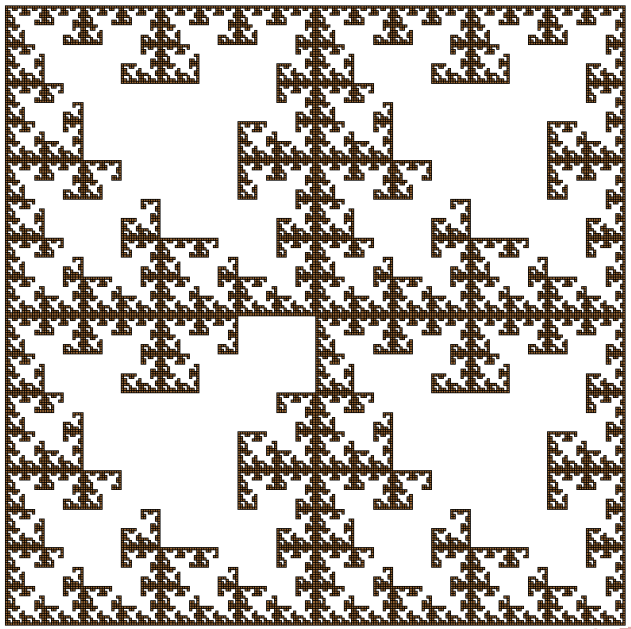
The same attractor is realized from this 1-IFS, $J(\mathcal{F}')$!



Reducing n-IFS to 1-IFS

Generalizing this procedure, any n-IFS (forbidden strings of length $\leq n+1$) can be reduced to a 1-IFS (forbidden pairs only):

1. List all allowed strings of length n
2. Populate transition matrix by following overlap conditions
3. Apply constructive procedure to this new IFS

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Theorem

The n-IFS $I(\mathcal{F})$ and the 1-IFS $J(\mathcal{F}')$ have the same attractor.

Proof.

Compare addresses of attractors, rewrite as I-strings. [11] □

Reducing n-IFS to 1-IFS

Observations:

- Procedure doesn't reduce 1-IFS to 0-IFS.
  Overlap conditions are vacuous.
- Previous results in [13] still helpful.
- Can now characterize all n-IFS as IFS-able / ∞-IFS-able / non-IFS-able

Reducing n-IFS to 1-IFS

Observations:

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- Previous results in [13] still helpful.
- Can now characterize all n-IFS as IFS-able / ∞-IFS-able / non-IFS-able
- If $\mathcal{F}$ contains strings of length $n+1$, alphabet J may have up to 4^n elements!

Question: What is the most *efficient* memory reduction procedure, yielding the least number of transformations?

Efficient memory reduction: $\mathcal{F} = \{14, 23, 32, 441\}$

Notice 441 is a *primary string*, i.e. it does not contain a forbidden substring.

Efficient memory reduction: $\mathcal{F} = \{14, 23, 32, 441\}$

Subdivide T_4 into four transformations. Define new set $\mathcal{S}$ by

$$S_1 = T_1$$
$$S_2 = T_2$$
$$S_3 = T_3$$
$$S_4 = T_4 \circ T_1$$
$$S_5 = T_4 \circ T_2$$
$$S_6 = T_4 \circ T_3$$
$$S_7 = T_4 \circ T_4$$

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$$S_7 = T_4 \circ T_4$$

$$\mathcal{F}_{\mathcal{S}} = \{14, 15, 16, 17, 23, 32, 44, 45, 46, 47, 53, 62, 71, 74\}$$

Efficient memory reduction: $\mathcal{F} = \{14, 23, 32, 441\}$

Subdivide T_4 into four transformations. Define new set $\mathcal{S}$ by

$$
\begin{aligned}
S_1 &= T_1 \\
S_2 &= T_2 \\
S_3 &= T_3 \\
S_4 &= T_4 \circ T_1 \\
S_5 &= T_4 \circ T_2 \\
S_6 &= T_4 \circ T_3 \\
S_7 &= T_4 \circ T_4
\end{aligned}
\qquad
M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}
\begin{array}{c}
\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\left[\begin{array}{ccccccc}
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 & 1
\end{array}\right]
\end{array}
$$

$$\mathcal{F}_{\mathcal{S}} = \{14, 15, 16, 17, 23, 32, 44, 45, 46, 47, 53, 62, 71, 74\}$$

Efficient memory reduction: 2-IFS to 1-IFS

Conjecture

Given 2-IFS, $I(\mathcal{F})$, efficiently equivalent 1-IFS is generated by

1. *Remove non-primary strings from $\mathcal{F}$*
2. *$\forall ijk \in \mathcal{F}$, subdivide i and j*
3. *Reduce forbidden ijk; 2 cases on whether k subdivided*
4. *Reduce forbidden ij; 4 cases on whether i, j subdivided*
5. *Reduce forbidden i; remove compositions*

Cases determine how many transformations needed in total.

To be proved and investigated for n-IFS in [15].

Results

- Can reduce any n-IFS to 1-IFS, perhaps efficiently.
- Can classify any n-IFS as IFS-able or not.
- Can apply method of [16] to calculate Hausdorff dimension.

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- Can classify any n-IFS as IFS-able or not.
- Can apply method of [16] to calculate Hausdorff dimension.
- Know that memory length is *not* a measure of fractal complexity.

Lingering questions

- Is the efficient procedure correct?
- Is the efficient procedure actually helpful in applications?

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- What exactly is the trade-off between memory length and # of transformations? Are certain formulations best for different applications?
- How many memory reductions are there with a fixed # of transformations?

Lingering questions

- Is the efficient procedure correct?
- Is the efficient procedure actually helpful in applications?
- What exactly is the trade-off between memory length and # of transformations? Are certain formulations best for different applications?
- How many memory reductions are there with a fixed # of transformations?
- What is the relationship between m-IFS and n-IFS? Are there embeddings? Is calculting Hausdorff dimension easier in certain settings? (Partially investigated in [12])

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THANK YOU