Allocation of Risk Capital via Intra-Firm Trading

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December 5, 2005

References

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Overview

- Value at Risk
- Coherent and Convex Measures of Risk
- Problem Definition
- Trading Algorithm
- Future Research

Modeling Risk

- Let Ω be the set of states of nature.

- Let random variable $X : \Omega \to \mathbb{R}$ be the final net worth of a financial position, normalized with respect to a risk-free asset.

- A measure of risk is mapping $\rho : \chi \to \mathbb{R}$, where χ is the set of all random variables on Ω.

- $\rho(X)$ specifies how much capital is required to make a position acceptable, i.e.

$$\rho(X) \leq 0 \Rightarrow X \text{ is acceptable.}$$

Value at Risk

VaR, Value at Risk, is a commonly used risk measure. For $X \in \chi$ with distribution $\mathbb{P}$ and $\alpha \in (0, 1)$,

$$VaR_\alpha(X) = -\inf\{x \mid \mathbb{P}[X \leq x] > \alpha\}.$$

The most significant drawback of VaR: it controls the frequency of failures but not their economic consequences.

In addition, VaR is not subadditive. It's easy to find examples where

$$VaR_\alpha(X_a + X_b) > VaR_\alpha(X_a) + VaR_\alpha(X_b).$$

Financial Engineering News, November/December 2004, *A Link Between Option Selling and Rogue Trading?*, based partly on research by Stephen Brown, professor of finance at NYU's Stern School of Business.

Rogue trading has caused significant losses at banks including: National Australia Bank, Allied Irish, Daiwa, Sumitomo and Barings.

The spiking and doubling trading strategies behind the losses are common.

VaR-based risk management tolerates these practices.

Coherent Measures of Risk

Monetary measure of risk ρ will be called *coherent* if it satisfies the following axioms.

1. For all $X, Y \in \chi$, $X \leq Y \Longrightarrow \rho(Y) \leq \rho(X)$.

2. For all $\alpha \in \mathbb{R}$, $\rho(X + \alpha) = \rho(X) - \alpha$.

3. For all $\lambda \geq 0$, $\rho(\lambda X) = \lambda \rho(X)$.

4. $\rho(X + Y) \leq \rho(X) + \rho(Y)$.

Convex Measures of Risk

Monetary measure of risk ρ will be called *convex* if it satisfies the following axioms.

1. For all $X, Y \in \chi, \;\; X \leq Y \Longrightarrow \rho(Y) \leq \rho(X)$.

2. For all $\alpha \in \mathbb{R}, \quad \rho(X + \alpha) = \rho(X) - \alpha$.

3. For any $\lambda \in [0, 1]$: $\;\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y)$.

Representation Theorem

Measure of risk ρ is convex if and only if there exists a family $\mathcal{S}$ of probability measures on Ω and risk limits $K_{\mathbb{S}}$ such that

$$\rho(X) = \sup_{\mathbb{S} \in \mathcal{S}} \left(E_{\mathbb{S}}[-X] + K_{\mathbb{S}} \right).$$

Coherent measures of risk are those convex measures for which the risk limits are zero.

Choose a set of meaningful scenarios and corresponding risk limits. Let a financial position X be acceptable if and only if for each scenario $\mathbb{S} \in \mathcal{S}$ and risk limit $K_{\mathbb{S}}$,

$$E_{\mathbb{S}}[X] \geq K_{\mathbb{S}}.$$

The resulting risk measure is coherent/convex.

Model

- Model a firm that invests in financial markets via trading desks.
- Manage firm-risk by generating a finite set of scenarios with corresponding risk limits.
- Decentralize risk management by allocating a portion of each risk limit to each desk.
- Require each desk to satisfy its portion of the risk limit for each scenario when optimizing its portfolio.

Model

Investment firm that deals on financial markets via D trading desks.

Manage firm risk using scenarios $\mathbb{S} \in \mathcal{S}$ and risk limits $\{K_{\mathbb{S}} \mid \mathbb{S} \in \mathcal{S}\}$.

Allocate risk capital so for each $\mathbb{S} \in \mathcal{S}$

$$\sum_{j=1}^{D} K_{j\mathbb{S}} = K_{\mathbb{S}}.$$

Desk j's problem is

$$\max_{x^{j,i}, 1 \le i \le n_j} \sum_{i=1}^{n_j} x^{j,i} \mathbf{E}_{\mathbb{P}}[X_{j,i}]$$

such that for all $\mathbb{S} \in \mathcal{S}$

$$\sum_{i=1}^{n_j} x^{j,i} \mathbf{E}_{\mathbb{S}}[X_{j,i}] \ge K_{j\mathbb{S}}.$$

The initial allocation of risk capital is arbitrary and may be extremely bad, the idea is to optimize it.

Idea from *Risk Management and Capital Allocation with Coherent Measures of Risk*, by ADEH: allow the desks to trade risk limits until the sum of the desk solutions is firm-optimal.

- Trading must be incentive-compatible.

- Trading mechanism must strictly maintain desk autonomy.

- Use tools from Optimal Partition Theory in Interior Point Methods for Linear Optimization.

Mathematical Tools

Rewrite the j'th desk problem in the following form:

Primal problem (P_j)

$$\min_{x_j}\{c_j^T x_j : A_j x_j = r_j, x_j \geq 0\}$$

and dual problem (D_j)

$$\max_{(y_j, s_j)}\{r_j^T y_j : A_j^T y_j + s_j = c_j, s_j \geq 0\}.$$

Assume each desk problem is feasible.

Also assume there is no arbitrage in the market, i.e. the firm problem is bounded.

The feasible regions for desk j's problem are

$$\begin{aligned}\mathcal{P}_j &= \{x_j : A_j x_j = r_j, x_j \geq 0\} \\ \mathcal{D}_j &= \{(y_j, s_j) : A_j^T y_j + s_j = c_j, s_j \geq 0\}\end{aligned}$$

with optimal solution sets $\mathcal{P}_j^*$ and $\mathcal{D}_j^*$.

Let $x_j^* \in \mathcal{P}_j^*$ and $(y_j^*, s_j^*) \in \mathcal{D}_j^*$. The optimal sets for desk j's problem may be expressed as

$$\begin{aligned}\mathcal{P}_j^* &= \{x_j : A_j x_j = r_j, x_j \geq 0, x_j^T s_j^* = 0\} \\ \mathcal{D}_j^* &= \{(y_j, s_j) : A_j^T y_j + s_j = c_j, s_j \geq 0, s_j^T x_j^* = 0\}.\end{aligned}$$

Examine the effect a perturbation Δr_j of size $\beta \geq 0$ will have on the optimal value of desk j's primal problem.

Define

$$f_j(\beta; r_j, \Delta r_j) = \min_{x_j} \{c_j^T x_j : A_j x_j = r_j + \beta \Delta r_j, x_j \geq 0\}.$$

Function $f_j(\cdot; r_j, \Delta r_j)$ has the following properties.

- $dom(f_j(\cdot; r_j, \Delta r_j))$ is a closed interval of $\mathbb{R}$.

- $f_j(\cdot; r_j, \Delta r_j)$ is continuous, convex and piecewise linear.

Given r_j and rhs-perturbation Δr_j, we would like to determine the linearity intervals and shadow prices of $f_j(\cdot; r_j, \Delta r_j)$ for all $\beta \geq 0$.

Let the optimal solution sets of the perturbed primal and dual problems be denoted $\mathcal{P}^*_{j\beta}$ and $\mathcal{D}^*_{j\beta}$.

Shadow prices: Let $\beta \in dom(f_j)$ and $x^*_j \in \mathcal{P}^*_{j\beta}$. Then

$$\begin{aligned} f'_j(\beta; r_j, \Delta r_j) &= \max_{(y_j,s_j)} \{\Delta r_j^T y_j : (y_j, s_j) \in \mathcal{D}^*_{j\beta}\} \\ &= \max_{(y_j,s_j)} \{\Delta r_j^T y_j : A_j^T y_j + s_j = c_j, s_j \geq 0, s_j^T x^*_j = 0\}. \end{aligned}$$

Extreme points of linearity intervals: Let $\overline{\beta} \in (\beta_1, \beta_2) \subset dom(f_j)$ and $(y^*_j, s^*_j) \in \mathcal{D}^*_{j\overline{\beta}}$. Then

$$\begin{aligned} \beta_2 &= \max_{(\beta,x_j)} \{\beta : x_j \in \mathcal{P}^*_{j\beta}\}. \\ &= \max_{(\beta,x_j)} \{\beta : A_j x_j = r_j + \beta \Delta r_j, x_j \geq 0, x_j^T s^*_j = 0\}. \end{aligned}$$

To preserve desk autonomy, it is useful to consider an alternative method of computing the shadow price.

For desk j let

$$w_j(r_j) = \min_{x_j}\{c_j^T x_j : A_j x_j = r_j, x_j \geq 0\}.$$

As shown earlier, the derivative of w_j in direction Δr_j is given by

$$Dw_j(r_j; \Delta r_j) = \max_{(y_j, s_j)}\{\Delta r_j^T y_j : y_j \in \mathcal{D}_j^*\}.$$

Optimal sets for linear programs have the form

$$\mathcal{D}_j^* = conv\{\tilde{y}_{j1}, \ldots, \tilde{y}_{jn_j}\},$$

so

$$Dw_j(r_j; \Delta r_j) = \max_{(y_j, s_j)}\{\Delta r_j^T y_j : y_j \in conv\{\tilde{y}_{j1}, \ldots, \tilde{y}_{jn_j}\}\}.$$

Writing the convex combinations explicitly gives

$$\begin{aligned} Dw_j(r_j; \Delta r_j) &= \max_{\lambda}\{\Delta r_j^T \sum_{i=1}^{n_j} \lambda_i \widetilde{y}_{ji} : \sum_{i=1}^{n_j} \lambda_i = 1, \lambda_i \geq 0\} \\ &= \max_{\lambda}\{\sum_{i=1}^{n_j} \lambda_i \Delta r_j^T \widetilde{y}_{ji} : \sum_{i=1}^{n_j} \lambda_i = 1, \lambda_i \geq 0\}. \end{aligned}$$

There is only one constraint in this problem, so the dual has only one variable. Writing the dual of this LP gives

$$Dw_j(r_j; \Delta r_j) = \min_{z_j}\{z_j : z_j \geq \Delta r_j^T \widetilde{y}_{ji} \text{ for } i = 1, \ldots, n_j\}.$$

Note that the computation of $Dw_j(r_j; \Delta r_j)$ is correct only if

$$Dw_j(r_j; \Delta r_j) = \max\{\Delta r_j^T \widetilde{y}_{ji} : i = 1, \ldots, n_j\}.$$

Trading Constraints

Create a central risk desk, virtual or physical, that will request and aggregate information to generate advantageous trades.

To generate a set of trades, the risk desk can use a steepest descent approach. Given a set of risk limits $r = (r_1, \ldots, r_D)$,

$$w(r) = \sum_{j=1}^{D} w_j(r_j),$$

where $w_j(r_j)$ is the optimal value of desk j's primal problem given risk capital r_j. One way to improve the allocation of risk capital is to choose a set of trades $\Delta r = (\Delta r_1, \ldots, \Delta r_D)$ that will minimize the derivative of the firm objective function,

$$\min Dw(r, \Delta r) = \min_{\Delta r} \sum_{j=1}^{D} Dw_j(r_j; \Delta r_j).$$

It is straightforward to show the directional derivatives are positively homogeneous, i.e.

$$Dw_j(r_j; \beta\Delta r_j) = \beta Dw_j(r_j; \Delta r_j) \text{ for } \beta \geq 0,$$

so the size of the trades must be normalized to be meaningful. Use the ∞-norm to maintain linearity,

$$\|\Delta r\|_\infty \leq 1.$$

To ensure the firm-level risk limits are satisfied,

$$\sum_{j=1}^{D} \Delta r_j = 0.$$

Trading Algorithm

The trading algorithm proceeds as follows.

1. Each desk j solves (P_j) and (D_j) and submits $\tilde{y}_{j1} \in D_j^*$ to the risk desk.

2. The risk desk solves LP

$$\min_{\Delta r, z} \sum_{j=1}^{D} z_j$$

subject to

$$\begin{aligned} \|\Delta r\|_\infty &\leq 1 \\ \sum_{j=1}^{D} \Delta r_j &= 0 \\ z_j &\geq \Delta r_j^T \tilde{y}_{j1} \text{ for all } j. \end{aligned}$$

3. The risk desk sends z_j and Δr_j^T to desk j for all j. The desks check acceptability of the trades by solving

$$\max(\Delta r_j^T \widetilde{y}_j - z_j)$$

subject to

$$\widetilde{y}_j \in \mathcal{D}_j^*.$$

4. If the optimal value is zero, the trade is accepted. If the optimal value is strictly positive, desk j submits the optimal solution $\widetilde{y}_{j2}$ to the risk desk to be added as a constraint to the trade-generation problem, and the risk desk generates a new set of trades. Repeat steps 2 to 4 until all trades are accepted.

5. When a set of trades is accepted by all desks, each desk submits linearity interval and shadow price data. The risk desk aggregates this information and computes a common step length. The trade is then executed, thus completing one iteration.

Implementation Issues

- Unlike futures and futures derivatives, there is no body of experience to guide scenario generation for equity and fixed income instruments.

- Optimal portfolio values are sensitive to changes in the expected values under the market measure.

- Bid/ask spreads must be introduced to ensure bounded problems.

- Further research needs to inform the choices of, for example, price and volatility ranges and other parameters to generate practical scenarios.

Risk Management Issues

- Value at Risk is still commonly used.
- Coherent risk measures like CVaR are neither widely used nor understood.
- Allowing desks to compute the expected values of their own assets for risk capital allocation purposes is not attractive to risk managers.
- The allocation of risk capital is currently a political process.

Future Possibilities

- Improve the allocation process by making people pay for risk capital. This would cause people to evaluate their need truthfully and would eliminate the political nature of allocation.

- People who use risk capital, for example traders and managers of business units, know fairly accurately what it is worth to them.

- Let the consumers of risk capital trade it. Post bid/ask prices in an internal market.

- Auction off risk capital.