Studies in topological combinatorics

Alan Lew

Studies in topological combinatorics

Research Thesis

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy

Alan Lew

Submitted to the Senate
of the Technion — Israel Institute of Technology
Elul 5781 Haifa August 2021

This research thesis was done under the supervision of Professor Roy Meshulam in the Department of Mathematics.

Some results in this thesis have been published as articles by the author and research collaborators in conferences and journals during the course of the author's doctoral research period, the most up-to-date versions of which being:

Minki Kim and Alan Lew. Complexes of graphs with bounded independence number. *Israel J. Math.* to appear. arXiv:1912.12605.

Minki Kim and Alan Lew. Complexes of graphs with bounded independence number (extended abstract). *Sém. Lothar. Combin.*, 84B:Art. 39, 12, 2020.

Alan Lew. Collapsibility of simplicial complexes of hypergraphs. *The Electronic Journal of Combinatorics*, 26(4):P4.10, 2019.

Alan Lew. Representability and boxicity of simplicial complexes. *Discrete Comput. Geom.*, 2021.

Acknowledgements

I thank Professor Meshulam for his guidance and constant encouragement.

The generous financial help of the Technion is gratefully acknowledged.

Contents

List of Figures

Abstract

Let X be a simplicial complex. X is called *d-Leray* if the homology groups of any induced subcomplex of X vanish in dimensions d and higher. X is called *d-collapsible* if it can be reduced to the void complex by sequentially removing a simplex of size at most d that is contained in a unique maximal face. X is called *d-representable* if it is the nerve of a family of convex sets in $\mathbb{R}^d$. It was shown by Wegner that any d-representable complex is d-collapsible and any d-collapsible complex is d-Leray. Moreover, many combinatorial properties of families of convex sets are known to follow from the d-Lerayness or d-collapsibility of the nerve of the family.

In this thesis we study different combinatorial, topological and geometric aspects of simplicial complexes and the relations between them. We focus in particular on the notions of d-Lerayness, d-collapsibility and d-representability.

First, we prove some general upper bounds on the *collapsibility* of a complex X (the minimum integer d such that X is d-collapsible). We then apply these bounds to several families of simplicial complexes related to different properties of graphs and hypergraphs. As an application, we obtain some old and new results concerning "rainbow independent sets" in graphs.

Inspired by results of Montejano and Oliveros, we study the *t-tolerance complex* of a complex X. This is the complex whose simplices are formed as the union of a simplex in X and a set of size at most t. We show that, for any t and d, there is a function $h(t,d)$ such that the t-tolerance complex of any d-collapsible complex is $h(t,d)$-Leray.

Next, we study the *d-boxicity* of a simplicial complex X, which is the minimal k such that X can be written as the intersection of k d-representable complexes. This is an extension of the classical notion of boxicity of graphs introduced by Roberts. We prove tight upper bounds and corresponding lower bounds on the d-boxicity of simplicial complexes with n vertices, improving upon previous work by Witsenhausen. We also present a related conjecture about the representability of complexes on n vertices.

Finally, we study certain complexes associated to linear and affine spaces over finite fields: we investigate the topology of the complex of line-free sets in a finite affine plane and its relation to blocking sets having certain stability properties, and we study the asymptotic behavior of the Laplacian eigenvalues of complexes of flags in $\mathbb{F}_q^n$, settling a special case of a conjecture of Papikian.

Abbreviations and Notations

$[n]$	:	the set $\{1, 2, \ldots, n\}$
$\binom{V}{k}$	:	the collection of all subsets of size k of the set V
2^V	:	the collection of all subsets of the set V
$X(k)$	:	the collection of all k-dimensional simplices of the complex X
$X^{(k)}$	:	the k-dimensional skeleton of the complex X
$X[U]$	:	the subcomplex of X induced by U
$\mathrm{st}(X, \sigma)$	:	the star of the simplex σ in the complex X
$\mathrm{lk}(X, \sigma)$	:	the link of the simplex σ in the complex X
$\dim(\sigma)$	:	the dimension of the simplex σ
$\dim(X)$	:	the dimension of the complex X
$X * Y$	:	the join of the complexes X and Y
$C_k(X; R)$	:	the space of R-valued k-chains of the complex X
$C^k(X; R)$	:	the space of R-valued k-cochains of the complex X
$\tilde{H}_k(X; R)$	:	the k-th reduced homology group of X with coefficients in R
$\tilde{H}^k(X; R)$	:	the k-th reduced cohomology group of X with coefficients in R
$C(X)$	:	the collapsibility of a complex X
$L(X)$	:	the Leray number of a complex X
$\mathrm{rep}(X)$	:	the representability of a complex X
$h(X)$	:	the maximal dimension of a missing face of X
$N(\mathcal{C})$	:	the nerve of a family of sets $\mathcal{C}$
$\mathrm{Cov}_{\mathcal{H},p}$	:	the complex of sub-hypergraphs of $\mathcal{H}$ with covering number at most p
$\mathrm{Int}_{\mathcal{H}}$	:	the complex of pairwise intersecting sub-hypergraphs of $\mathcal{H}$
$I_n(G)$	:	the complex whose missing faces are the independent sets of size n in G
$f_G(n)$	:	minimal integer k such that any family of k independent sets of size n in G have a rainbow independent set of size n
$\eta(r, t)$	:	the maximum number of vertices in an r-uniform t-critical hypergraph
$\mathcal{T}_t(K)$	:	the t-tolerance complex of K

$\mathrm{box}(G)$	:	the boxicity of the graph G
$\mathrm{box}_d(X)$	:	the d-boxicity of the complex X
$\mathbb{F}_q$	:	the finite field of order q
X_q	:	the complex of line-free subsets of $\mathbb{F}_q^2$
$\hat{X}_q$	:	the complex of line-free subsets of $\mathbb{F}_q^2 \setminus \{0\}$
$\mathrm{Fl}_{n,q}$	:	the complex of flags in $\mathbb{F}_q^n$
$L_k^+(X)$	:	the k-dimensional weighted upper Laplacian on X

Chapter 1

Introduction

A simplicial complex is a topological space formed as the union of simple building blocks, called simplices. Simplicial complexes can be naturally associated to various combinatorial or geometric objects, and the topological structure of these complexes often sheds light on combinatorial properties of the original object.

In this thesis we study different topological, combinatorial and geometric aspects of simplicial complexes and the relations between them. We focus on the properties of d-Lerayness, d-collapsibility and d-representability, which are defined as follows:

Let X be a simplicial complex on vertex set V. For $U \subset V$, the subcomplex of X induced by U is the complex $X[U] = \{\sigma \in X : \sigma \subset U\}$. Let $\mathbb{F}$ be a field. X is called *d-Leray* if $\tilde{H}_k(X[U]; \mathbb{F}) = 0$ for any $U \subset V$ and any $k \geq d$. The *Leray number* of X, denoted by $L(X)$, is the minimum d such that X is d-Leray.

Let η be a simplex of X of size at most d that is contained in a unique maximal face $\tau \in X$. Then, we say that the complex $X' = X \setminus \{\sigma \in X : \eta \subset \sigma \subset \tau\}$ is obtained from X by an *elementary d-collapse*. The complex X is called *d-collapsible* if there exists a sequence of elementary d-collapses from X to the void complex $\emptyset$. The *collapsibility* of X, denoted by $C(X)$, is the minimum d such that X is d-collapsible.

Let $\mathcal{C} = \{C_1, \ldots, C_m\}$ be a family of sets. The *nerve* of $\mathcal{C}$ is the simplicial complex

$$N(\mathcal{C}) = \{I \subset [m] : \cap_{i \in I} C_i \neq \emptyset\}.$$

A complex X is called *d-representable* if it is isomorphic to the nerve of a family of convex sets in $\mathbb{R}^d$. The *representability* of X, denoted by $\text{rep}(X)$, is the minimum d such that X is d-representable.

The notions of d-collapsibility and d-Lerayness were introduced by Wegner in the seminal paper [Weg75], where he showed that any d-representable complex is d-collapsible, and any d-collapsible complex is d-Leray. In the following years, further research was done on the relations between these properties (see e.g. [MT09, Tan10b]) and on combinatorial applications, in particular in the context of Helly-type problems (see e.g. [Kal84, AK85, KM05]).

Here, we first develop some tools for bounding the collapsibility of a simplicial complex. We then apply these tools for studying various families of complexes associated to different properties of graphs, hypergraphs and matrices.

Next, we study the topology of "tolerance complexes", a family of simplicial complexes related to a "tolerant version" of Helly's theorem due to Montejano and Oliveros ([MO11]).

We then study the d-boxicity of a complex, a notion related to d-representability, which generalizes the classical notion of boxicity of a graph due to Roberts ([Rob69]).

Finally, we study two families of complexes associated to vector spaces over finite fields. The first ones are the complexes of line-free sets in finite affine planes. In order to determine the homology of these complexes, we study certain stability properties of affine blocking sets. The second family that we study is that of the complexes of flags in $\mathbb{F}_q^n$. We study the Laplacian eigenvalues of these complexes, solving a special case of a conjecture of Papikian ([Pap16]).

In the following sections we give a detailed account of our results.

1.1 Minimal exclusion sequences and collapsibility of complexes of hypergraphs

Let $\mathcal{H}$ be a finite hypergraph. We identify $\mathcal{H}$ with its edge set. The *rank* of $\mathcal{H}$ is the maximal size of an edge of $\mathcal{H}$.

A set C is a *cover* of $\mathcal{H}$ if $A \cap C \neq \emptyset$ for all $A \in \mathcal{H}$. The *covering number* of $\mathcal{H}$, denoted by $\tau(\mathcal{H})$, is the minimal size of a cover of $\mathcal{H}$.

For $p \in \mathbb{N}$, let

$$\mathrm{Cov}_{\mathcal{H},p} = \{\mathcal{F} \subset \mathcal{H} : \tau(\mathcal{F}) \leq p\}.$$

That is, $\mathrm{Cov}_{\mathcal{H},p}$ is a simplicial complex whose vertices are the edges of $\mathcal{H}$ and whose simplices are the hypergraphs $\mathcal{F} \subset \mathcal{H}$ that can be covered by a set of size at most p. Some topological properties of the complex $\mathrm{Cov}_{\binom{[n]}{r},p}$ were studied by Jonsson in [Jon08].

The hypergraph $\mathcal{H}$ is called *pairwise intersecting* if $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{H}$. Let

$$\mathrm{Int}_{\mathcal{H}} = \{\mathcal{F} \subset \mathcal{H} : A \cap B \neq \emptyset \text{ for all } A, B \in \mathcal{F}\}.$$

So, $\mathrm{Int}_{\mathcal{H}}$ is a simplicial complex whose vertices are the edges of $\mathcal{H}$ and whose simplices are the hypergraphs $\mathcal{F} \subset \mathcal{H}$ that are pairwise intersecting.

Our main results are the following:

Theorem 1.1.1. *Let $\mathcal{H}$ be a hypergraph of rank r. Then $\mathrm{Cov}_{\mathcal{H},p}$ is $\left(\binom{r+p}{r} - 1\right)$-collapsible.*

Theorem 1.1.2. *Let $\mathcal{H}$ be a hypergraph of rank r. Then $\mathrm{Int}_{\mathcal{H}}$ is $\frac{1}{2}\binom{2r}{r}$-collapsible.*

The following examples show that these bounds are sharp:

- Let $\mathcal{H} = \binom{[r+p]}{r}$ be the complete r-uniform hypergraph on $r+p$ vertices. The covering number of $\mathcal{H}$ is $p+1$, but for any $A \in \mathcal{H}$ the hypergraph $\mathcal{H} \setminus \{A\}$ can be covered by a set of size p, namely by $[r+p] \setminus A$. Therefore the complex $\mathrm{Cov}_{\binom{[r+p]}{r},p}$ is the boundary of the $\left(\binom{r+p}{r}-1\right)$-dimensional simplex, so it is homeomorphic to a $\left(\binom{r+p}{r}-2\right)$-dimensional sphere. Hence, $\mathrm{Cov}_{\binom{[r+p]}{r},p}$ is not $\left(\binom{r+p}{r}-2\right)$-Leray, and therefore it is not $\left(\binom{r+p}{r}-2\right)$-collapsible.

- Let $\mathcal{H} = \binom{[2r]}{r}$ be the complete r-uniform hypergraph on $2r$ vertices. Any $A \in \mathcal{H}$ intersects all the edges of $\mathcal{H}$ except the edge $[2r] \setminus A$. Therefore the complex $\mathrm{Int}_{\binom{[2r]}{r}}$ is the boundary of the $\frac{1}{2}\binom{2r}{r}$-dimensional cross-polytope, so it is homeomorphic to a $\left(\frac{1}{2}\binom{2r}{r}-1\right)$-dimensional sphere. Hence, $\mathrm{Int}_{\binom{[2r]}{r}}$ is not $\left(\frac{1}{2}\binom{2r}{r}-1\right)$-Leray, and therefore it is not $\left(\frac{1}{2}\binom{2r}{r}-1\right)$-collapsible.

A related problem was studied by Aharoni, Holzman and Jiang in [AHJ19], where they show that for any r-uniform hypergraph $\mathcal{H}$ and $p \in \mathbb{Q}$, the complex of hypergraphs $\mathcal{F} \subset \mathcal{H}$ with fractional matching number (or equivalently, fractional covering number) smaller than p is $(\lceil rp \rceil - 1)$-collapsible.

Our proofs rely on two main ingredients. The first one is the following theorem:

Theorem 1.1.3. *Let X be a simplicial complex on vertex set V. Let $S(X)$ be the collection of all sets $\{v_1, \ldots, v_k\} \subset V$ satisfying the following condition:*

There exist maximal faces $\sigma_1, \sigma_2, \ldots, \sigma_{k+1}$ of X such that:

- *$v_i \notin \sigma_i$ for all $i \in [k]$,*
- *$v_i \in \sigma_j$ for all $1 \leq i < j \leq k+1$.*

Let $d'(X)$ be the maximum size of a set in $S(X)$. Then X is $d'(X)$-collapsible.

Theorem 1.1.3 is a special case of a more general result, due essentially to Matoušek and Tancer (who stated it in the special case where the complex is the nerve of a family of finite sets, and used it to prove the case $p = 1$ of Theorem 1.1.1; see [MT09]).

The second ingredient is the following combinatorial lemma, proved independently by Frankl and Kalai.

Lemma 1.1.4 (Frankl [Fra82], Kalai [Kal84])**.** *Let $\{A_1, \ldots, A_k\}$ and $\{B_1, \ldots, B_k\}$ be families of sets such that:*

- *$|A_i| \leq r$, $|B_i| \leq p$ for all $i \in [k]$,*
- *$A_i \cap B_i = \emptyset$ for all $i \in [k]$,*
- *$A_i \cap B_j \neq \emptyset$ for all $1 \leq i < j \leq k$.*

Then

$$k \leq \binom{r+p}{r}.$$

Finally, we present some additional applications of Theorem 1.1.3. In particular, we obtain the following result:

Let $\mathbb{F}$ be a field. Let $\mathcal{A}$ be a finite set of matrices in $\mathbb{F}^{m \times n}$. Let

$$\rho(\mathcal{A}) = \max\{\text{rank}(A) : A \in \text{span}(\mathcal{A})\}.$$

For $r \in \mathbb{N}$, define the simplicial complex

$$\text{M}_{\mathcal{A},r} = \{\mathcal{B} \subset \mathcal{A} : \rho(\mathcal{B}) \leq r\}.$$

Theorem 1.1.5. *Assume that $\mathbb{F}$ is infinite. Then, the complex $M_{\mathcal{A},r}$ is $r(r+1)$-collapsible.*

1.2 Complexes of graphs with bounded independence number

Let $G = (V, E)$ be a (simple) graph. A set $I \subset V$ is called an *independent set* in G if no two vertices in I are adjacent in G. The *independence number* of G, denoted by $\alpha(G)$, is the maximal size of an independent set in G. For $U \subset V$, we denote by $G[U]$ the subgraph of G induced by U. For every integer $n \geq 1$, we define the simplicial complex

$$I_n(G) = \{U \subset V : \alpha(G[U]) < n\}.$$

For example, $I_2(G)$ is the clique complex of G, i.e. $U \in I_2(G)$ if and only if $G[U]$ is a complete graph. For any graph G, the complex $I_1(G)$ is just the empty complex $\{\emptyset\}$.

Here, we study the collapsibility of the complexes $I_n(G)$, for several classes of graphs. Our main motivation is the following problem, presented by Aharoni, Briggs, Kim and Kim in [ABKK19]:

Let $\mathcal{F} = \{A_1, \ldots, A_m\}$ be a family of (not necessarily distinct) non-empty subsets of some finite set V. For a positive integer $n \leq m$, a *rainbow set* of size n for $\mathcal{F}$ is a set of n distinct elements in V of the form $\{a_{i_1}, \ldots, a_{i_n}\}$, where $1 \leq i_1 < i_2 < \cdots < i_n \leq m$ and $a_{i_j} \in A_{i_j}$ for each $j \leq n$.

Let G be a graph, and let $\mathcal{F}$ be a finite family of independent sets in G. A *rainbow independent set* in G with respect to $\mathcal{F}$ is a rainbow set for $\mathcal{F}$ that forms an independent set in G. For a positive integer n, let $f_G(n)$ be the minimum integer t such that every collection of t independent sets of size n in G has a rainbow independent set of size n. For a graph class $\mathcal{G}$ and a positive integer n, let

$$f_{\mathcal{G}}(n) = \sup_{G \in \mathcal{G}} f_G(n).$$

The connection between the complexes $I_n(G)$ and the parameters $f_G(n)$ is given by the following version of Kalai and Meshulam's "topological colorful Helly theorem":

Theorem 1.2.1 (Kalai and Meshulam [KM05])**.** *Let X be a d-collapsible simplicial complex on vertex set V, and let $X^c = \{\sigma \subset V : \sigma \notin X\}$. Then, every collection of $d+1$ sets in X^c has a rainbow set belonging to X^c.*

Theorem 1.2.1 is a special case of Theorem 2.1 in [KM05] (see Section 2.4 for a detailed derivation). An immediate application of Theorem 1.2.1 gives us:

Proposition 1.2.2. *Let G be a graph and $n \geq 1$ an integer. Then,*

$$f_G(n) \leq C(I_n(G)) + 1.$$

Proof. Let $G = (V, E)$. Recall that $A \subset V$ does not belong to $I_n(G)$ if and only if A contains an independent set of size n in G. Therefore, by Theorem 1.2.1, every family of $C(I_n(G)) + 1$ independent sets of size n in G has a rainbow set that contains an independent set of size n. ☐

The study of rainbow independent sets originated as a generalization of the "rainbow matching problem" in graphs (note that a matching in a graph is an independent set in its line graph); see e.g. [AB09, ABC$^+$19, BGS17]. The application of collapsibility numbers in the study of rainbow matchings was initiated in [AHJ19], and further developed in [BK19]. In [HL20], the Leray number of complexes of graphs with bounded matching number was studied, and some applications to rainbow matching problems were found.

In [ABKK19], Aharoni et al. proved some results about $f_G(n)$ for different classes of graphs. One of the main conjectures in [ABKK19] is the following.

Conjecture 1.2.3 (Aharoni, Briggs, Kim, Kim [ABKK19])**.** *Let $\mathcal{D}(\Delta)$ be the class of graphs with maximum degree at most Δ, and let n be a positive integer. Then,*

$$f_{\mathcal{D}(\Delta)}(n) = \left\lceil \frac{\Delta+1}{2} \right\rceil (n-1) + 1.$$

It was shown in [ABKK19] that Conjecture 1.2.3 is true for $\Delta \leq 2$ and for $n \leq 3$. In the general case, the best bounds observed by Aharoni et al. are given by

$$\left\lceil \frac{\Delta+1}{2} \right\rceil (n-1) + 1 \leq f_{\mathcal{D}(\Delta)}(n) \leq \Delta(n-1) + 1.$$

It is natural to ask whether the following extension of Conjecture 1.2.3 holds:

Question 1.2.4 (Aharoni [Aha19])**.** *Let G be a graph with maximum degree at most Δ, and let n be a positive integer. Does the following bound hold?*

$$C(I_n(G)) \leq \left\lceil \frac{\Delta+1}{2} \right\rceil (n-1).$$

Our main results are the following:

Theorem 1.2.5. *Let $G = (V, E)$ be a chordal graph and $n \geq 1$ an integer. Then,*

$$C(I_n(G)) \leq n - 1.$$

Moreover, if $\alpha(G) \geq n$, then $C(I_n(G)) = n - 1$.

Theorem 1.2.6. *Let $G = (V, E)$ be a graph with maximum degree at most Δ and $n \geq 1$ an integer. Then,*

$$C(I_n(G)) \leq \Delta(n - 1).$$

The bound in Theorem 1.2.6 is tight only for $\Delta \leq 2$. In the case $n \leq 3$ we can prove the following tight bounds, for general Δ:

Theorem 1.2.7. *Let $G = (V, E)$ be a graph with maximum degree at most Δ. Then,*

$$C(I_2(G)) \leq \left\lceil \frac{\Delta + 1}{2} \right\rceil.$$

Theorem 1.2.8. *Let $G = (V, E)$ be a graph with maximum degree at most Δ. Then,*

$$C(I_3(G)) \leq \begin{cases} \Delta + 2 & \text{if } \Delta \text{ is even,} \\ \Delta + 1 & \text{if } \Delta \text{ is odd.} \end{cases}$$

Theorems 1.2.6, 1.2.7 and 1.2.8 settle Question 1.2.4 affirmatively in the special cases where $\Delta \leq 2$ or $n \leq 3$. Unfortunately, the bound in Question 1.2.4 does not hold in general: In Section 4.5 we present a family of counterexamples to the case $\Delta = 3$.

Combining these results with Proposition 1.2.2, we obtain corresponding upper bounds for $f_G(n)$, thus recovering several results first proved in [ABKK19]. The following bound, however, is new:

Theorem 1.2.9. *Let G be a claw-free graph with maximum degree at most Δ, and let $n \geq 1$ be an integer. Then,*

$$f_G(n) \leq \left\lfloor \left(\frac{\Delta}{2} + 1 \right) (n - 1) \right\rfloor + 1.$$

Theorem 1.2.9 shows that Conjecture 1.2.3 holds for the subclass of claw-free graphs with maximum degree at most Δ, in the case where Δ is even. The proof of Theorem 1.2.9 relies on bounding the collapsibility of certain subcomplexes of the complex $I_n(G)$.

1.3 Leray numbers of tolerance complexes

Let $\mathcal{H}$ be an r-uniform hypergraph on vertex set V. Recall that the *covering number* of $\mathcal{H}$, denoted by $\tau(\mathcal{H})$, is the minimum size of a set $U \subset V$ such that U intersects all the edges of $\mathcal{H}$. The hypergraph $\mathcal{H}$ is called t-critical if $\tau(\mathcal{H}) = t$ and $\tau(\mathcal{H}') < t$ for every hypergraph $\mathcal{H}'$ that is obtained from $\mathcal{H}$ be removing an edge. The *Erdős-Gallai number* $\eta(r,t)$ is the maximum number of vertices in an r-uniform t-critical hypergraph. Erdős and Gallai showed in [EG61] that $\eta(2,t) = 2t$ and $\eta(r,2) = \left\lfloor \left(\frac{r+2}{2}\right)^2 \right\rfloor$. For general r and t, Tuza proved in [Tuz85] the bound

$$\eta(r,t) < \binom{r+t-1}{r-1} + \binom{r+t-2}{r-1},$$

which is tight up to a constant factor. In particular, we have $\eta(r,t) = O(t^{r-1})$ for r fixed and $t \to \infty$, and $\eta(r,t) = O(r^t)$ for t fixed and $r \to \infty$.

Let $\mathcal{F}$ be a family of sets. We say that $\mathcal{F}$ has a *point in common with tolerance* t if there is a subfamily $\mathcal{F}' \subset \mathcal{F}$ such that $|\mathcal{F}'| \geq |\mathcal{F}| - t$ and $\cap_{A \in \mathcal{F}'} A \neq \emptyset$. In [MO11], Montejano and Oliveros proved the following Helly-type theorem.

Theorem 1.3.1 (Montejano-Oliveros [MO11, Theorem 3.1])**.** *Let $\mathcal{F}$ be a family of convex sets in $\mathbb{R}^d$. If every subfamily $\mathcal{F}' \subset \mathcal{F}$ of size at most $\eta(d+1, t+1)$ has a point in common with tolerance t, then $\mathcal{F}$ has a point in common with tolerance t.*

In fact, it was shown in [MO11] that any family of sets satisfying a Helly property satisfies also a corresponding "tolerant Helly property". In terms of simplicial complexes, this may be stated as follows:

Let K be a simplicial complex on vertex set V, and let $t \geq 0$ be an integer. A *missing face* of K is a set $\tau \subset V$ such that $\tau \notin K$ but $\sigma \in K$ for any $\sigma \subsetneq \tau$. Let $h(K)$ be the maximal dimension of a missing face of K.

Define the simplicial complex

$$\begin{aligned} \mathcal{T}_t(K) &= \{\eta \cup \tau : \eta \in K, \tau \subset V, |\tau| \leq t\} \\ &= \{\sigma \subset V : \exists \eta \subset \sigma, |\sigma \setminus \eta| \leq t, \eta \in K\}. \end{aligned}$$

We call $\mathcal{T}_t(K)$ the *t-tolerance complex* of K. Note that $\mathcal{T}_0(K) = K$ for every complex K.

Theorem 1.3.2 (Montejano-Oliveros [MO11, Theorem 1.1])**.** *Let K be a simplicial complex with $h(K) \leq d$, and let $t \geq 0$ be an integer. Then, $h(\mathcal{T}_t(K)) \leq \eta(d+1, t+1) - 1$.*

It is known that any d-Leray complex K satisfies $h(K) \leq d$ (see e.g. [Weg75]). By replacing the $h(K)$ with the collapsibility or Leray number of K, the following conjectures arise:

Conjecture 1.3.3. *Let K be a d-Leray simplicial complex. Then, $\mathcal{T}_t(K)$ is $(\eta(d+1, t+1) - 1)$-Leray.*

Conjecture 1.3.4. *Let K be a d-collapsible simplicial complex. Then, $\mathcal{T}_t(K)$ is $(\eta(d+1, t+1) - 1)$-collapsible.*

Let $t \geq 1$, and let A, B be two disjoint sets of size $t+1$ each. Let K be the simplicial complex on vertex set $A \cup B$ whose maximal faces are the sets A and B. It is easy to check that K is 1-collapsible, and therefore 1-Leray (in fact, it is easy to show that it is even 1-representable). On the other hand, the complex $\mathcal{T}_t(K)$ is the boundary of the simplex $A \cup B$. That is, $\mathcal{T}_t(K)$ is a $2t$-dimensional sphere. In particular, it is not $2t$-Leray. Therefore, for $d = 1$, the bound $\eta(2, t+1) - 1 = 2t + 1$ in Conjectures 1.3.3 and 1.3.4 cannot be improved.

For $t = 1$, it was shown in [MO11, Theorem 3.2] that there exists a d-representable complex K such that $\mathcal{T}_1(K)$ is the boundary of a $\left(\left\lfloor \left(\frac{d+3}{2}\right)^2 \right\rfloor - 1\right)$-dimensional simplex. In particular, $\mathcal{T}_1(K)$ is not $\left(\left\lfloor \left(\frac{d+3}{2}\right)^2 \right\rfloor - 2\right)$-Leray. Therefore, for $t = 1$, the bound $\eta(d+1, 2) - 1 = \left\lfloor \left(\frac{d+3}{2}\right)^2 \right\rfloor - 1$ in Conjectures 1.3.3 and 1.3.4 cannot be improved.

Our main result is the following weak version of Conjectures 1.3.3 and 1.3.4:

Theorem 1.3.5. *Let K be a d-collapsible complex. Let $t \geq 0$. Then, $\mathcal{T}_t(K)$ is $h(t,d)$-Leray, where $h(0,d) = d$ for all $d \geq 0$, and for $t > 0$,*

$$h(t,d) = \left(\sum_{s=1}^{\min\{t,d\}} \binom{d}{s} (h(t-s,d)+1) \right) + d.$$

Note that we require the stronger property (collapsibility) for K, and obtain only the weaker property (Leray) for the tolerance complex. For $d = 1$, we obtain the sharp bound $h(t,1) = 2t + 1 = \eta(2, t+1) - 1$. For $d > 1$, $h(t,d)$ is larger than the conjectural bound $\eta(d+1, t+1) - 1$. However, for fixed t, we have $h(t,d) = O(d^{t+1})$, which is of the same order of magnitude as that of $\eta(d+1, t+1) - 1$.

In the special case $d = 2, t = 1$, we can prove the following stronger bound:

Theorem 1.3.6. *Let K be a 2-collapsible complex. Then, $\mathcal{T}_1(K)$ is 5-Leray.*

Note that $5 = \eta(3,2) - 1$, so the bound in Theorem 1.3.6 is tight.

1.4 Representability and boxicity of simplicial complexes

Let $\mathcal{F} = \{F_1, \ldots, F_n\}$ be a family of sets. The *intersection graph* of $\mathcal{F}$ is the graph on vertex set $[n]$, whose edges are the pairs $\{i,j\}$ for $1 \leq i < j \leq n$ such that $F_i \cap F_j \neq \emptyset$. A graph $G = (V, E)$ is called an *interval graph* if it is isomorphic to the intersection graph of a family of compact intervals in the real line.

Let G be a graph. The *boxicity* of G, denoted by $\text{box}(G)$, is the minimal integer k such that G can be written as the intersection of k interval graphs. Equivalently, $\text{box}(G)$ is the minimal k such that G is isomorphic to the intersection graph of a family of axis-parallel boxes in $\mathbb{R}^k$.

The notion of boxicity was introduced by Roberts in [Rob69]. The following result was first proved by Roberts in [Rob69], and later rediscovered by Witsenhausen in [Wit80]:

Theorem 1.4.1 (Roberts [Rob69], Witsenhausen [Wit80, Theorem 1]). *Let G be a graph with n vertices. Then*

$$box(G) \leq \left\lfloor \frac{n}{2} \right\rfloor.$$

Moreover, $box(G) = \frac{n}{2}$ if and only if G is the complete $\frac{n}{2}$-partite graph with sides of size 2.

We extend the notion of boxicity from graphs to simplicial complexes as follows:

Let X be a simplicial complex. For every $d \geq 1$, we define the *d-boxicity* of X, denoted by $\text{box}_d(X)$, as the minimal k such that X can be written as the intersection of k d-representable simplicial complexes.

Let $G = (V, E)$ be a graph. The *clique complex* of G, denoted by $X(G)$, is the simplicial complex on vertex set V whose simplices are the cliques in G, that is, the sets $U \subset V$ satisfying $\{u, w\} \in E$ for all $u, w \in U$ such that $u \neq w$.

Let $\mathcal{B} = \{B_1, \ldots, B_n\}$ be a family of axis-parallel boxes in $\mathbb{R}^k$. It is well known that any t boxes $B_{i_1}, \ldots, B_{i_t}$ have a point in common if and only if $B_{i_j} \cap B_{i_r} \neq \emptyset$ for every $1 \leq j < r \leq t$. Therefore, the nerve $N(\mathcal{B})$ is exactly the clique complex of the intersection graph of $\mathcal{B}$. So, for any graph G, we have $\text{box}(G) = \text{box}_1(X(G))$. Thus, we can see the parameters $\text{box}_d(X)$ as higher dimensional generalizations of the boxicity of a graph.

Let X be a simplicial complex on vertex set V. Recall that a missing face of X is a set $\tau \subset V$ such that $\tau \notin X$ but $\sigma \in X$ for any $\sigma \subsetneq \tau$, and that $h(X)$ is the maximal dimension of a missing face of X. Note that a complex X satisfies $h(X) = 0$ if and only if it is a simplex, and it satisfies $h(X) = 1$ if and only if it is the clique complex of some graph G (the missing faces of $X(G)$ are the edges of the complement graph of G).

A family $\mathcal{F}$ of subsets of size k of a set V of size n is called a *Steiner (t, k, n)-system* if any subset of V of size t is contained in exactly one set of $\mathcal{F}$. If any subset of V of size t is contained in *at most* one set of $\mathcal{F}$, then $\mathcal{F}$ is called a *partial Steiner (t, k, n)-system.* A Steiner $(2, 3, n)$-system is also called a *Steiner triple system.*

In [Wit80, Theorem 2], Witsenhausen extended Theorem 1.4.1, proving that any simplicial complex X with n vertices whose missing faces are all of dimension exactly d has d-boxicity at most $\frac{1}{2}\binom{n}{d}$. On the other hand, he showed in [Wit80, Theorem 3] that a complex X whose missing faces form a Steiner triple system (in particular, $h(X) = 2$) has 2-boxicity at least $\frac{1}{3}\binom{n}{2}$.

Here, we extend Witsenhausen's lower bound to all values of d, and prove an improved upper bound, matching the lower bound.

Theorem 1.4.2. *Let X be a simplicial complex with n vertices, satisfying $h(X) \leq d$. Then*

$$box_d(X) \leq \left\lfloor \frac{1}{d+1}\binom{n}{d} \right\rfloor.$$

Moreover, if $h(X) = d$, then $box_d(X) = \frac{1}{d+1}\binom{n}{d}$ if and only if the missing faces of X form a Steiner $(d, d+1, n)$-system.

To prove the equality case in Theorem 1.4.2 we will need the following result:

Theorem 1.4.3. *Let X be a complex whose set of missing faces is a partial Steiner $(d, d+1, n)$-system $\mathcal{M}$. Then, X cannot be written as the intersection of less than $|\mathcal{M}|$ d-Leray complexes. On the other hand, the d-boxicity of X is at most $|\mathcal{M}|$. As a consequence,*

$$box_d(X) = |\mathcal{M}|.$$

It was proved by Rödl in [Röd85] that, for any $d \geq 1$, there exist partial Steiner $(d, d+1, n)$-systems of size $(1 - o(1))\frac{1}{d+1}\binom{n}{d}$. Therefore, the bound in Theorem 1.4.2 is asymptotically tight. Moreover, by a well known result of Keevash ([Kee14]), there exist Steiner $(d, d+1, n)$-systems for infinitely many values of n. Thus, the equality case in Theorem 1.4.2 is achieved for infinitely many values of n.

The upper bound in Theorem 1.4.2 follows as a consequence of the next result:

Theorem 1.4.4. *Let X be a simplicial complex on vertex set V. Let $V_1, \ldots, V_k$ be subsets of V satisfying $V_i \notin X$ for all $i \in [k]$, such that for any missing face τ of X there exists some $i \in [k]$ satisfying $|\tau \setminus V_i| \leq 1$. Then, X can be written as an intersection*

$$X = \cap_{i=1}^{k} X_i,$$

where, for all $i \in [k]$, X_i is a $(|V_i| - 1)$-representable complex. In particular, X is $\left(\sum_{i=1}^{k}(|V_i| - 1)\right)$-representable.

Finally, we present the following conjecture related to the representability of complexes without large missing faces:

Conjecture 1.4.5. *Let X be simplicial complex with n vertices, satisfying $h(X) \leq d$. Then,*

$$rep(X) \leq \left\lfloor \frac{dn}{d+1} \right\rfloor.$$

Moreover, $rep(X) = \frac{dn}{d+1}$ if and only if the missing faces of X consist of $\frac{n}{d+1}$ pairwise disjoint sets of size $d+1$.

The $d = 2$ case of the conjecture follows from Robert's theorem (Theorem 1.4.1) and the $d = n - 1$ case follows from a result of Wegner (Theorem 6.3.3). Moreover, the analogous bound is known to hold for Leray numbers (see [Ada14, Proposition 5.4]) and for collapsibility ([KL19, Proposition 3.5]).

1.5 Complexes of line-free sets in finite affine planes

Let q be a prime power and let $\mathbb{F}_q$ be the finite field of order q. A set $\sigma \subset \mathbb{F}_q^2$ is called *line-free* if it does not contain any affine line.

We define the following simplicial complexes:

$$X_q = \left\{\sigma \subset \mathbb{F}_q^2 : \sigma \text{ is line-free}\right\},$$

$$\hat{X}_q = \left\{\sigma \subset \mathbb{F}_q^2 \setminus \{0\} : \sigma \text{ is line-free}\right\} = X_q \setminus 0.$$

A *blocking set* in $\mathbb{F}_q^2$ is a set of points that intersects all the affine lines. One can build a blocking set of size $2q-1$ by taking the union of any two non-parallel lines. The following theorem shows that there are no smaller blocking sets:

Theorem 1.5.1 (Jamison [Jam77], Brouwer-Schrijver [BS78])**.** *The minimum size of a blocking set in $\mathbb{F}_q^2$ is $2q-1$.*

Note that a set $\sigma \subset \mathbb{F}_q^2$ is line-free if and only if its complement is a blocking set. Similarly, a set $\eta \subset \mathbb{F}_q^2 \setminus \{0\}$ is line-free if and only if its complement is a blocking set containing the origin. Therefore, by Theorem 1.5.1, we have

$$\dim(X_q) = \dim(\hat{X}_q) = q^2 - 2q.$$

The homology of the complexes X_q and $\hat{X}_q$ seems to be quite "rich" (see Figures 1.1 and 1.2). We chose to focus on the top-dimensional homology groups $\tilde{H}_{q^2-2q}(X_q)$ and $\tilde{H}_{q^2-2q}(\hat{X}_q)$.

$$\tilde{H}_i(X_2) = \begin{cases} \mathbb{Z}^3 & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$\tilde{H}_i(X_3) = \begin{cases} \mathbb{Z} & \text{if } i = 2, \\ \mathbb{Z}^{11} & \text{if } i = 3, \\ 0 & \text{otherwise.} \end{cases}$$

$$\tilde{H}_i(X_4) = \begin{cases} \mathbb{Z}_3 & \text{if } i = 6, \\ \mathbb{Z}^{45} & \text{if } i = 7, \\ \mathbb{Z}^{20} & \text{if } i = 8, \\ 0 & \text{otherwise.} \end{cases}$$

Figure 1.1: The homology of the complexes X_q, for $q \leq 4$.

Our main result is the following:

Theorem 1.5.2.

$$\tilde{H}_{q^2-2q}(X_q) = \begin{cases} \mathbb{Z}^3 & \text{if } q = 2, \\ \mathbb{Z}^{11} & \text{if } q = 3, \\ \mathbb{Z}^{q(q+1)} & \text{if } q > 3, \end{cases}$$

and

$$\tilde{H}_i(\hat{X}_2) = \begin{cases} \mathbb{Z}^2 & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$\tilde{H}_i(\hat{X}_3) = \begin{cases} \mathbb{Z} & \text{if } i = 2, \\ \mathbb{Z}^4 & \text{if } i = 3, \\ 0 & \text{otherwise.} \end{cases}$$

$$\tilde{H}_i(\hat{X}_4) = \begin{cases} \mathbb{Z}_3 & \text{if } i = 6, \\ \mathbb{Z}^{13} & \text{if } i = 7, \\ \mathbb{Z}^5 & \text{if } i = 8, \\ 0 & \text{otherwise.} \end{cases}$$

Figure 1.2: The homology of the complexes $\hat{X}_q$, for $q \leq 4$.

$$\tilde{H}_{q^2-2q}(\hat{X}_q) = \begin{cases} \mathbb{Z}^2 & \textit{if } q = 2, \\ \mathbb{Z}^{q+1} & \textit{if } q > 2. \end{cases}$$

Let B be a blocking set in $\mathbb{F}_q^2$ of size $2q-1$. We call B *stable* if for every point $v \notin B$ there is some $u \in B$ such that $B \cup \{v\} \setminus \{u\}$ is also a blocking set. We call B *strongly stable* if $0 \in B$ and for every point $v \notin B$ there is some $u \in B \setminus \{0\}$ such that $B \cup \{v\} \setminus \{u\}$ is also a blocking set.

One of the main tools in the proof of Theorem 1.5.2 is the following characterization of stable and strongly stable blocking sets:

Theorem 1.5.3. *Let B be a blocking set in $\mathbb{F}_q^2$ of size $2q-1$. Then, B is stable if and only if B contains an affine line, and it is strongly stable if and only if it contains a line through the origin.*

1.6 Laplacian eigenvalues of complexes of flags

Let $n \geq 3$ be an integer and q be a prime power. Let $\mathrm{Fl}_{n,q}$ be the simplicial complex whose vertices correspond to non-trivial linear subspaces of $\mathbb{F}_q^n$, and whose simplices correspond to flags; that is, the simplices are the sets of subspaces $\{V_1, \ldots, V_k\}$ such that $V_1 \subset \cdots \subset V_k$.

A flag of length $n-1$ is called a *complete flag.* Note that the complete flags are exactly the maximal faces of $\mathrm{Fl}_{n,q}$. In particular, for any prime power q, $\mathrm{Fl}_{n,q}$ is a pure $(n-2)$-dimensional complex.

Let $C^k(\mathrm{Fl}_{n,q})$ be the space of real k-cochains on $\mathrm{Fl}_{n,q}$, and let $d_k : C^k(\mathrm{Fl}_{n,q}) \to C^{k+1}(\mathrm{Fl}_{n,q})$ be the k-th *coboundary operator.*

Let $\mathrm{Fl}_{n,q}(k)$ be the set of k-dimensional simplices of $\mathrm{Fl}_{n,q}$. For $\sigma = \{V_1, \ldots, V_{k+1}\} \in \mathrm{Fl}_{n,q}(k)$, let $w(\sigma)$ be the number of complete flags extending σ. That is, $w(\sigma)$ is the number of maximal faces of $\mathrm{Fl}_{n,q}$ containing σ. We define an inner product on the vector space $C^k(\mathrm{Fl}_{n,q})$ by

$$\langle \phi, \psi \rangle = \sum_{\sigma \in \mathrm{Fl}_{n,q}(k)} w(\sigma)\phi(\sigma)\psi(\sigma).$$

Let d_k^* be the operator adjoint to d_k with respect to this inner product.

We define the *weighted upper k-Laplacian* $L_k^+(\mathrm{Fl}_{n,q}) : C^k(\mathrm{Fl}_{n,q}) \to C^k(\mathrm{Fl}_{n,q})$ by

$$L_k^+(\mathrm{Fl}_{n,q}) = d_k^* d_k.$$

In [Pap16], Papikian conjectured the following:

Conjecture 1.6.1 (Papikian [Pap16])**.** *Let $n \geq 3$ and let $q > 1$ be a prime power. Let $0 \leq k \leq n-3$. Then,*

1. *The number of distinct eigenvalues of $L_k^+(Fl_{n,q})$ does not depend on q.*
2. *As q tends to infinity, the positive (i.e nonzero) eigenvalues of $L_k^+(Fl_{n,q})$ tend to the integers*
$$n-k-2, n-k-1, n-k, \ldots, n-1.$$
Or, more formally: for any $\epsilon > 0$ there exists an integer q_0 such that, for $q \geq q_0$, for any eigenvalue λ of $L_k^+(Fl_{n,q})$ there is some $m \in \{n-k-2, n-k-1, \ldots, n-1\}$ such that
$$|\lambda - m| < \epsilon.$$

Here, we prove the $k = 0$ case of the second part of Papikian's conjecture:

Theorem 1.6.2. *Let $n \geq 3$ and let q be a prime power. Then, for any $\epsilon > 0$ there is an integer q_0 such that, for $q \geq q_0$, any eigenvalue $\lambda \neq 0, n-1$ of $L_0^+(Fl_{n,q})$ satisfies*

$$|\lambda - (n-2)| < \epsilon.$$

That is, as q tends to infinity, all nonzero eigenvalues of $L_0^+(Fl_{n,q})$ either are equal to $n-1$ or tend to $n-2$.

Chapter 2

Background

2.1 Simplicial complexes

Let V be a finite set and let $X \subset 2^V$ be a family of sets. X is called a *simplicial complex* if $\sigma \in X$ for all $\tau \in X$ and $\sigma \subset \tau$. The set V is called the *vertex set* of X. Unless otherwise stated, we always assume that $V = \cup_{\sigma \in X} \sigma$. A set $\sigma \in X$ is called a *simplex* or a *face* of X. The *dimension* of a simplex $\sigma \in X$ is $\dim(\sigma) = |\sigma| - 1$. For short, we call a k-dimensional simplex a *k-simplex*. Let $X(k)$ be the set of all k-simplices.

The *dimension* of the complex X, denoted by $\dim(X)$, is the maximal dimension of a simplex in X.

A *missing face* of a complex X is a set $\tau \subset V$ such that $\tau \notin X$ but $\sigma \in X$ for any $\sigma \subsetneq \tau$. We denote by $h(X)$ the maximum dimension of a missing face of X. If all the missing faces of X are of size 2 (that is, if $h(X) = 1$), then X is called a *flag complex*.

A *subcomplex* of X is a simplicial complex Y such that each simplex of Y is also a simplex of X. The *k-dimensional skeleton* of X, denoted by $X^{(k)}$, is the subcomplex of X consisting of all the faces of X of dimension k or less.

Let $U \subset V$. The subcomplex of X *induced* by U is the complex

$$X[U] = \{\sigma \in X : \sigma \subset U\}.$$

Let $\tau \in X$. We define the *link* of τ in X to be the subcomplex

$$\operatorname{lk}(X, \tau) = \{\sigma \in X : \sigma \cap \tau = \emptyset, \sigma \cup \tau \in X\},$$

the *star* of τ in X to be the subcomplex

$$\operatorname{st}(X, \tau) = \{\sigma \in X : \sigma \cup \tau \in X\}$$

and the *costar* of τ in X to be the subcomplex

$$\operatorname{cost}(X, \tau) = \{\sigma \in X : \tau \not\subset \sigma\}.$$

If $\tau = \{v\}$, we write $\mathrm{lk}(X, v) = \mathrm{lk}(X, \{v\})$, $\mathrm{st}(X, v) = \mathrm{st}(X, \{v\})$ and $X \setminus v = \mathrm{cost}(X, \{v\}) = X[V \setminus \{v\}]$.

Let $X_1, \ldots, X_m$ be simplicial complexes on pairwise disjoint vertex sets. We define the *join* of $X_1, \ldots, X_m$ to be the simplicial complex

$$*_{i=1}^m X_i = X_1 * X_2 * \cdots * X_m = \{\sigma_1 \cup \sigma_2 \cup \cdots \cup \sigma_m : \sigma_i \in X_i \text{ for all } i \in [m]\}.$$

Let $v \in V$. If $v \in \tau$ for every maximal face $\tau \in X$ we say that X is a *cone over* v.

For $U \subset V$, we denote by $2^U = \{\sigma : \sigma \subset U\}$ the *complete complex* on vertex set U.

The *complete* k*-dimensional complex* on vertex set U is the complex

$$(2^U)^{(k)} = \{\sigma \subset U : |\sigma| \leq k + 1\}.$$

2.2 Simplicial homology

Let X be a simplicial complex on vertex set V and let $k \geq -1$. An *ordered* k*-simplex* $[v_0, \ldots, v_k]$ is a k-dimensional simplex $\{v_0, \ldots, v_k\} \in X$ together with an order of its vertices.

Let R be a commutative ring with unit element. Let $C_k(X; R)$ be the free R-module generated by the ordered k-simplices of X, under the relations

$$[v_0, \ldots, v_k] = \mathrm{sgn}(\pi)[v_{\pi(0)}, \ldots, v_{\pi(k)}],$$

for every k-simplex $\{v_0, \ldots, v_k\} \in X$ and permutation $\pi : \{0, \ldots, k\} \to \{0, \ldots, k\}$ (where $\mathrm{sgn}(\pi) \in \{1, -1\}$ is the sign of the permutation).

The elements of $C_k(X; R)$ are called k*-chains*.

We define a homomorphism $\partial_k : C_k(X; R) \to C_{k-1}(X; R)$ that acts on the spanning set as follows:

$$\partial_k[v_0, \ldots, v_k] = \sum_{i=0}^{k} (-1)^i [v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k].$$

The operator ∂_k is called the *boundary operator*.

We define the group of k*-cycles* as $Z_k(X; R) = \mathrm{Ker}(\partial_k)$ and the group of k*-boundaries* as $B_k(X; R) = \mathrm{Im}(\partial_{k+1})$. For any k, we have $B_k(X; R) \subset Z_k(X; R)$, so we can define the quotient

$$\tilde{H}_k(X; R) = Z_k(X; R)/B_k(X; R).$$

The group $\tilde{H}_k(X; R)$ is called the k*-th reduced homology group of* X *with coefficients in* R.

For $R = \mathbb{Z}$, we denote $\tilde{H}_k(X; \mathbb{Z}) = \tilde{H}_k(X)$.

If $\tilde{H}_k(X; R) = 0$ for all $k \geq -1$, we call X *acyclic* (over R).

A useful tool for computing homology is the Mayer-Vietoris long exact sequence:

Theorem 2.2.1 (Mayer-Vietoris)**.** *Let X, Y be simplicial complexes. Then, the following sequence is exact*

$$\cdots \to \tilde{H}_k(X \cap Y; R) \to \tilde{H}_k(X; R) \bigoplus \tilde{H}_k(Y; R) \to \tilde{H}_k(X \cup Y; R) \to \tilde{H}_{k-1}(X \cap Y; R) \to \cdots$$

A useful special case is the following:

Theorem 2.2.2. *Let X be a simplicial complex on vertex set V, and let $v \in V$. Then, the following sequence is exact*

$$\cdots \to \tilde{H}_k(\mathrm{lk}(X, v); R) \to \tilde{H}_k(X \setminus v; R) \to \tilde{H}_k(X; R) \to \tilde{H}_{k-1}(\mathrm{lk}(X, v); R) \to \cdots.$$

Proof. Let $A = X \setminus v$ and $B = \mathrm{st}(X, v)$. By Theorem 2.2.1, we have a long exact sequence

$$\cdots \to \tilde{H}_k(A \cap B; R) \to \tilde{H}_k(A; R) \bigoplus \tilde{H}_k(B; R) \to \tilde{H}_k(A \cup B; R) \to \tilde{H}_{k-1}(A \cap B; R) \to \cdots.$$

Note that B is a cone over v, and therefore $\tilde{H}_k(B; R) = 0$ for all k. Moreover, $A \cup B = X$ and $A \cap B = \mathrm{lk}(X, v)$. So, we obtain a long exact sequence

$$\cdots \to \tilde{H}_k(\mathrm{lk}(X, v); R) \to \tilde{H}_k(X \setminus v; R) \to \tilde{H}_k(X; R) \to \tilde{H}_{k-1}(\mathrm{lk}(X, v); R) \to \cdots,$$

as wanted. □

The homology with field coefficients of a join of complexes can be computed by the following simple formula:

Theorem 2.2.3 (Künneth Theorem)**.** *Let $X = X_1 * X_2 * \cdots * X_m$. Then,*

$$\tilde{H}_i(X; \mathbb{F}) \cong \bigoplus_{\substack{i_1+\cdots+i_m=i-m+1,\\ -1\le i_j\le \dim(X_j)\ \forall j\in[m]}} \tilde{H}_{i_1}(X_1; \mathbb{F}) \otimes \cdots \otimes \tilde{H}_{i_m}(X_m; \mathbb{F}).$$

2.2.1 Nerve theorems

Another tool we will need is the Nerve Theorem: Let $X_1, \ldots, X_m$ be simplicial complexes. The *nerve* of the family $\{X_1, \ldots, X_m\}$ is the simplicial complex

$$N(\{X_1, \ldots, X_m\}) = \{I \subset [m] : \cap_{i \in I} X_i \neq \{\emptyset\}\}.$$

Theorem 2.2.4 (Leray's Nerve Theorem)**.** *Let $X_1, \ldots, X_m$ be simplicial complexes, and let $X = \cup_{i=1}^m X_i$. If for every $\emptyset \neq I \subset [m]$, $\cap_{i \in I} X_i$ is either empty or acyclic, then*

$$\tilde{H}_k(X; R) \cong \tilde{H}_k(N(\{X_1, \ldots, X_m\}); R)$$

for all $k \geq -1$.

We present a proof of Theorem 2.2.4 and some generalizations of it in Appendix 2.A.

The following special case of the Nerve Theorem will be useful: Let X be a simplicial complex, and let $\sigma_1, \ldots, \sigma_m$ be the maximal faces of X. Let

$$N(X) = N(\{2^{\sigma_1}, \ldots, 2^{\sigma_m}\}).$$

Then, we have

Corollary 2.2.5. *For all $k \geq -1$,*

$$\tilde{H}_k\left(X; R\right) \cong \tilde{H}_k\left(N(X); R\right).$$

Proof. For every $I \subset [m]$, we have

$$\cap_{i\in I} 2^{\sigma_i} = 2^{\cap_{i\in I}\sigma_i}.$$

In particular, if $\cap_{i\in I} 2^{\sigma_i} \neq \{\emptyset\}$, then

$$\tilde{H}_k\left(\cap_{i\in I} 2^{\sigma_i}; R\right) = 0$$

for all $k \geq -1$. Therefore, by Theorem 2.2.4, we have

$$\tilde{H}_k\left(X; R\right) \cong \tilde{H}_k\left(N(X); R\right)$$

for $k \geq -1$, as wanted. □

2.2.2 Relative homology

Let X be a simplicial complex and let Y be a subcomplex of X. Let $C_k(X, Y; R)$ be the free R-module generated by the ordered k-simplices in $X \setminus Y$, under the relations

$$[v_0, \ldots, v_k] = \operatorname{sgn}(\pi)[v_{\pi(0)}, \ldots, v_{\pi(k)}],$$

for every k-simplex $\{v_0, \ldots, v_k\} \in X \setminus Y$ and permutation $\pi : \{0, \ldots, k\} \to \{0, \ldots, k\}$. We define a homomorphism $\partial_k : C_k(X, Y; R) \to C_{k-1}(X, Y; R)$ that acts on the spanning set by

$$\partial_k[v_0, \ldots, v_k] = \sum_{\substack{i\in\{0,\ldots,k\}:\\ \{v_0,\ldots,v_{i-1},v_{i+1},\ldots,v_k\}\notin Y}} (-1)^i [v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k].$$

We define the group of *k-cycles* as $Z_k(X, Y; R) = \operatorname{Ker}(\partial_k)$ and the group of *k-boundaries* as $B_k(X, Y; R) = \operatorname{Im}(\partial_{k+1})$. For any k, we have $B_k(X, Y; R) \subset Z_k(X, Y; R)$, so we can define the quotient

$$H_k(X, Y; R) = Z_k(X, Y; R)/B_k(X, Y; R).$$

For $R = \mathbb{Z}$, we denote $H_k(X, Y; \mathbb{Z}) = H_k(X, Y)$.

The relative homology of the pair $Y \subset X$ is related to the homology of the two complexes via the following result:

Theorem 2.2.6 (Long exact sequence of a pair)**.** *Let $Y \subset X$ be simplicial complexes. Then, the following sequence is exact:*

$$\cdots \to \tilde{H}_k(Y; R) \to \tilde{H}_k(X; R) \to H_k(X, Y; R) \to \tilde{H}_{k-1}(Y; R) \to \cdots$$

The following relative version of the Mayer-Vietoris exact sequence will be useful:

Theorem 2.2.7 (Relative Mayer-Vietoris)**.** *Let $Y \subset X$ be simplicial complexes. Let $C \subset A$, $D \subset B$ be simplicial complexes, such that $X = A \cup B$ and $Y = C \cup D$. Then, the following sequence is exact*

$$\begin{aligned} \cdots \to H_k(A \cap B, C \cap D; R) \to H_k(A, C; R) \bigoplus H_k(B, D; R) \to \\ \to H_k(X, Y; R) \to H_{k-1}(A \cap B, C \cap D; R) \to \cdots \end{aligned}$$

2.2.3 Cohomology and Alexander duality

Let X be a simplicial complex. Let R be a commutative ring with unit element.

Let $k \geq -1$. A *k-cochain* is an R-valued skew-symmetric function on the ordered k-simplices. That is, ϕ is a k-cochain if for any two ordered k-simplices $\sigma, \tilde{\sigma}$ in X that are equal as sets, it satisfies $\phi(\tilde{\sigma}) = \text{sgn}(\pi)\phi(\sigma)$, where π is the permutation that maps σ to $\tilde{\sigma}$.

Let $C^k(X)$ denote the space of k-cochains on X. We define a homomorphism $d_k : C^k(X) \to C^{k+1}(X)$ by

$$d_k(\phi)([v_0, \ldots, v_k]) = \sum_{i=0}^{k} (-1)^i \phi([v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k])$$

for any k-cochain ϕ and any ordered k-simplex $[v_0, \ldots, v_k]$. The homomorphism d_k is called the *coboundary operator*.

We define the group of *k-cocycles* as $Z^k(X; R) = \text{Ker}(d_k)$ and the group of *k-coboundaries* as $B^k(X; R) = \text{Im}(d_{k-1})$. For any k, we have $B^k(X; R) \subset Z^k(X; R)$, so we can define the quotient

$$\tilde{H}^k(X; R) = Z^k(X; R)/B^k(X; R).$$

The group $\tilde{H}^k(X;R)$ is called the *k-th reduced cohomology group of X with coefficients in R*. For $R=\mathbb{Z}$, we denote $\tilde{H}^k(X;\mathbb{Z})=\tilde{H}^k(X)$.

Let $\mathbb{F}$ be a field. The following result is a simple corollary of the universal coefficient theorem (see e.g [Hat02]).

Theorem 2.2.8. *Let X be a simplicial complex. Then,*

$$\tilde{H}_k(X;\mathbb{F}) \cong \tilde{H}^k(X;\mathbb{F})$$

for all $k \geq -1$.

Another simple consequence of the universal coefficient theorem is the following:

Lemma 2.2.9. *Let X be a simplicial complex. For any $k \geq -1$, we can write $\tilde{H}_k(X) \cong \mathbb{Z}^{\beta_k} \oplus T_k$, where T_k is a finite abelian group (the torsion subgroup of $\tilde{H}_k(X)$). Then, we have for all $k \geq -1$*

$$\tilde{H}^k(X) \cong \mathbb{Z}^{\beta_k} \oplus T_{k-1}.$$

Let X be a simplicial complex on vertex set V. Let

$$X^V = \{\sigma \subset V : V \setminus \sigma \notin X\}.$$

Note that X^V is also a simplicial complex. X^V is called the *Alexander dual* of X.

It is easy to check that the maximal faces of X^V are the complements of the missing faces of X. Similarly, the missing faces of X^V are the complements of the maximal faces of X.

The homology of X is related to the cohomology of its dual by the following result:

Theorem 2.2.10 (Alexander duality)**.** *Let X be a simplicial complex on vertex set V. If $V \notin X$, then for all $-1 \leq k \leq |V|-2$, we have*

$$\tilde{H}_k(X;R) \cong \tilde{H}^{|V|-k-3}(X^V;R).$$

In particular, if $R=\mathbb{F}$ is a field, we obtain from Theorem 2.2.8:

Corollary 2.2.11. *Let X be a simplicial complex on vertex set V. If $V \notin X$, then for all $-1 \leq k \leq |V|-2$, we have*

$$\tilde{H}_k(X;\mathbb{F}) \cong \tilde{H}_{|V|-k-3}\left(X^V;\mathbb{F}\right).$$

Let X be a simplicial complex, and let $\mathcal{M}$ be the set of missing faces of X. Let

$$\Gamma(X) = \left\{ \mathcal{N} \subset \mathcal{M} : \bigcup_{\tau \in \mathcal{N}} \tau \neq V \right\}.$$

Note that $\Gamma(X)$ is a simplicial complex on vertex set $\mathcal{M}$. The homology groups of X and $\Gamma(X)$ are related as follows:

Theorem 2.2.12 (Björner, Butler, Matveev [BBM97, Theorem 2]). *Let X be a simplicial complex on vertex set V. If X is not the complete complex on V, then for all $-1 \le k \le |V| - 2$,*

$$\tilde{H}_k(X; \mathbb{F}) \cong \tilde{H}_{|V|-k-3}(\Gamma(X); \mathbb{F}).$$

Proof. Note that $\Gamma(X) = N(X^V)$. Therefore, the claim follows from Corollary 2.2.11 and Corollary 2.2.5. □

2.2.4 Leray numbers

Let X be a simplicial complex on vertex set V, and let $\mathbb{F}$ be a field. We say that X is *d-Leray* if

$$\tilde{H}_k(X[U]; \mathbb{F}) = 0$$

for all $U \subset V$ and $k \ge d$.

The *Leray number* of X, denoted by $L(X)$, is the minimal d such that X is d-Leray.

The following result on the Leray numbers of union of complexes will be of use later:

Theorem 2.2.13 (Kalai, Meshulam ([KM06]). *Let $X = \cup_{i=1}^m X_i$. Then*

$$L(X) \le \left(\sum_{i=1}^m (L(X_i) + 1) \right) - 1.$$

2.2.5 Weighted Laplacians

Let X be a simplicial complex on vertex set V. Given an ordered k-simplex $\sigma = [v_0, \ldots, v_k]$ and a vertex $v \in \text{lk}(X, \sigma)$, denote by $v\sigma$ the ordered simplex $[v, v_0, \ldots, v_k]$. For ordered simplices σ, τ in X such that $\tau \subset \sigma$ and $\sigma \setminus \tau = \{v\}$ for some vertex $v \in V$, let $(\sigma : \tau)$ be the sign of the permutation mapping σ to $v\tau$. For ordered simplices σ, τ such that $\sigma = \tau$ as sets, let $(\sigma : \tau)$ be the sign of the permutation mapping σ to τ.

We will consider the simplices in $X(k)$ as ordered simplices, each given an arbitrary fixed order.

We can write the coboundary operator $d_k : C^k(X; \mathbb{R}) \to C^{k+1}(X; \mathbb{R})$ as

$$d_k(\phi)(\sigma) = \sum_{\tau \in \sigma(k)} (\sigma : \tau)\phi(\tau),$$

for any k-cochain ϕ and ordered k-simplex σ, where $\sigma(k) \subset X(k)$ is the set of k-dimensional faces contained in the $(k+1)$-dimensional simplex σ.

Let $w : X \to \mathbb{R}^+$ be a weight function on the simplices. We define an inner product on the vector space $C^k(X; \mathbb{R})$ by

$$\langle \phi, \psi \rangle = \sum_{\sigma \in X(k)} w(\sigma)\phi(\sigma)\psi(\sigma).$$

Let d_k^* be the operator adjoint to d_k with respect to this inner product. Then, we have

Lemma 2.2.14.
$$d_k^*(\psi)(\tau) = \sum_{v\in \mathrm{lk}(X,\tau)} \frac{w(v\tau)}{w(\tau)}\psi(v\tau).$$

Proof. Let $\phi \in C^k(X;\mathbb{R})$ and $\psi \in C^{k+1}(X;\mathbb{R})$. Then,

$$\begin{aligned}
\langle d_k\phi, \psi\rangle &= \sum_{\sigma\in X(k+1)} w(\sigma)d_k\phi(\sigma)\psi(\sigma) = \sum_{\sigma\in X(k+1)}\sum_{\tau\in\sigma(k)} w(\sigma)(\sigma:\tau)\phi(\tau)\psi(\sigma)\\
&= \sum_{\tau\in X(k)}\sum_{\sigma\in X(k+1),\,\tau\subset\sigma} w(\sigma)(\sigma:\tau)\phi(\tau)\psi(\sigma) = \sum_{\tau\in X(k)}\sum_{v\in\mathrm{lk}(X,\tau)} w(v\tau)(v\tau:\tau)\phi(\tau)\psi(v\tau)\\
&= \sum_{\tau\in X(k)} w(\tau)\phi(\tau)\left(\sum_{v\in\mathrm{lk}(X,\tau)}\frac{w(v\tau)}{w(\tau)}\psi(v\tau)\right),
\end{aligned}$$

where we used the facts that, since ψ is a cochain, we have $(\sigma:\tau)\psi(\sigma) = (v\tau:\tau)\psi(v\tau)$, and that $(v\tau:\tau)=1$. Thus, we obtain

$$d_k^*(\psi)(\tau) = \sum_{v\in \mathrm{lk}(X,\tau)} \frac{w(v\tau)}{w(\tau)}\psi(v\tau),$$

as wanted. □

We define the *weighted upper k-Laplacian* $L_k^+ : C^k(X;\mathbb{R}) \to C^k(X;\mathbb{R})$ by

$$L_k^+ = d_k^* d_k.$$

Let $k \geq 0$ and $\sigma \in X(k)$. We define the k-cochain 1_σ by

$$1_\sigma(\tau) = \begin{cases} (\sigma:\tau) & \text{if } \sigma=\tau \text{ (as sets)},\\ 0 & \text{otherwise.}\end{cases}$$

The set $\{1_\sigma\}_{\sigma\in X(k)}$ forms a basis of the space $C^k(X;\mathbb{R})$, that we will call the standard basis.

We will identify the operator L_k^+ with its matrix representation in the standard basis. For $\sigma,\tau\in X(k)$, we will denote by $L_k^+(\sigma,\tau)$ the matrix element at row indexed by 1_σ and column indexed by 1_τ. That is, $L_k^+(\sigma,\tau) = L_k^+ 1_\tau(\sigma)$.

Lemma 2.2.15. *Let $\sigma,\tau\in X(k)$. Then,*

$$L_k^+(\sigma,\tau) = \begin{cases} \sum_{v\in\mathrm{lk}(X,\sigma)}\frac{w(v\sigma)}{w(\sigma)} & \text{if } \sigma=\tau,\\ -\frac{w(\sigma\cup\tau)}{w(\sigma)}(\sigma:\sigma\cap\tau)(\tau:\sigma\cap\tau) & \text{if } |\sigma\cap\tau|=k,\ \sigma\cup\tau\in X(k+1),\\ 0 & \text{otherwise.}\end{cases}$$

Proof. Let $\phi \in C^k(X;\mathbb{R})$ and $\sigma \in X(k)$. Then, we have

$$\begin{aligned}
L_k^+\phi(\sigma) = d_k^* d_k \phi(\sigma) &= \sum_{v\in \mathrm{lk}(X,\sigma)} \frac{w(v\sigma)}{w(\sigma)} d_k\phi(v\sigma) = \sum_{v\in \mathrm{lk}(X,\sigma)} \sum_{\substack{\theta\in X(k),\\ \theta\subset v\sigma}} \frac{w(v\sigma)}{w(\sigma)} (v\sigma:\theta)\phi(\theta)\\
&= \sum_{v\in \mathrm{lk}(X,\sigma)} \left((v\sigma:\sigma)\frac{w(v\sigma)}{w(\sigma)}\phi(\sigma) + \sum_{\eta\in\sigma(k-1)} \frac{w(v\sigma)}{w(\sigma)}(v\sigma:v\eta)\phi(v\eta)\right).
\end{aligned}$$

Using the fact that $(v\sigma:\sigma)=1$ and $(v\sigma:v\eta)=-(\sigma:\eta)$, we obtain

$$\begin{aligned}
L_k^+\phi(\sigma) &= \sum_{v\in \mathrm{lk}(X,\sigma)} \frac{w(v\sigma)}{w(\sigma)}\phi(\sigma) - \sum_{v\in \mathrm{lk}(X,\sigma)} \sum_{\eta\in\sigma(k-1)} \frac{w(v\sigma)}{w(\sigma)}(\sigma:\eta)\phi(v\eta)\\
&= \sum_{v\in \mathrm{lk}(X,\sigma)} \frac{w(v\sigma)}{w(\sigma)}\phi(\sigma) - \sum_{\substack{\theta\in X(k),\\ |\sigma\cap\theta|=k,\\ \sigma\cup\theta\in X}} \frac{w(\sigma\cup\theta)}{w(\sigma)}(\sigma:\sigma\cap\theta)(\theta:\sigma\cap\theta)\phi(\theta),
\end{aligned}$$

where we used the fact that, since ϕ is a k-cochain, we have, for $\sigma\in X(k)$, $v\in \mathrm{lk}(X,\sigma)$, $\eta\in\sigma(k-1)$ and $\theta=\eta\cup\{v\}$,

$$\begin{aligned}
&(\sigma:\eta)\phi(v\eta) = (\sigma:\eta)(\theta:v\eta)\phi(\theta) = (\sigma:\eta)(\theta:\eta)\phi(\theta)\\
&\quad = (\sigma:\sigma\cap\theta)(\sigma\cap\theta:\eta)(\theta:\sigma\cap\theta)(\sigma\cap\theta:\eta)\phi(\theta) = (\sigma:\sigma\cap\theta)(\theta:\sigma\cap\theta)\phi(\theta).
\end{aligned}$$

Finally, setting $\phi = 1_\tau$ for $\tau\in X(k)$, we obtain

$$\begin{aligned}
&L_k^+(\sigma,\tau) = L_k^+ 1_\tau(\sigma)\\
&\quad = \begin{cases} \sum_{v\in \mathrm{lk}(X,\sigma)} \frac{w(v\sigma)}{w(\sigma)} & \text{if } \sigma=\tau,\\ -\frac{w(\sigma\cup\tau)}{w(\sigma)}(\sigma:\sigma\cap\tau)(\tau:\sigma\cap\tau) & \text{if } |\sigma\cap\tau|=k,\ \sigma\cup\tau\in X(k+1),\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

□

In particular, for $k=0$, we obtain:

Corollary 2.2.16. *Let $u,u'\in V$. Then,*

$$L_0^+(u,u') = \begin{cases} \sum_{v\in V:\,\{u,v\}\in X(1)} \frac{w(uv)}{w(u)} & \textit{if } u=u',\\ -\frac{w(\{u,u'\})}{w(u)} & \textit{if } \{u,u'\}\in X(1),\\ 0 & \textit{otherwise.}\end{cases}$$

2.3 Collapsibility

Let X be a finite simplicial complex. Let η be a simplex of X of size at most d that is contained in a unique maximal face $\tau \in X$. We say that the complex

$$X' = X \setminus \{\sigma \in X : \eta \subset \sigma \subset \tau\}$$

is obtained from X by an *elementary d-collapse*, and we write $X \xrightarrow{\eta} X'$.

The complex X is called *d-collapsible* if there exists a sequence of elementary d-collapses from X to the void complex $\emptyset$. The sequence

$$X = X_0 \xrightarrow{\eta_1} X_1 \xrightarrow{\eta_2} \cdots \xrightarrow{\eta_m} X_m = \emptyset$$

is called a *d-collapsing sequence* for X. The *collapsibility* of X (or *collapsibility number*), denoted by $C(X)$, is the minimal d such that X is d-collapsible.

The notion of d-collapsibility was introduced by Wegner in [Weg75]. He proved the following simple properties of collapsibility:

Lemma 2.3.1 (Wegner [Weg75]). *Let X be a simplicial complex on vertex set V and $U \subset V$. Then,*

$$C(X[U]) \leq C(X).$$

Lemma 2.3.2 (Wegner [Weg75]). *Let X be a d-collapsible complex. Then, X is homotopy equivalent to a complex of dimension at most $d-1$. In particular, $\tilde{H}_k(X) = 0$ for $k \geq d$.*

Most importantly, he showed the following relation between collapsibility and representability:

Let $\mathcal{F} = \{F_1, \ldots, F_m\}$ be a familiy of sets. We define the *nerve* of $\mathcal{F}$ to be the simplicial complex

$$N(\mathcal{F}) = \{I \subset [m] : \cap_{i \in I} F_i \neq \emptyset\}.$$

If $X = N(\mathcal{C})$, where $\mathcal{C}$ is a family of convex sets in $\mathbb{R}^d$, X is called *d-representable*. The *representability* of X, denoted by $\text{rep}(X)$, is the minimal d such that X is d-representable.

Theorem 2.3.3 (Wegner [Weg75]). *Let X be a d-representable complex. Then, X is d-collapsible.*

From Lemma 2.3.1, Lemma 2.3.2 and Theorem 2.3.3, we obtain that for any complex X,

$$\text{rep}(X) \leq C(X) \leq L(X).$$

It will be convenient to extend the notion of d-collapsibility from complexes to general families of sets:

Let V be a finite set. An *interval* in 2^V is a family of sets

$$[\sigma, \tau] = \{\eta \subset V : \sigma \subset \eta \subset \tau\},$$

for some $\sigma \subset \tau \subset V$.

Let $\mathcal{F} \subset 2^V$ be a family of sets. Let $\sigma \in \mathcal{F}$ such that

- $|\sigma| \leq d$,
- σ is contained in a unique maximal set $\tau \in \mathcal{F}$, and
- $[\sigma, \tau] \subset \mathcal{F}$.

Then, we say that the family

$$\mathcal{F}' = \mathcal{F} \setminus [\sigma, \tau]$$

is obtained from $\mathcal{F}$ by an *elementary d-collapse*. The family $\mathcal{F}$ is called *d-collapsible* if there is a sequence of elementary d-collapses from $\mathcal{F}$ to the void family $\emptyset$. Let $C(\mathcal{F})$ be the minimal d such that $\mathcal{F}$ is d-collapsible.

The following equivalent definition of d-collapsibility will be useful to us:

Lemma 2.3.4. *The family $\mathcal{F}$ is d-collapsible if and only if it can be written as a union of intervals $\mathcal{F} = \cup_{i=1}^m [\sigma_i, \tau_i]$, such that*

- *$|\sigma_i| \leq d$ for all $1 \leq i \leq m$,*
- *$\sigma_i \not\subset \tau_j$ for $1 \leq i < j \leq m$.*

We will call such a partition $\mathcal{F} = \cup_{i=1}^m [\sigma_i, \tau_i]$ a *d-collapsing partition* of $\mathcal{F}$. Note that this is indeed a partition of $\mathcal{F}$: For $1 \leq i < j \leq m$, $[\sigma_i, \tau_i] \cap [\sigma_j, \tau_j] = \emptyset$ (otherwise, we would obtain $\sigma_i \subset \tau_j$, a contradiction).

Proof of Lemma 2.3.4. Assume $\mathcal{F}$ is d-collapsible. Let

$$\mathcal{F} = \mathcal{F}_0 \to \mathcal{F}_1 \to \cdots \mathcal{F}_m = \emptyset$$

be a d-collapsing sequence for $\mathcal{F}$, where for each $1 \leq i \leq m$, $\mathcal{F}_i = \mathcal{F}_{i-1} \setminus [\sigma_i, \tau_i]$.

We have $\mathcal{F} = \cup_{i=1}^m [\sigma_i, \tau_i]$ and $|\sigma_i| \leq d$ for all $1 \leq i \leq m$. Let $1 \leq i < j \leq m$. Then, we have $\sigma_i, \tau_j \in \mathcal{F}_{i-1}$. The set σ_i is contained in the unique maximal set τ_i in $\mathcal{F}_{i-1}$; therefore, if $\sigma_i \subset \tau_j$, then $\tau_j \in [\sigma_i, \tau_i]$. But then, $\tau_j \notin \mathcal{F}_i$, a contradiction to the fact that $\tau_j \in \mathcal{F}_{j-1}$.

The other direction is similar: Given a d-collapsing partition $\mathcal{F} = \cup_{i=1}^m [\sigma_i, \tau_i]$, we obtain a d-collapsing sequence

$$\mathcal{F} = \mathcal{F}_0 \to \mathcal{F}_1 \to \cdots \to \mathcal{F}_m = \emptyset,$$

where

$$\mathcal{F}_i = \cup_{j=i+1}^m [\sigma_j, \tau_j] = \mathcal{F}_{i-1} \setminus [\sigma_i, \tau_i]$$

for all $1 \leq i \leq m$.

Indeed, we have, for all $1 \leq i \leq m$, $|\sigma_i| \leq d$ and $[\sigma_i, \tau_i] \subset \mathcal{F}_{i-1}$. It is left to show that for each $1 \leq i \leq m$, τ_i is the unique maximal set of $\mathcal{F}_{i-1}$ containing σ_i: Assume for contradiction that $\sigma_i \subset \tau$ for some $\tau \in \mathcal{F}_{i-1}$ such that $\tau \not\subset \tau_i$. Then, we must have $\tau \subset \tau_j$ for some $j > i$. But then we obtain $\sigma_i \subset \tau \subset \tau_j$, a contradiction. $\square$

Remark. In [Mat09], a similar approach was applied for studying ($\geq d$)-collapsibility, a variant of d-collapsibility (See [Mat09, Lemma 4.2]).

2.3.1 Basic properties

Next, we present several useful properties of d-collapsible families. Most of these results were previously known (in the context of simplicial complexes), but we present here new short proofs, based on Lemma 2.3.4.

Lemma 2.3.5. *Let $\mathbb{F} \subset 2^V$. Then,*

$$C(\mathcal{F}) \leq \max\{|\sigma| : \sigma \in \mathcal{F}\}.$$

Proof. Let $d = \max\{|\sigma| : \sigma \in \mathcal{F}\}$. Let $\sigma_1, \ldots, \sigma_m$ be the sets in $\mathcal{F}$, ordered by decreasing size. In particular, $\sigma_i \not\subset \sigma_j$ for $i < j$. Thus, $\mathcal{F} = \cup_{i=1}^m [\sigma_i, \sigma_i]$ is a d-collapsing partition for $\mathcal{F}$, as wanted. $\square$

In particular, for a simplicial complex X, we obtain $C(X) \leq \dim(X) + 1$.

Let $\mathcal{F} \subset 2^V$, and let $U \subset V$. Let

$$\mathcal{F}[U] = \{\sigma \subset U : \sigma \in \mathcal{F}\}$$

be the subfamily of $\mathcal{F}$ induced by U.

Lemma 2.3.6 (Wegner [Weg75]). *Let $\mathcal{F} \subset 2^V$, and let $U \subset V$. Then,*

$$C(\mathcal{F}[U]) \leq C(\mathcal{F}).$$

Proof. Let $\mathcal{F} = \cup_{i=1}^m [\sigma_i, \tau_i]$ be a d-collapsing partition of $\mathcal{F}$. Then, we can write

$$\mathcal{F}[U] = \bigcup_{\substack{i \in [m]: \\ \sigma_i \subset U}} [\sigma_i, \tau_i \cap U].$$

Let $1 \leq i < j \leq m$. Then $\sigma_i \not\subset \tau_j$; therefore, $\sigma_i \not\subset \tau_j \cap U$. Hence, we obtain a d-collapsing partition of $\mathcal{F}[U]$. $\square$

Lemma 2.3.7 (Khmelnitsky [Khm18]). *Let $\mathcal{F}, \mathcal{G} \subset 2^V$. Then*

$$C(\mathcal{F} \cap \mathcal{G}) \leq C(\mathcal{F}) + C(\mathcal{G}).$$

Proof. Let $\mathcal{F} = \cup_{i=1}^m [\sigma_i, \tau_i]$ be a $C(\mathcal{F})$-collapsing partition for $\mathcal{F}$, and $\mathcal{G} = \cup_{j=1}^t [\eta_j, \theta_j]$ be a $C(\mathcal{G})$-collapsing partition for $\mathcal{G}$.

Let $\mathcal{I} = \{(i,j) : \sigma_i \cup \eta_j \subset \tau_i \cap \theta_j\}$. For $(i,j), (r,s) \in \mathcal{I}$, we say that $(i,j) < (r,s)$ if $i < r$ or $i = r$ and $j < s$.

We can write

$$\mathcal{F} \cap \mathcal{G} = \bigcup_{(i,j) \in \mathcal{I}} [\sigma_i \cup \eta_j, \tau_i \cap \theta_j].$$

Now, let $(i,j) < (r,s)$. If $i < r$, then $\sigma_i \not\subset \tau_r$; hence, $\sigma_i \cup \eta_j \not\subset \tau_r \cap \theta_s$. If $i = r$ and $j < s$, then $\eta_j \not\subset \theta_s$; therefore, $\sigma_i \cup \eta_j \not\subset \tau_r \cap \theta_s$. So, this is a $(C(\mathcal{F}) + C(\mathcal{G}))$-collapsing partition for $\mathcal{F} \cap \mathcal{G}$, as wanted. $\square$

Remark. Note that Lemma 2.3.6 (and its proof) is a special case of Lemma 2.3.7 (where $\mathcal{G} = 2^U$).

Let V, W be disjoint finite sets. Let $\mathcal{F} \subset 2^V$ and $\mathcal{G} \subset 2^W$. The *join* of $\mathcal{F}$ and $\mathcal{G}$ is the family

$$\mathcal{F} * \mathcal{G} = \{\sigma \cup \tau : \sigma \in \mathcal{F}, \tau \in \mathcal{G}\}.$$

Note that, if $\mathcal{F}$ and $\mathcal{G}$ are simplicial complexes, this corresponds to the definition of join stated in Section 2.1.

Lemma 2.3.8. *Let V, W be disjoint finite sets. Let $\mathcal{F} \subset 2^V$. Then,*

$$C(\mathcal{F} * 2^W) = C(\mathcal{F}).$$

Proof. Since $\mathcal{F} = (\mathcal{F} * 2^W)[V]$, we have by Lemma 2.3.6,

$$C(\mathcal{F}) \leq C(\mathcal{F} * 2^W).$$

Now, let $\mathcal{F} = \cup_{i=1}^m [\sigma_i, \tau_i]$ be a d-collapsing partition of $\mathcal{F}$. Then,

$$\mathcal{F} * 2^W = \cup_{i=1}^m [\sigma_i, \tau_i \cup \{W\}]$$

is a d-collapsing partition of $\mathcal{F} * 2^W$. Thus,

$$C(\mathcal{F}) \geq C(\mathcal{F} * 2^W),$$

as wanted. $\square$

A useful special case of Lemma 2.3.8 is the following:

Lemma 2.3.9 (Tancer [Tan11, Prop. 3.1])**.** *Let X be a simplicial complex on vertex set V, and let $v \in V$ such that X is a cone over v. Then,*

$$C(X) = C(X \setminus v).$$

Lemma 2.3.10 (Khmelnitsky [Khm18]). *Let V, W be disjoint finite sets. Let $\mathcal{F} \subset 2^V$ and $\mathcal{G} \subset 2^W$. Then,*

$$C(\mathcal{F} * \mathcal{G}) \leq C(\mathcal{F}) + C(\mathcal{G}).$$

Proof. We can write

$$\mathcal{F} * \mathcal{G} = (\mathcal{F} * 2^W) \cap (\mathcal{G} * 2^V).$$

By Lemma 2.3.8 and Lemma 2.3.7,

$$C(\mathcal{F} * \mathcal{G}) \leq C(\mathcal{F}) + C(\mathcal{G}),$$

as wanted. □

Another useful special case is the following:

Lemma 2.3.11. *Let V, W be disjoint finite sets. Let $\mathcal{F} \subset 2^V$. Then,*

$$C(\mathcal{F} * \{W\}) = C(\mathcal{F}) + |W|.$$

Proof. Let $d \geq 0$. It is easy to check that $\mathcal{F} = \cup_{i=1}^m [\sigma_i, \tau_i]$ is a d-collapsing partition for $\mathcal{F}$ if and only if $\mathcal{F} * \{W\} = \cup_{i=1}^m [\sigma_i \cup W, \tau_i \cup W]$ is a $(d + |W|)$-collapsing partition for $\mathcal{F} * \{W\}$. □

Lemma 2.3.12. *Let $\mathcal{F} \subset 2^V$ and let $(P, \leq)$ be a poset. Let $p : \mathcal{F} \to P$ that satisfies*

$$\sigma \subset \sigma' \implies p(\sigma) \leq p(\sigma').$$

Assume that for each $x \in P$ the family $p^{-1}(x)$ is d-collapsible. Then, $\mathcal{F}$ is d-collapsible.

Proof. We argue by induction on $|P|$. If $|P| = 1$, then $P = \{x\}$ and $\mathcal{F} = p^{-1}(x)$; therefore, $\mathcal{F}$ is d-collapsible.

Assume $|P| > 1$. Let x be a maximal element in P. By Lemma 2.3.4, we can write

$$p^{-1}(x) = \cup_{i=1}^m [\sigma_i, \tau_i],$$

where $|\sigma_i| \leq d$ for all $1 \leq i \leq m$, and $\sigma_i \not\subset \tau_j$ for $1 \leq i < j \leq m$.

By the induction hypothesis, $p^{-1}(P \setminus \{x\})$ is also d-collapsible; hence, we can write

$$p^{-1}(P \setminus \{x\}) = \cup_{i=m+1}^t [\sigma_i, \tau_i],$$

where $|\sigma_i| \leq d$ for all $m + 1 \leq i \leq t$, and $\sigma_i \not\subset \tau_j$ for $m + 1 \leq i < j \leq t$.

Thus, we can write

$$\mathcal{F} = p^{-1}(x) \cup p^{-1}(P \setminus \{x\}) = \cup_{i=1}^t [\sigma_i, \tau_i].$$

For all $1 \leq i \leq t$, we have $|\sigma_i| \leq d$. If $1 \leq i < j \leq m$ or $m+1 \leq i < j \leq t$, then $\sigma_i \not\subset \tau_j$. Let $1 \leq i \leq m$ and $m+1 \leq j \leq t$, and assume for contradiction that $\sigma_i \subset \tau_j$. Then, $p(\sigma_i) \leq p(\tau_j)$. But $p(\sigma_i) = x$, and x is maximal in P; therefore, $p(\tau_j) = x$. But this is a contradiction to $\tau_j \in p^{-1}(P \setminus \{x\})$.

Therefore, $\sigma_i \not\subset \tau_j$ for all $1 \leq i < j \leq t$; so, by Lemma 2.3.4, $\mathcal{F}$ is d-collapsible. □

Remark. Lemma 2.3.12 can be seen as an analogue of the "Cluster Lemma" from discrete Morse theory (see [Jon08, Lemma 4.2]).

Let $\mathcal{F} \subset 2^V$. For $\sigma \in \mathcal{F}$, define

$$\begin{aligned}\mathrm{lk}(\mathcal{F}, \sigma) &= \{\tau \setminus \sigma : \tau \in \mathcal{F}, \sigma \subset \tau\} \\ &= \{\eta \subset V : \sigma \cap \eta = \emptyset, \sigma \cup \eta \in \mathcal{F}\}\end{aligned}$$

and

$$\mathrm{cost}(\mathcal{F}, \sigma) = \{\eta \in \mathcal{F} : \sigma \not\subset \eta\}.$$

Note that, if $\mathcal{F}$ is a simplicial complex, these definitions coincide with the definitions of the link and costar of a simplex presented in Section 2.1.

Lemma 2.3.13. *Let $\mathcal{F} \subset 2^V$ and let $\sigma \in \mathcal{F}$. Then,*

$$C(\mathcal{F}) \leq \max\{C(\mathrm{cost}(\mathcal{F}, \sigma)), C(\mathrm{lk}(\mathcal{F}, \sigma)) + |\sigma|\}.$$

Proof. Let $P = \{0, 1\}$, and let $p : \mathcal{F} \to P$ be defined by

$$p(\eta) = \begin{cases} 0 & \text{if } \sigma \not\subset \eta, \\ 1 & \text{if } \sigma \subset \eta. \end{cases}$$

for all $\eta \in \mathcal{F}$.

Now let $\eta, \tau \in \mathcal{F}$ such that $\eta \subset \tau$. If $\sigma \not\subset \eta$ then $p(\eta) = 0$, therefore $p(\eta) \leq p(\tau)$. If $\sigma \subset \eta$, then $\sigma \subset \tau$, therefore $p(\eta) = p(\tau) = 1$. In all cases, $p(\eta) \leq p(\tau)$. Moreover, we have

$$p^{-1}(0) = \mathrm{cost}(\mathcal{F}, \sigma)$$

and

$$p^{-1}(1) = \{\eta \in \mathcal{F} : \sigma \subset \eta\} = \mathrm{lk}(\mathcal{F}, \sigma) * \{\sigma\}.$$

By Lemma 2.3.11,

$$C(\mathrm{lk}(\mathcal{F}, \sigma) * \{\sigma\}) = C(\mathrm{lk}(\mathcal{F}, \sigma)) + |\sigma|.$$

Therefore, by Lemma 2.3.12, we obtain

$$C(\mathcal{F}) \leq \max\{C(\mathrm{cost}(\mathcal{F}, \sigma)), C(\mathrm{lk}(\mathcal{F}, \sigma)) + |\sigma|\}.$$

For $v \in V$, we write $\mathrm{lk}(\mathcal{F}, v) = \mathrm{lk}(\mathcal{F}, \{v\})$ and $\mathcal{F} \setminus v = \mathrm{cost}(\mathcal{F}, \{v\}) = \mathcal{F}[V \setminus \{v\}]$. As an immediate consequence of Lemma 2.3.13, we obtain

Lemma 2.3.14 (Tancer [Tan11, Prop 1.2]). *Let $\mathcal{F} \subset 2^V$ and let $v \in V$. Then,*

$$C(\mathcal{F}) \leq \max\{C(\mathcal{F} \setminus v), C(\mathrm{lk}(\mathcal{F}, v)) + 1\}.$$

Another useful relation between the collapsibility of a family of sets and that of their links is the following result:

Lemma 2.3.15 (Khmelnitsky [Khm18]). *Let $\mathcal{F} \subset 2^V$, and let $\sigma \in \mathcal{F}$. Then,*

$$C(\mathrm{lk}(\mathcal{F}, \sigma)) \leq C(\mathcal{F}).$$

Proof. Let $d = C(\mathcal{F})$. Let $\mathcal{F} = \cup_{i=1}^m [\sigma_i, \tau_i]$ be a d-collapsing partition of $\mathcal{F}$. Then, we have

$$\mathrm{lk}(\mathcal{F}, \sigma) = \bigcup_{\substack{i \in [m] \\ \sigma \subset \tau_i}} [\sigma_i \setminus \sigma, \tau_i \setminus \sigma].$$

Indeed, let $\eta \in \mathrm{lk}(\mathcal{F}, \sigma)$. Then, $\eta \cap \sigma = \emptyset$ and $\eta \cup \sigma \in \mathcal{F}$. Let $[\sigma_i, \tau_i]$ be the interval containing $\eta \cup \sigma$. Note that $\sigma \subset \tau_i$. Then, $\eta \in [\sigma_i \setminus \sigma, \tau_i \setminus \sigma]$. Hence,

$$\mathrm{lk}(\mathcal{F}, \sigma) \subset \bigcup_{\substack{i \in [m] \\ \sigma \subset \tau_i}} [\sigma_i \setminus \sigma, \tau_i \setminus \sigma].$$

On the other direction, let $\eta \in [\sigma_i \setminus \sigma, \tau_i \setminus \sigma]$ for some $i \in [m]$ such that $\sigma \subset \tau_i$. Then, $\eta \cap \sigma = \emptyset$ and $\eta \cup \sigma \in [\sigma_i, \tau_i] \subset \mathcal{F}$. Therefore, $\eta \in \mathrm{lk}(\mathcal{F}, \sigma)$. Thus,

$$\mathrm{lk}(\mathcal{F}, \sigma) \supset \bigcup_{\substack{i \in [m] \\ \sigma \subset \tau_i}} [\sigma_i \setminus \sigma, \tau_i \setminus \sigma].$$

It is left to show that $\mathrm{lk}(\mathcal{F}, \sigma) = \bigcup_{\substack{i \in [m] \\ \sigma \subset \tau_i}} [\sigma_i \setminus \sigma, \tau_i \setminus \sigma]$ is indeed a d-collapsing partition. First, note that $|\sigma_i \setminus \sigma| \leq |\sigma_i| \leq d$ for all $i \in [m]$. Now, let $1 \leq i < j \leq m$ such that $\sigma \subset \tau_i$ and $\sigma \subset \tau_j$. We have to show that $\sigma_i \setminus \sigma \not\subset \tau_j \setminus \sigma$. Assume for contradiction that

$$\sigma_i \setminus \sigma \subset \tau_j \setminus \sigma.$$

Then, we obtain

$$\sigma_i \subset (\sigma_i \setminus \sigma) \cup \sigma \subset (\tau_j \setminus \sigma) \cup \sigma = \tau_j,$$

a contradiction to the fact that $\mathcal{F} = \cup_{i=1}^m [\sigma_i, \tau_i]$ is a d-collapsing partition.

Thus, $C(\mathrm{lk}(\mathcal{F}, \sigma)) \leq d$. □

Let V be a finite set, and let $\mathcal{F} \subset 2^V$. Let

$$\mathcal{F}^{(\geq d)} = \{\sigma \in \mathcal{F} : |\sigma| \geq d\}.$$

Lemma 2.3.16. *Let V be a finite set. Then, the family $(2^V)^{(\geq d)}$ is d-collapsible.*

Proof. We argue by induction on $|V|$. If $|V| = 0$ the claim holds trivially for all d.

Assume $|V| > 0$. Let $u \in V$. Note that

$$\mathrm{lk}((2^V)^{(\geq d)}, u) = (2^{V\setminus\{u\}})^{(\geq d-1)},$$

and

$$(2^V)^{(\geq d)} \setminus u = (2^{V\setminus\{u\}})^{(\geq d)}.$$

So, by the induction hypothesis,

$$C(\mathrm{lk}((2^V)^{(\geq d)}, u)) \leq d - 1$$

and

$$C((2^V)^{(\geq d)} \setminus u) \leq d.$$

Therefore, by Lemma 2.3.14, we obtain $C((2^V)^{(\geq d)}) \leq d$. □

Proposition 2.3.17. *Let $\mathcal{F} \subset 2^V$. Then, $\mathcal{F}$ is d-collapsible if and only if $\mathcal{F}^{(\geq d)}$ is d-collapsible.*

Proof. Assume that $\mathcal{F}$ is d-collapsible. Let $\mathcal{F} = \cup_{i=1}^m [\sigma_i, \tau_i]$ be a d-collapsing partition of $\mathcal{F}$. Then, we can write

$$\mathcal{F}^{(\geq d)} = \bigcup_{i=1}^m [\sigma_i, \tau_i]^{(\geq d)}$$

For each $\sigma \in \mathcal{F}^{(\geq d)}$, let $p(\sigma) = m - i$, where i is the unique index such that $\sigma \in [\sigma_i, \tau_i]^{(\geq d)}$.

Let $\sigma, \sigma' \in \mathcal{F}^{(\geq d)}$ such that $\sigma \subset \sigma'$. Note that if $p(\sigma) = m - i$, then $\sigma \in [\sigma_i, \tau_i]$, and therefore $\sigma' \notin [\sigma_j, \tau_j]$ for $j > i$ (otherwise we would obtain $\sigma_i \subset \sigma \subset \sigma' \subset \tau_j$, a contradiction). That is, $\sigma' \in [\sigma_j, \tau_j]$ for some $j \leq i$. Hence, $p(\sigma') \geq m - i = p(\sigma)$.

For each $1 \leq i \leq m$, we have

$$[\sigma_i, \tau_i]^{(\geq d)} = (\{\sigma_i\} * 2^{\tau_i \setminus \sigma_i})^{(\geq d)} = \{\sigma_i\} * (2^{\tau_i \setminus \sigma_i})^{(\geq d - |\sigma_i|)}$$

By Lemma 2.3.11 and Lemma 2.3.16, we have

$$\begin{aligned} C([\sigma_i, \tau_i]^{(\geq d)}) &= C(\{\sigma_i\} * (2^{\tau_i \setminus \sigma_i})^{(\geq d - |\sigma_i|)}) \\ &= C((2^{\tau_i \setminus \sigma_i})^{(\geq d - |\sigma_i|)}) + |\sigma_i| = (d - |\sigma_i|) + |\sigma_i| = d. \end{aligned}$$

So, by Lemma 2.3.12, $C(\mathcal{F}^{(\geq d)}) \leq d$.

On the other direction, assume that $\mathcal{F}^{(\geq d)}$ is d-collapsible. Let $p : \mathcal{F} \to \{0, 1\}$ be defined by

$$p(\eta) = \begin{cases} 0 & \text{if } |\eta| \leq d - 1, \\ 1 & \text{if } |\eta| \geq d. \end{cases}$$

Note that for any $\eta, \tau \in \mathcal{F}$ such that $\eta \subset \tau$, we have $p(\eta) \leq p(\tau)$. Also, we have

$$p^{-1}(0) = \{\eta \in \mathcal{F} : |\eta| \leq d-1\}$$

and

$$p^{-1}(1) = \mathcal{F}^{(\geq d)}.$$

By Lemma 2.3.5, $p^{-1}(0)$ is $(d-1)$-collapsible. Therefore, by Lemma 2.3.12, $\mathcal{F}$ is d-collapsible. $\square$

As a consequence of Proposition 2.3.17, we obtain the following equivalent definition for d-collapsibility. For simplicity, we state the result only for simplicial complexes:

Lemma 2.3.18 (Tancer [Tan10a, Lemma 5.2])**.** *Let X be a simplicial complex. Then, X is d-collapsible if and only if one of the following holds:*

- $\dim(X) < d$, *or*
- *There exists some $\sigma \in X$ such that $|\sigma| = d$, σ is contained in a unique maximal face $\tau \neq \sigma$ of X, and $\text{cost}(X, \sigma)$ is d-collapsible.*

Proof. If $\dim(X) < d$, then X is d-collapsible by Lemma 2.3.5. If there is some $\sigma \in X$ such that $|\sigma| = d$, σ is contained in a unique maximal face of X, and $\text{cost}(X, \sigma)$ is d-collapsible, then X is d-collapsible by definition.

On the other direction, assume that X is d-collapsible. If $\dim(X) < d$, we are done. Otherwise, assume that $\dim(X) \geq d$. By Proposition 2.3.17, $X^{(\geq d)}$ is d-collapsible. Let $X^{(\geq d)} = \cup_{i=1}^m [\sigma_i, \tau_i]$ be a d-collapsing partition. Note that, for any $i \in [m]$, since $\sigma_i \in X^{(\geq d)}$, we have $|\sigma_i| = d$. Let i be the minimal index in $[m]$ such that $\sigma_i \neq \tau_i$ (there is such an index i since $\dim(X) \geq d$).

Note that $\text{cost}(X^{(\geq d)}, \sigma_i) = \cup_{j \neq i} [\sigma_j, \tau_j]$. Indeed, we have

$$\text{cost}(X^{(\geq d)}, \sigma_i) \subset X^{(\geq d)} \setminus [\sigma_i, \tau_i] = \cup_{j \neq i} [\sigma_j, \tau_j].$$

On the other direction, note that for $j > i$, since $\cup_{k=1}^m [\sigma_k, \tau_k]$ is a d-collapsing partition, we have $\sigma_i \not\subset \tau_j$; hence $[\sigma_j, \tau_j] \subset \text{cost}(X^{(\geq d)}, \sigma_i)$. For $i < j$, we have $\sigma_j = \tau_j$, and therefore $\sigma_i \not\subset \tau_j$ (since $|\sigma_i| = |\tau_j|$ but $\sigma_i \neq \tau_j$). So, $[\sigma_j, \tau_j] \subset \text{cost}(X^{(\geq d)}, \sigma_i)$.

In particular, $\text{cost}(X^{(\geq d)}, \sigma_i) = \text{cost}(X, \sigma_i)^{(\geq d)}$ is d-collapsible. So, by Proposition 2.3.17, $\text{cost}(X, \sigma_i)$ is d-collapsible, as wanted. Finally, since $\sigma_i \not\subset \tau_j$ for $j \neq i$, τ_i is the unique maximal face in $X^{(\geq d)}$ containing σ_i, and therefore it is the unique maximal face in X containing σ_i. $\square$

Lemma 2.3.19 (see e.g. [AHJ19, Prop. 2.1])**.** *Let $\mathcal{F} \subset 2^V$ be a family of sets. Let $\tilde{V}$ be a finite set, and let $\pi : \tilde{V} \to V$ be surjective. Let*

$$\pi^{-1}(\mathcal{F}) = \{\sigma \subset \tilde{V} : \pi(\sigma) \in \mathcal{F}\}.$$

Then, $C(\pi^{-1}(\mathcal{F})) = C(\mathcal{F})$.

Proof. Note that $\mathcal{F}$ is isomorphic to an induced subfamily of $\pi^{-1}(\mathcal{F})$; therefore, $C(\pi^{-1}(\mathcal{F})) \geq C(\mathcal{F})$.

On the other direction, assume that $\mathcal{F}$ is d-collapsible, and let $\mathcal{F} = \cup_{i=1}^{m}[\sigma_i, \tau_i]$ be a d-collapsing partition. Then, we can write

$$\pi^{-1}(\mathcal{F}) = \cup_{i=1}^{m}\pi^{-1}([\sigma_i, \tau_i]).$$

For $\sigma \in \pi^{-1}(\mathcal{F})$, let $p(\sigma) = m-i$, where i is the unique index such that $\sigma \in \pi^{-1}([\sigma_i, \tau_i])$. Let $\sigma, \sigma' \in \pi^{-1}(\mathcal{F})$ such that $\sigma \subset \sigma'$. If $p(\sigma) = m-i$, then $\pi(\sigma) \in [\sigma_i, \tau_i]$, and therefore $\pi(\sigma') \notin [\sigma_j, \tau_j]$ for $j > i$ (otherwise, $\sigma_i \subset \pi(\sigma) \subset \pi(\sigma') \subset \tau_j$, a contradiction). Thus, $\pi(\sigma') \in [\sigma_j, \tau_j]$ for some $j \leq i$. So, $p(\sigma') \geq m - i = p(\sigma)$.

Let $1 \leq i \leq m$. Let $\sigma_i = \{v_1, \ldots, v_k\}$ (for some $k \leq d$). Then,

$$\pi^{-1}([\sigma_i, \tau_i]) = (2^{\pi^{-1}(v_1)})^{(\geq 1)} * (2^{\pi^{-1}(v_2)})^{(\geq 1)} * \cdots * (2^{\pi^{-1}(v_k)})^{(\geq 1)} * 2^{\pi^{-1}(\tau_i \setminus \sigma_i)}$$

By Lemma 2.3.10 and Lemma 2.3.16,

$$C(\pi^{-1}([\sigma_i, \tau_i])) \leq k \cdot 1 + 0 = k \leq d.$$

Thus, by Lemma 2.3.12,

$$C(\pi^{-1}(\mathcal{F})) \leq d.$$

Therefore, $C(\pi^{-1}(\mathcal{F})) = C(\mathcal{F})$. □

2.4 Helly-type theorems

The well known Helly Theorem states that for any family of convex sets in $\mathbb{R}^d$, if any $d+1$ of the sets have non-empty intersection, then the whole family has non-empty intersection.

Recall that a missing face of a complex X is a set $\tau \subset V$ such that $\tau \notin X$ but $\sigma \in X$ for any $\sigma \subsetneq \tau$, and $h(X)$ is the maximal dimension of a missing face of X.

Helly's Theorem may be stated in terms of simplicial complexes as follows:

Theorem 2.4.1 (Helly's Theorem)**.** *Let X be d-representable. Then, $h(X) \leq d$.*

Helly's Theorem holds also for the larger classes of d-collapsible and d-Leray complexes:

Theorem 2.4.2 (Topological Helly's Theorem)**.** *Let X be d-Leray. Then, $h(X) \leq d$.*

The following "colorful" extension of Helly's Theorem was proved by Lovász (see [Bár82]):

Theorem 2.4.3 (Colorful Helly Theorem (Lovász)). *Let $\mathcal{C}_1, \ldots, \mathcal{C}_{d+1}$ be non-empty families of compact convex sets in $\mathbb{R}^d$, and suppose that for any $C_1 \in \mathcal{C}_1, \ldots, C_{d+1} \in \mathcal{C}_{d+1}$, the intersection $\cap_{i=1}^{d+1} C_i$ is not empty. Then there there is some $i \in [d+1]$ such that the intersection of all the sets in $\mathcal{C}_i$ is not empty.*

In [KM05], the following generalization of Lovász's theorem was presented:

Theorem 2.4.4 (Kalai-Meshulam [KM05]). *Let X be a d-collapsible complex on vertex set V. Let M be a matroid on vertex set V with rank function ρ. If $M \subset X$ then there exists a simplex $\tau \in X$ such that $\rho(\tau) = \rho(V)$ and $\rho(V \setminus \tau) \leq d$.*

For d-Leray complexes, a slightly weaker result was proved:

Theorem 2.4.5 (Kalai-Meshulam [KM05]). *Let X be a d-Leray complex on vertex set V. Let M be a matroid on vertex set V with rank function ρ. If $M \subset X$ then there exists a simplex $\tau \in X$ such that $\rho(V \setminus \tau) \leq d$.*

We will need the following version of the Colorful Helly Theorem for d-collapsible complexes:

Theorem 1.2.1 (Kalai-Meshulam [KM05], see also [AHJ19]). *Let X be a d-collapsible simplicial complex on vertex set V. Let $V_1, \ldots, V_{d+1} \subset V$ such that $V_i \notin X$ for all $i \in [d+1]$. Then, there exists distinct vertices $v_1 \in V_{i_1}, \ldots, v_k \in V_{i_k}$, where $1 \leq i_1 < i_2 < \cdots < i_k \leq d+1$, such that $\{v_1, \ldots, v_k\} \notin X$.*

Proof. Let $U = V_1 \cup \cdots \cup V_{d+1}$ and let $X' = X[U]$. By Lemma 2.3.6, X' is d-collapsible. Let

$$\tilde{U} = \{(v,i) : i \in [d+1], v \in V_i\}$$

and let $\pi : \tilde{U} \to U$ be defined by

$$\pi((v,i)) = v$$

for all $(v,i) \in \tilde{U}$. By Lemma 2.3.19, $\pi^{-1}(X')$ is also d-collapsible.

For $i \in [d+1]$, let $\tilde{U}_i = \{(v,i) : v \in V_i\}$. We define a matroid M on vertex set $\tilde{U}$ by

$$M = (2^{\tilde{U}_1})^{(0)} * \cdots * (2^{\tilde{U}_{d+1}})^{(0)}.$$

That is, M consists of all the sets of the form $\{(v_1, i_1), \ldots, (v_k, i_k)\}$, where $i_j \neq i_s$ for all $1 \leq j < s \leq k$. Let ρ be the rank function of M. Note that for any set $\sigma \subset \tilde{U}$, we have

$$\rho(\tilde{U} \setminus \sigma) = d + 1 - |\{i \in [d+1] : \tilde{U}_i \subset \sigma\}|.$$

Now, assume for contradiction that for any choice of distinct vertices $v_1 \in V_{i_1}, \ldots, v_k \in V_{i_k}$, where $1 \leq i_1 < i_2 < \cdots < i_k \leq d+1$, we have $\{v_1, \ldots, v_k\} \in X$. Then, we have $M \subset \pi^{-1}(X')$. So, by Theorem 2.4.4, there is a simplex $\tau \in \pi^{-1}(X')$ such that

$\rho(\tilde{U} \setminus \tau) \leq d$. So, there is some $i \in [d+1]$ such that $\tilde{U}_i \subset \tau$. In particular, we obtain $\tilde{U}_i \in \pi^{-1}(X')$, and therefore $V_i = \pi(\tilde{U}_i) \in X'$. But this contradicts the assumption that $V_i \notin X$ for all $i \in [d+1]$. $\square$

2.A Nerve theorems from double complexes

In this section we present a proof of the Nerve Theorem (Theorem 2.2.4). In fact, we prove the following stronger versions of the theorem:

Theorem 2.A.1. *Let $X_1, \ldots, X_m$ be non-empty simplicial complexes, and let $X = \cup_{i=1}^m X_i$. Then, there exists an homomorphism $h : \tilde{H}_k(X; R) \to \tilde{H}_k(N(\{X_1, \ldots, X_m\}); R)$ such that*

- *If, for all $I \subset [m]$ of size $1 \leq |I| \leq k$, $\tilde{H}_{k-|I|}(\cap_{i \in I} X_i; R) = 0$, then h is surjective.*
- *If, for all $I \subset [m]$ of size $1 \leq |I| \leq k+1$, $\tilde{H}_{k+1-|I|}(\cap_{i \in I} X_i; R) = 0$, then h is injective.*

Theorem 2.A.1 was proved, in a slightly different form, by Meshulam in [Mes01]. A version of this theorem more similar to the one presented here is proved in [Mon17].

Theorem 2.A.2. *Let $X_1, \ldots, X_m$ be non-empty simplicial complexes, and let $X = \cup_{i=1}^m X_i$. Let $N = N(\{X_1, \ldots, X_m\})$. Assume that, for all $I \subset [m]$ of size at least 2, $\cap_{i \in I} X_i$ is either empty or acyclic. Then, the following sequence is exact:*

$$\cdots \to \tilde{H}_{k+1}(N; R) \to \bigoplus_{i=1}^m \tilde{H}_k(X_i; R) \to \tilde{H}_k(X; R) \to \tilde{H}_k(N; R) \to \cdots$$

The proofs presented here are based on the proof of Theorem 2.2.4 appearing in [BT82, Section 8], and on ideas from [HS10].

Let R be a commutative ring with unit element. A *chain complex* is a sequence of R-modules $\mathcal{C} = \{C_k\}_{k=-\infty}^{\infty}$ together with a family of homomorphisms $\partial_k : C_k \to C_{k-1}$ satisfying $\partial_k \partial_{k+1} = 0$ for all k. The operator ∂_k is called the *boundary operator*. We define the homology groups of $\mathcal{C}$ by

$$H_k(\mathcal{C}) = \frac{\operatorname{Ker} \partial_k}{\operatorname{Im} \partial_{k+1}}.$$

For example, if X is a simplicial complex, then, for $\mathcal{C} = \{C_k(X; R)\}_{k=-\infty}^{\infty}$ (taking $C_k(X; R) = 0$ whenever $X(k) = \emptyset$), we obtain $H_k(\mathcal{C}) = \tilde{H}_k(X; R)$.

Let $(\mathcal{C}, \partial)$ and $(\mathcal{D}, \partial')$ be two chain complexes. A *chain map* $f = \{f_k\}_{k=-\infty}^{\infty}$ is a family of homomorphisms $f_k : C_k \to D_k$ satisfying $f_{k-1} \partial_k = \partial'_k f_k$ for all k.

Given a chain map $f : \mathcal{C} \to \mathcal{D}$, we define the *mapping cone* of f to be the chain complex

$$\operatorname{Cone}(f)_k = C_{k-1} \oplus D_k,$$

with boundary operator

$$\partial_k^{\text{cone}}(c, d) = (-\partial_{k-1} c, -f_{k-1} c + \partial'_k d)$$

for any $(c, d) \in \mathrm{Cone}(f)_k = C_{k-1} \oplus D_k$. It is easy to check that $\partial_k^{\mathrm{cone}} \partial_{k+1}^{\mathrm{cone}} = 0$ for all k, so $\mathrm{Cone}(f)$ is indeed a chain complex.

The homology of the mapping cone is related to the homology of the complexes $\mathcal{C}$ and $\mathcal{D}$ by the following result:

Theorem 2.A.3. *Let $f : \mathcal{C} \to \mathcal{D}$ be a chain map. Then, the following sequence is exact:*

$$\cdots \to H_k(\mathcal{C}) \to H_k(\mathcal{D}) \to H_k(\mathit{Cone}(f)) \to H_{k-1}(\mathcal{C}) \to \cdots$$

A *double complex* is a family of R-modules $\mathcal{C} = \{C_{i,j}\}_{i,j=-\infty}^{\infty}$ with commuting boundary operators

$$\partial_{i,j}^h : C_{i,j} \to C_{i-1,j}$$

and

$$\partial_{i,j}^v : C_{i,j} \to C_{i,j-1}$$

for all i, j. That is, for all i, j, we have

$$\partial_{i,j}^h \partial_{i+1,j}^h = 0,$$

$$\partial_{i,j}^v \partial_{i,j+1}^v = 0$$

and

$$\partial_{i,j-1}^h \partial_{i,j}^v = \partial_{i-1,j}^v \partial_{i,j}^h.$$

For any i, $\{C_{i,j}\}_{j=-\infty}^{\infty}$ is a chain complex with boundary operator $\partial_{i,j}^v$. Denote by $H_{i,j}^v(\mathcal{C})$ the j-th homology group of this complex. Moreover, for any j, $\{H_{i,j}^v(\mathcal{C})\}_{i=-\infty}^{\infty}$ is a chain complex with boundary operator ∂_h. Denote by $H_{i,j}^h H^v(\mathcal{C})$ the i-th homology group of this complex.

The *total complex* of the double complex $\mathcal{C}$ is the chain complex

$$\mathrm{Tot}(\mathcal{C})_k = \bigoplus_{\substack{i,j: \\ i+j=k}} C_{i,j}$$

with boundary operator

$$\partial_k^{\mathrm{tot}} : \mathrm{Tot}(\mathcal{C})_k \to \mathrm{Tot}(\mathcal{C})_{k-1}$$

defined by

$$\partial_k^{\mathrm{tot}} z = \partial_{i,j}^h z + (-1)^i \partial_{i,j}^v z$$

for any i, j such that $i + j = k$ and $z \in C_{i,j} \subset \mathrm{Tot}(\mathcal{C})_k$. It is easy to check that $(\{\mathrm{Tot}(\mathcal{C})_k\}_{k=-\infty}^{\infty}, \partial^{\mathrm{tot}})$ is indeed a chain complex. Let $H_k^{\mathrm{tot}}(\mathcal{C})$ be the k-th homology group of the total complex $\mathrm{Tot}(\mathcal{C})$.

From now on, we will assume that $C_{i,j} = 0$ whenever $i < 0$ or $j < 0$.

Lemma 2.A.4. *Let $k \geq 1$. Assume that $H_{i,k-i}^v(\mathcal{C}) = 0$ for all $\leq i \leq k-1$. Let $z = z_0 + \cdots + z_{k-1} \in \mathrm{Ker}(\partial_k^{tot})$, where $z_i \in C_{i,k-i}$ for all $0 \leq i \leq k-1$. Then, there is* *some* $y = y_0 + \cdots + y_{k-1} \in \mathit{Tot}(\mathcal{C})_{k+1}$*, where* $y_i \in C_{i,k+1-i}$ *for all* $0 \leq i \leq k-1$*, such that* $z = \partial_{k+1}^{tot} y$*.*

Proof. Let $n \in \{-1, 0, \ldots, k-1\}$ be the minimal index such that $z_i = 0$ for all $i > n$. We argue by induction on n. If $n = -1$, then $z = 0$, and we can take $y = 0 \in \mathrm{Tot}(\mathcal{C})_{k+1}$.

Assume $n \geq 0$. Since $\partial_k^{\mathrm{tot}} z = 0$ and $z_i = 0$ for $i > n$, we must have $\partial_{n,k-n}^v z_n = 0$. Since $H_{n,k-n}^v = 0$, there is some $x \in C_{n,k-n+1}$ such that $z_n = \partial_{n,k-n+1}^v x$.

Let $z' = z + (-1)^{n+1} \partial_{k+1}^{\mathrm{tot}} x$. Write $z' = z'_0 + \cdots + z'_{k-1}$, where $z'_0 \in C_{i,k-i}$ for all $0 \leq i \leq k-1$. Then, we have $z'_i = z_i$ for $i \leq n-2$, $z'_{n-1} = z_{n-1} + (-1)^{n+1} \partial_{n,k-n+1}^h x$ and $z'_i = 0$ for $i > n-1$. So, since $\partial_k^{\mathrm{tot}} z' = \partial_k^{\mathrm{tot}} z = 0$, by the induction hypothesis there is some $y' \in \mathrm{Tot}(\mathcal{C})_{k+1}$ such that $z' = \partial_{k+1}^{\mathrm{tot}} y'$. Setting $y = y' + (-1)^n x$, we obtain $z = \partial_{k+1}^{\mathrm{tot}} y$. Moreover, by the induction hypothesis we can write $y' = y'_0 + \cdots + y'_{k-1}$, where $y'_i \in C_{i,k+1-i}$ for $0 \leq i \leq k-1$. Hence, we can write $y = y_0 + \cdots + y_{k-1}$, where $y_i = y'_i \in C_{i,k+1-i}$ for $i \neq n$, and $y_n = y'_n + (-1)^n x \in C_{n,k-n+1}$. $\square$

Lemma 2.A.5. *Let* $k \geq 0$*. Then, there is a homomorphism* $h : H_k^{tot}(\mathcal{C}) \to H_{k,0}^h H^v(\mathcal{C})$ *such that:*

- *If* $H_{i,k-1-i}^v(\mathcal{C}) = 0$ *for all* $0 \leq i \leq k-2$*, then* h *is surjective.*
- *If* $H_{i,k-i}^v(\mathcal{C}) = 0$ *for all* $0 \leq i \leq k-1$*, then* h *is injective.*

Proof. Let $z = z_0 + \cdots + z_k \in \mathrm{Ker}(\partial_k^{\mathrm{tot}})$, where $z_i \in C_{i,k-i}$ for all $0 \leq i \leq k$.

Note that $\partial_{k,0}^v z_k = 0$ and $\partial_{k,0}^h z_k = \partial_{k-1,1}^v ((-1)^k z_{k-1})$. Therefore, z_k represents an homology class $[z_k] \in H_{k,0}^h H^v(\mathcal{C})$, and we can define $h : H_k^{\mathrm{tot}}(\mathcal{C}) \to H_{k,0}^h H^v(\mathcal{C})$ by

$$h([z]) = [z_k].$$

Note that h is well defined: Let $y = y_0 + \cdots + y_{k+1}$, where $y_i \in C_{i,k+1-i}$ for all $0 \leq i \leq k+1$. Then, we have

$$h([z + \partial_{k+1}^{\mathrm{tot}} y]) = [z_k + \partial_{k+1,0}^h y_{k+1} + (-1)^k \partial_{k,1}^v y_k] = [z_k] \in H_{k,0}^h H^v(\mathcal{C}),$$

since $[\partial_{k,1}^v y_k] = 0 \in H_{k,0}^v(\mathcal{C})$ and $[\partial_{k+1,0}^h y_{k+1}] = 0 \in H_{k,0}^h H^v(\mathcal{C})$. So, $h([z + \partial_{k+1}^{\mathrm{tot}} y]) = h([z])$.

Now, assume that $H_{i,k-1-i}^v(\mathcal{C}) = 0$ for all $0 \leq i \leq k-2$. Let $[z_k] \in H_{k,0}^h H^v(\mathcal{C})$. Then, we have $\partial_{k,0}^h z_k = \partial_{k-1,1}^v x$ for some $x \in C_{k-1,1}$.

Let $z' = \partial_{k-1,1}^h x$. Note that $\partial_{k-2,1}^h z' = 0$ and

$$\partial_{k-2,1}^v z' = \partial_{k-2,1}^v \partial_{k-1,1}^h x = \partial_{k-1,0}^h \partial_{k-1,1}^v x = \partial_{k-1,0}^h \partial_{k,0}^h z_k = 0.$$

So, $z' \in \mathrm{Ker}(\partial_{k-1}^{\mathrm{tot}})$. Therefore, by Lemma 2.A.4, there is some $y' = y'_0 + \cdots + y'_{k-2} \in \mathrm{Tot}(\mathcal{C})_k$, where $y'_i \in C_{i,k-i}$ for all $0 \leq i \leq k-2$, such that $z' = \partial_k^{\mathrm{tot}} y'$.

Let $z = z_k + (-1)^k(x - y')$. Then, we have

$$\partial_k^{\text{tot}} z = \partial_{k,0}^h z_k + (-1)^k(\partial_{k-1,1}^h x + (-1)^{k-1}\partial_{k-1,1}^v x - \partial_{k-1,1}^h x) = 0.$$

So, $z \in \operatorname{Ker}(\partial_k^{\text{tot}})$ and $h([z]) = z_k$. Thus, h is surjective.

Finally, assume that $H_{i,k-i}^v(\mathcal{C}) = 0$ for all $0 \le i \le k-1$. Let $z = z_0 + \cdots + z_k \in \operatorname{Ker}(\partial_k^{\text{tot}})$, where $z_i \in C_{i,k-i}$ for all $0 \le i \le k$, such that $h([z]) = [z_k] = 0 \in H_{k,0}^h H^v(\mathcal{C})$. That is, there exist $x \in C_{k+1,0}$ and $w \in C_{k,1}$ such that $\partial_{k+1,0}^h x = z_k + \partial_{k,1}^v w$.

Let $z' = z + \partial_{k+1}^{\text{tot}}((-1)^k w - x)$. We have

$$\begin{aligned} z' &= z_0 + \cdots + z_k + (-1)^k \partial_{k,1}^h w + \partial_{k,1}^v w - \partial_{k+1,0}^h x \\ &= z_0 + \cdots + z_{k-2} + \left(z_{k-1} + (-1)^k \partial_{k,1}^h w\right). \end{aligned}$$

Since $\partial_k^{\text{tot}} z' = \partial_k^{\text{tot}} z = 0$, then, by Lemma 2.A.4, there is some $y \in \operatorname{Tot}(\mathcal{C})_{k+1}$ such that $z' = \partial_{k+1}^{\text{tot}} y$. So, we have

$$z = z' - \partial_{k+1}^{\text{tot}}((-1)^k w - x) = \partial_{k+1}^{\text{tot}}(y + (-1)^{k+1} w + x).$$

That is, $[z] = 0 \in H_k^{\text{tot}}(\mathcal{C})$. Hence, h is injective. □

The following special case of Lemma 2.A.5 will be useful:

Corollary 2.A.6. *If $H_{i,j}^v(\mathcal{C}) = 0$ for all i and all $j \ge 1$, then $H_k^{tot}(\mathcal{C}) \cong H_{k,0}^h H^v(\mathcal{C})$ for all k.*

For $a \in \mathbb{Z}$, let $\mathcal{C}^{\le a} = \{C_{i,j}^{\le a}\}_{i,j=-\infty}^{\infty}$ be the double complex

$$C_{i,j}^{\le a} = \begin{cases} C_{i,j} & \text{if } i \le a, \\ 0 & \text{otherwise,} \end{cases}$$

with vertical boundary operators $\partial_{i,j}^v$ for $i \le a$ and 0 for $i > a$, and horizontal boundary operators $\partial_{i,j}^h$ for $i \le a$ and 0 for $i > a$.

Similarly, let $\mathcal{C}^{\ge a} = \{C_{i,j}^{\ge a}\}_{i,j=-\infty}^{\infty}$ be the double complex

$$C_{i,j}^{\ge a} = \begin{cases} C_{i,j} & \text{if } i \ge a, \\ 0 & \text{otherwise,} \end{cases}$$

with vertical boundary operators $-\partial_{i,j}^v$ for $i \ge a$ and 0 for $i < a$, and horizontal boundary operators $-\partial_{i,j}^h$ for $i > a$ and 0 for $i \le a$.

The following Lemma relates between the total homology groups of the complexes $\mathcal{C}^{\le a}$, $\mathcal{C}^{\ge a+1}$ and $\mathcal{C}$:

Lemma 2.A.7. *The following sequence is exact*

$$\cdots \to H_k^{tot}(\mathcal{C}^{\le a}) \to H_k^{tot}(\mathcal{C}) \to H_k^{tot}(\mathcal{C}^{\ge a+1}) \to H_{k-1}^{tot}(\mathcal{C}^{\le a}) \to \cdots$$

Proof. For $k \in \mathbb{Z}$, we define a map $f_k : \mathrm{Tot}_{k+1}(\mathcal{C}^{\geq a+1}) \to \mathrm{Tot}_k(\mathcal{C}^{\leq a})$ as follows: For $z \in \mathrm{Tot}_{k+1}(C^{\geq a+1})$, let

$$f_k(z) = -\partial^h_{a+1,k-a} z',$$

where z' is the component of z in $C_{a+1,k-a}$.

It is easy to check that $f = \{f_k\}_{k\in\mathbb{Z}}$ is a chain map between the complexes $\mathrm{Tot}_{*+1}(\mathcal{C}^{\geq a+1})$ and $\mathrm{Tot}_*(\mathcal{C}^{\leq a})$. We have

$$\mathrm{Cone}(f)_k = \mathrm{Tot}_k(\mathcal{C}^{\geq a+1}) \oplus \mathrm{Tot}_k(\mathcal{C}^{\leq a}) = \mathrm{Tot}_k(\mathcal{C}).$$

Moreover, for $z \in \mathrm{Tot}_k(\mathcal{C})$, write $z = z_1 + z_2$, where $z_1 \in \mathrm{Tot}_k(\mathcal{C}^{\geq a+1})$ and $z_2 \in \mathrm{Tot}_k(\mathcal{C}^{\leq a})$. Let z' be the component of z_1 in $C_{a+1,k-a-1}$. Then, we have

$$\partial^{\mathrm{cone}}_k(z) = (\partial^{\mathrm{tot}}_k(z_1) - \partial^h_{a+1,k-a-1}(z')) + (-f_{k-1}(z_1) + \partial^{\mathrm{tot}}_k(z_2)) = \partial^{\mathrm{tot}}_k(z).$$

So, $H_k(\mathrm{Cone}(f)) \cong H^{\mathrm{tot}}_k(\mathcal{C})$. Thus, by Theorem 2.A.3, we obtain the long exact sequence

$$\cdots \to H^{\mathrm{tot}}_{k+1}(\mathcal{C}^{\geq a+1}) \to H^{\mathrm{tot}}_k(\mathcal{C}^{\leq a}) \to H^{\mathrm{tot}}_k(\mathcal{C}) \to H^{\mathrm{tot}}_k(\mathcal{C}^{\geq a+1}) \to \cdots$$

as wanted. □

Remark. In [AY21], some variants of Lemma 2.A.7 are applied to the study of double complexes arising from problems in combinatorial commutative algebra.

2.A.1 A double complex from a partition

Let K be a simplicial complex on vertex set V. Let $V = A \cup B$ be a partition of V. Following [HS10], we define a double complex as follows:

For any i, j, let

$$K(i,j) = \{\sigma \in K : |\sigma \cap A| = i,\ |\sigma \cap B| = j\}.$$

Let $K_{i,j}$ be the free R-module generated by the ordered $(i+j-1)$-dimensional simplices of the form $[v_0, \ldots, v_{i+j-1}]$, where $\{v_0, \ldots, v_{i+j-1}\} \in K(i,j)$, under the relations

$$[v_0, \ldots, v_{i+j-1}] = \mathrm{sgn}(\pi)[v_{\pi(0)}, \ldots, v_{\pi(i+j-1)}],$$

for every simplex $\{v_0, \ldots, v_{i+j-1}\} \in K(i,j)$ and permutation $\pi : \{0, \ldots, i+j-1\} \to \{0, \ldots, i+j-1\}$.

Let $\mathcal{K} = \{K_{i,j}\}_{i,j=-\infty}^{\infty}$. Let $\sigma = [v_0, \ldots, v_{i-1}, u_0, \ldots, u_{j-1}]$ be an ordered simplex in K such that $v_0, \ldots, v_{i-1} \in A$ and $u_0, \ldots, u_{j-1} \in B$. We define boundary operators

$$\partial^h_{i,j}\sigma = \sum_{k=0}^{i-1} (-1)^k [v_0, \ldots, v_{k-1}, v_{k+1}, \ldots, v_{i-1}, u_0, \ldots, u_{j-1}]$$

and

$$\partial^v_{i,j}\sigma = \sum_{k=0}^{j-1}(-1)^k[v_0,\ldots,v_{i-1},u_0,\ldots,u_{k-1},u_{k+1},\ldots,u_{j-1}].$$

It is easy to check that these are indeed boundary operators and that they commute.

For any k, we have

$$\mathrm{Tot}(\mathcal{K})_k = \bigoplus_{i+j=k} K_{i,j} = C_{k-1}(K;R).$$

Moreover, it is easy to check that $\partial^{\mathrm{tot}}_k$ is exactly the boundary operator ∂_{k-1} of the complex K. That is, we have for all k,

$$H^{\mathrm{tot}}_k(\mathcal{K}) = \tilde{H}_{k-1}\left(K;R\right).$$

Lemma 2.A.8. *For all i,j, we have*

$$H^v_{i,j}(\mathcal{K}) \cong \bigoplus_{\substack{\eta\in K(i-1),\\ \eta\subset A}} \tilde{H}_{j-1}\left(\mathrm{lk}(K,\eta)[B];R\right).$$

Proof. For any i,j, define

$$f: K_{i,j} \to \bigoplus_{\substack{\eta\in K(i-1),\\ \eta\subset A}} C_{j-1}(\mathrm{lk}(K,\eta)[B];R)$$

by

$$f([v_0,\ldots,v_{i-1},u_0,\ldots,u_{j-1}]) = [u_0,\ldots,u_{j-1}] \in C_{j-1}(\mathrm{lk}(K,\{v_0,\ldots,v_{i-1}\});R)$$

for any $\{v_0,\ldots,v_{i-1},u_0,\ldots,u_{j-1}\}\in K$, where $\{v_0,\ldots,v_{i-1}\}\subset A$ and $\{u_0,\ldots,u_{j-1}\}\subset B$. It is easy to check that f is a chain map (where we take the boundary operator of $K_{i,j}$ to be $\partial^v_{i,j}$ and the boundary operator of $\bigoplus_{\substack{\eta\in K(i-1)\\ \eta\subset A}} C_{j-1}(\mathrm{lk}(K,\eta)[B];R)$ to be the direct sum of the boundary operators of its summands), and that f is bijective. Therefore, we obtain

$$H^v_{i,j}(\mathcal{K}) \cong \bigoplus_{\substack{\eta\in K(i-1),\\ \eta\subset A}} \tilde{H}_{j-1}\left(\mathrm{lk}(K,\eta)[B];R\right)$$

as wanted. □

Let

$$N_A(K) = \{\eta\in K[A]:\ \mathrm{lk}(K,\eta)[B]\neq\{\emptyset\}\}.$$

Note that $N_A(K)$ is a subcomplex of $K[A]$.

Lemma 2.A.9. *For all i, we have*

$$H^v_{i,0}(\mathcal{K}) \cong C_{i-1}(K[A], N_A(K); R)$$

and

$$H^h_{i,0}H^v(\mathcal{K}) \cong H_{i-1}\left(K[A], N_A(K); R\right).$$

Proof. By Lemma 2.A.8, there is an isomorphism $f^* : H^v_{i,0} \to \bigoplus_{\substack{\eta\in K(i-1),\\ \eta\subset A}} \tilde{H}_{-1}\left(\mathrm{lk}(K,\eta)[B]; R\right)$ defined by

$$f^*([\sigma]) = [\emptyset] \in \tilde{H}_{-1}\left(\mathrm{lk}(K,\sigma)[B]; R\right)$$

for all $\sigma \in K[A](i-1)$.

Note that, for any complex X, we have $\tilde{H}_{-1}\left(X; R\right) = 0$ if $X \neq \{\emptyset\}$, and $\tilde{H}_{-1}\left(X; R\right) \cong R$ if $X = \{\emptyset\}$. Hence, we have

$$\bigoplus_{\substack{\eta\in K(i-1),\\ \eta\subset A}} \tilde{H}_{-1}\left(\mathrm{lk}(K,\eta)[B]; R\right) \cong \bigoplus_{\substack{\eta\in K[A](i-1),\\ \eta\notin N_A(K)}} R \cong C_{i-1}(K[A], N_A(K); R).$$

Thus , we obtain an isomorphism

$$\tilde{f} : H^v_{i,0}(\mathcal{K}) \to C_{i-1}(K[A], N_A(K); R)$$

defined by

$$\tilde{f}([\sigma]) = \begin{cases} \sigma & \text{if } \sigma \notin N_A(K), \\ 0 & \text{if } \sigma \in N_A(K) \end{cases}$$

for any $\sigma \in K[A](i-1)$. It is easy to check that $\tilde{f}$ is a chain map between $(\{H^v_{i,0}(\mathcal{K})\}_{i=-\infty}^{\infty}, \partial^h)$ and $(\{C_{i-1}(K[A], N_A(K); R)\}_{i=-\infty}^{\infty}, \partial)$; therefore, we obtain

$$H^h_{i,0}H^v(\mathcal{K}) \cong H_{i-1}\left(K[A], N_A(K); R\right),$$

as wanted. □

Lemma 2.A.10. *Assume that $A \in K$. Then, for all i, we have*

$$H^h_{i,0}H^v(\mathcal{K}) \cong \tilde{H}_{i-2}\left(N_A(K); R\right).$$

Proof. Since $A \in K$, we have $K[A] = 2^A$. In particular, $K[A]$ is acyclic. Hence, by Lemma 2.A.9 and Theorem 2.2.6, we obtain

$$H^h_{i,0}H^v(\mathcal{K}) \cong H_{i-1}\left(2^A, N_A(K); R\right) \cong \tilde{H}_{i-2}\left(N_A(K); R\right),$$

as wanted. □

By applying Lemma 2.A.5 to the double complex $\mathcal{K}$, we obtain the following result:

Theorem 2.A.11. *Let K be a simplicial complex on vertex set $V = A \uplus B$. Assume that $A \in K$. Then, there is a homomorphism $h : \tilde{H}_k(K; R) \to \tilde{H}_{k-1}(N_A(K); R)$ such that*

- *If, for all $0 \leq i \leq k-1$, $\tilde{H}_{k-1-i}(\mathrm{lk}(K, \sigma)[B]; R) = 0$ for all $\sigma \subset A$ of size i, then h is surjective.*
- *If, for all $0 \leq i \leq k$, $\tilde{H}_{k-i}(\mathrm{lk}(K, \sigma)[B]; R) = 0$ for all $\sigma \subset A$ of size i, then h is injective.*

Proof. For all k we have $H^{\mathrm{tot}}_{k+1}(\mathcal{K}) \cong \tilde{H}_k(K; R)$, and, by Lemma 2.A.10, $H^h_{k+1,0}H^v(\mathcal{K}) \cong \tilde{H}_{k-1}(N_A(K); R)$. By Lemma 2.A.8, we have for all i and j

$$H^v_{i,j}(\mathcal{K}) \cong \bigoplus_{\substack{\sigma \subset A \\ |\sigma| = i}} \tilde{H}_{j-1}(\mathrm{lk}(K, \sigma)[B]; R).$$

Thus, the claim follows immediately from Lemma 2.A.5. □

As an immediate consequence, we obtain:

Corollary 2.A.12. *Let K be a simplicial complex on vertex set $V = A \uplus B$. Assume that $A \in K$. If*

$$\tilde{H}_k(\mathrm{lk}(K, \sigma)[B]; R) = 0$$

for all $\sigma \subset A$ and all $k \geq 0$, then $\tilde{H}_k(K; R) \cong \tilde{H}_{k-1}(N_A(K); R)$ for all k.

Theorem 2.A.13. *Let K be a simplicial complex on vertex set $V = A \uplus B$. Assume that $A, B \in K$ and that, for any $v \in A$, $\mathrm{lk}(K, v)[B] \neq \{\emptyset\}$. If, for all $\sigma \subset A$ of size $|\sigma| \geq 2$ and all $k \geq 0$ we have*

$$\tilde{H}_k(\mathrm{lk}(K, \sigma)[B]; R) = 0,$$

then the following sequence is exact

$$\cdots \to \tilde{H}_{k+1}(N_A(K); R) \to \bigoplus_{v \in A} \tilde{H}_k(\mathrm{lk}(K, v)[B]; R) \to \tilde{H}_{k+1}(K; R) \to \tilde{H}_k(N_A(K); R) \to \cdots$$

Proof. Let $\mathcal{C} = \mathcal{K}^{\leq 1}$. Since $B \in K$, we have $K[B] = 2^B$. In particular, $\tilde{H}_k(K[B]; R) = 0$ for all k. Since $\mathcal{C}^{\leq 0}$ consists of only one non-zero column, we have, using Lemma 2.A.8,

$$H^{\mathrm{tot}}_k(\mathcal{C}^{\leq 0}) = H^v_{0,k}(\mathcal{C}^{\leq 0}) = H^v_{0,k}(\mathcal{K}) \cong \tilde{H}_{k-1}(K[B]; R) = 0$$

for all k. By Lemma 2.A.7, we have an exact sequence

$$\cdots \to H^{\mathrm{tot}}_k(\mathcal{C}^{\leq 0}) \to H^{\mathrm{tot}}_k(\mathcal{C}) \to H^{\mathrm{tot}}_k(\mathcal{C}^{\geq 1}) \to H^{\mathrm{tot}}_{k-1}(\mathcal{C}^{\leq 0}) \to \cdots$$

Therefore,

$$H_k^{\mathrm{tot}}(\mathcal{K}^{\leq 1}) = H_k^{\mathrm{tot}}(\mathcal{C}) \cong H_k^{\mathrm{tot}}(\mathcal{C}^{\geq 1})$$

for all k. Since $\mathcal{C}^{\geq 1}$ consists of only one non-zero column, we obtain

$$H_k^{\mathrm{tot}}(\mathcal{K}^{\leq 1}) \cong H_k^{\mathrm{tot}}(\mathcal{C}^{\geq 1}) = H_{1,k-1}^v(\mathcal{C}^{\geq 1}) = H_{1,k-1}^v(\mathcal{K}) \cong \bigoplus_{v \in A} \tilde{H}_{k-2}\left(\mathrm{lk}(K,v)[B]; R\right)$$

for all k (where the last isomorphism follows from Lemma 2.A.8).
By Lemma 2.A.10, we have for $i \geq 3$

$$H_{i,0}^h H^v(\mathcal{K}^{\geq 2}) = H_{i,0}^h H^v(\mathcal{K}) \cong \tilde{H}_{i-2}\left(N_A(K); R\right).$$

Moreover, since $\mathrm{lk}(K,v)[B] \neq \{\emptyset\}$ for all $v \in A$, we have $v \in N_A(K)$ for all $v \in A$. Therefore, by Lemma 2.A.9,

$$H_{1,0}^v(\mathcal{K}) \cong C_0(2^A, N_A(K); R) = 0 = H_{1,0}^v(\mathcal{K}^{\geq 2}).$$

So,

$$H_{2,0}^h H^v(\mathcal{K}^{\geq 2}) = H_{2,0}^h H^v(\mathcal{K}) \cong \tilde{H}_0\left(N_A(K); R\right).$$

Since for $i < 2$ we have $H_{i,0}^h H^v(\mathcal{K}^{\geq 2}) = 0 = \tilde{H}_{i-2}\left(N_A(K); R\right)$, we obtain

$$H_{i,0}^h H^v(\mathcal{K}^{\geq 2}) \cong \tilde{H}_{i-2}\left(N_A(K); R\right)$$

for all i. By Lemma 2.A.8, we have for all $i \geq 2$ and $j \geq 1$

$$H_{i,j}^v(\mathcal{K}^{\geq 2}) = H_{i,j}^v(\mathcal{K}) \cong \bigoplus_{\substack{\sigma \subset A \\ |\sigma| = i}} \tilde{H}_{j-1}\left(\mathrm{lk}(K,\sigma)[B]; R\right) = 0.$$

Moreover, by definition of $\mathcal{K}^{\geq 2}$, we have $H_{i,j}^v(\mathcal{K}^{\geq 2}) = 0$ for $i < 2$ and all j. Hence, by Corollary 2.A.6, we obtain

$$H_k^{\mathrm{tot}}(\mathcal{K}^{\geq 2}) \cong H_{k,0}^h H^v(\mathcal{K}^{\geq 2}) \cong \tilde{H}_{k-2}\left(N_A(K); R\right)$$

for all k. Recall that $H_k^{\mathrm{tot}}(\mathcal{K}) \cong \tilde{H}_{k-1}\left(K; R\right)$ for all k. Therefore, by Lemma 2.A.7, the sequence

$$\cdots \to \tilde{H}_{k+1}\left(N_A(K); R\right) \to \bigoplus_{v \in A} \tilde{H}_k\left(\mathrm{lk}(K,v)[B]; R\right) \to \tilde{H}_{k+1}\left(K; R\right) \to \tilde{H}_k\left(N_A(K); R\right) \to \cdots$$

is exact. $\square$

2.A.2 The Mayer-Vietoris double complex

Let $X_1, \ldots, X_m$ be non-empty simplicial complexes on vertex set V, and let $X = \cup_{i=1}^m X_i$. Let K be the simplicial complex on vertex set $V \uplus [m]$ whose simplices are the sets of the form $\sigma \cup I$, where $I \subset [m]$ and $\sigma \in \cap_{i \in I} X_i$ (where, for $I = \emptyset$, we define $\cap_{i\in I} X_i = 2^V$).

Let $A = V$ and $B = [m]$. Note that $V \in K$. For any $\sigma \in K[V] = 2^V$, we have

$$\mathrm{lk}(K, \sigma)[B] = \{I \subset [m] : \sigma \in \cap_{i \in I} X_i\} = 2^{\{i \in [m]:\, \sigma \in X_i\}}.$$

Thus, $\mathrm{lk}(K, \sigma)[B] = \{\emptyset\}$ if $\sigma \notin X$ and $\mathrm{lk}(K, \sigma)[B]$ is a complete complex if $\sigma \in X$. Therefore, $N_A(K) = X$. Moreover, we have $\tilde{H}_k\left(\mathrm{lk}(K, \sigma)[B]; R\right) = 0$ for all $k \geq 0$ and $\sigma \subset V$. Hence, by Corollary 2.A.12, we have

$$\tilde{H}_k\left(X; R\right) \cong \tilde{H}_{k+1}\left(K; R\right)$$

for all k.

Now, let $A = [m]$ and $B = V$. Note that $[m] \in K$ (since $\emptyset \in \cap_{i \in [m]} X_i$). For any $I \subset A = [m]$, we have

$$\mathrm{lk}(K, I)[B] = \{\sigma \subset V : \sigma \in \cap_{i \in I} X_i\} = \cap_{i \in I} X_i$$

and

$$N_A(K) = \{I \subset [m] : \cap_{i \in I} X_i \neq \{\emptyset\}\} = N(\{X_1, \ldots, X_m\}).$$

Now we can complete the proofs of Theorems 2.A.1 and 2.A.2:

Proof of Theorem 2.A.1. Let $A = [m] \in K$ and $B = V$. For $\emptyset \neq I \subset [m]$, we have

$$\mathrm{lk}(K, I)[V] = \cap_{i \in I} X_i.$$

Moreover, for $I = \emptyset$ we have

$$\mathrm{lk}(K, \emptyset)[V] = K[V] = 2^V.$$

In particular, $\mathrm{lk}(K, \emptyset)[V]$ is acyclic. Since $\tilde{H}_{k+1}\left(K; R\right) \cong \tilde{H}_k\left(X; R\right)$ and $N_A(K) = N(\{X_1, \ldots, X_m\})$, the claim follows from Theorem 2.A.11. □

Proof of Theorem 2.A.2. Let $A = [m] \in K$ and $B = V \in K$. For all $i \in A = [m]$, we have

$$\mathrm{lk}(K, i)[V] = X_i \neq \{\emptyset\}.$$

For $I \subset [m]$ of size $|I| \geq 2$,

$$\mathrm{lk}(K, I)[V] = \cap_{i \in I} X_i$$

is either empty or acyclic. So, by Theorem 2.A.13, the sequence

$$\cdots \to \tilde{H}_{k+1}\left(N_A(K); R\right) \to \bigoplus_{i\in[m]} \tilde{H}_k\left(\mathrm{lk}(K,i)[V]; R\right) \to \tilde{H}_{k+1}\left(K; R\right) \to \tilde{H}_k\left(N_A(K); R\right) \to \cdots$$

is exact. Since $N_A(K) = N$ and $\tilde{H}_k\left(X; R\right) \cong \tilde{H}_{k+1}\left(K; R\right)$ for all k, we obtain the long exact sequence

$$\cdots \to \tilde{H}_{k+1}\left(N; R\right) \to \bigoplus_{i\in[m]} \tilde{H}_k\left(X_i; R\right) \to \tilde{H}_k\left(X; R\right) \to \tilde{H}_k\left(N; R\right) \to \cdots ,$$

as wanted. ∎

Chapter 3

Minimal exclusion sequences and collapsibility of complexes of hypergraphs

This chapter is organized as follows. In Section 3.1 we present our generalization of Matoušek and Tancer's bound on the collapsibility of a simplicial complex, and we prove Theorem 1.1.3. In Section 3.2 we present some results on the collapsibility of independence complexes of graphs. In Section 3.3 we prove our main results on the collapsibility of complexes of hypergraphs. Section 3.4 contains some generalizations of Theorems 1.1.1 and 1.1.2, which are obtained by applying different known variants of the Frankl-Kalai Lemma (Lemma 1.1.4). Section 3.5 contains more applications of the minimal exclusion sequence method. In particular, we prove Theorem 1.1.5 about the collapsibility of complexes of matrices with bounded maximal rank. We also present some conjectures about the collapsibility of the complexes $\mathrm{M}_{\mathcal{A},r}$ for different classes of matrices.

3.1 A bound on the collapsibility of a complex

Let X be a (non-void) simplicial complex on vertex set V. Fix a linear order $<$ on V. Let $\mathcal{A} = (\sigma_1, \ldots, \sigma_m)$ be a sequence of faces of X such that, for any $\sigma \in X$, $\sigma \subset \sigma_i$ for some $i \in [m]$. For example, we may take $\sigma_1, \ldots, \sigma_m$ to be the set of maximal faces of X (given some fixed ordering).

For a simplex $\sigma \in X$, let $m_{X,\mathcal{A},<}(\sigma) = \min\{i \in [m] : \sigma \subset \sigma_i\}$. Let $i \in [m]$ and $\sigma \in X$ such that $m_{X,\mathcal{A},<}(\sigma) = i$. We define the *minimal exclusion sequence*

$$\mathrm{mes}_{X,\mathcal{A},<}(\sigma) = (v_1, \ldots, v_{i-1})$$

as follows: If $i = 1$ then $\mathrm{mes}_{X,\mathcal{A},<}(\sigma)$ is the empty sequence. If $i > 1$ we define the sequence recursively as follows:

Since $i > 1$, we must have $\sigma \not\subset \sigma_1$; hence, there is some $v \in \sigma$ such that $v \notin \sigma_1$. Let v_1 be the minimal such vertex (with respect to the order $<$).

Let $1 < j < i$ and assume that we already defined $v_1, \ldots, v_{j-1}$. Since $i > j$, we must have $\sigma \not\subset \sigma_j$; hence, there exists some $v \in \sigma$ such that $v \notin \sigma_j$.

- If there is a vertex $v_k \in \{v_1, \ldots, v_{j-1}\}$ such that $v_k \notin \sigma_j$, let v_j be such a vertex of minimal index k. In this case we call v_j *old at* j.

- If $v_k \in \sigma_j$ for all $k < j$, let v_j be the minimal vertex $v \in \sigma$ (with respect to the order $<$) such that $v \notin \sigma_j$. In this case we call v_j *new at* j.

Let $M_{X,\mathcal{A},<}(\sigma) \subset \sigma$ be the simplex consisting of all the vertices appearing in the sequence $\mathrm{mes}_{X,\mathcal{A},<}(\sigma)$. Let

$$d(X, \mathcal{A}, <) = \max\{|M_{X,\mathcal{A},<}(\sigma)| : \sigma \in X\}.$$

The following result was stated and proved in [MT09, Prop. 1.3] in the special case where X is the nerve of a finite family of sets (in our notation, $X = \mathrm{Cov}_{\mathcal{H},1}$ for some hypergraph $\mathcal{H}$).

Theorem 3.1.1. *The simplicial complex X is $d(X, \mathcal{A}, <)$-collapsible.*

The proof given in [MT09] can be easily modified to hold in this more general setting. Here we present a different proof, based on the application of Lemma 2.3.14.

Proof of Theorem 3.1.1. First, we deal with the case where X is a complete complex (i.e. a simplex). Then X is 0-collapsible; therefore, the claim holds.

For a general complex X, we argue by induction on the number of vertices of X. If $|V| = 0$, then $X = \{\emptyset\}$. In particular, it is a complete complex; hence, the claim holds.

Let $|V| > 0$, and assume that the claim holds for any complex with less than $|V|$ vertices. If $\sigma_1 = V$, then X is the complete complex on vertex set V, and the claim holds. Otherwise, let v be the minimal vertex (with respect to $<$) in $V \setminus \sigma_1$.

In order to apply Lemma 2.3.14, we will need the following two claims:

Claim 3.1.2. *The complex $X \setminus v$ is $d(X, \mathcal{A}, <)$-collapsible.*

Proof. For every $i \in [m]$, let $\sigma_i' = \sigma_i \setminus \{v\}$, and let $\mathcal{A}' = (\sigma_1', \ldots, \sigma_m')$. Let $\sigma \in X \setminus v$. Since $v \notin \sigma$, then, for any $i \in [m]$, $\sigma \subset \sigma_i$ if and only if $\sigma \subset \sigma_i'$. Hence, every simplex $\sigma \in X \setminus v$ is contained in σ_i' for some $i \in [m]$ (since, by the definition of $\mathcal{A}$, $\sigma \subset \sigma_i$ for some $i \in [m]$). So, by the induction hypothesis, $X \setminus v$ is $d(X \setminus v, \mathcal{A}', <)$-collapsible.

Let $\sigma \in X \setminus v$. We will show that $\mathrm{mes}_{X,\mathcal{A},<}(\sigma) = \mathrm{mes}_{X \setminus v,\mathcal{A}',<}(\sigma)$. Since for any $i \in [m]$, $\sigma \subset \sigma_i$ if and only if $\sigma \subset \sigma_i'$, then the two sequences are of the same length. Let

$$\mathrm{mes}_{X,\mathcal{A},<}(\sigma) = (v_1, \ldots, v_k)$$

and

$$\mathrm{mes}_{X\setminus v,\mathcal{A}',<}(\sigma) = (v'_1,\ldots,v'_k).$$

We will show that $v_i = v'_i$ for all $i \in [k]$. We argue by induction on i. Let $i \in [k]$, and assume that $v_j = v'_j$ for all $j < i$. Since $v \notin \sigma$, then $\sigma \setminus \sigma_i = \sigma \setminus \sigma'_i$. Therefore, for any $j < i$, $v_j \in \sigma \setminus \sigma_i$ if and only if $v'_j = v_j \in \sigma \setminus \sigma'_i$. Hence, v_i is old at i if and only if v'_i is old at i, and if v_i and v'_i are both old at i, then $v_i = v'_i$. Otherwise, both v_i and v'_i are new at i. Then, v_i is the minimal vertex in $\sigma \setminus \sigma_i$, and v'_i is the minimal vertex in $\sigma \setminus \sigma'_i = \sigma \setminus \sigma_i$. Thus, $v_i = v'_i$.

Therefore, $|M_{X\setminus v,\mathcal{A}',<}(\sigma)| = |M_{X,\mathcal{A},<}(\sigma)|$ for any $\sigma \in X \setminus v$; hence,

$$d(X \setminus v, \mathcal{A}', <) \leq d(X, \mathcal{A}, <).$$

So, $X \setminus v$ is $d(X,\mathcal{A},<)$-collapsible. $\square$

Claim 3.1.3. *The complex* $\mathrm{lk}(X, v)$ *is* $(d(X,\mathcal{A},<) - 1)$*-collapsible.*

Proof. Let $I = \{i \in [m] : v \in \sigma_i\}$. For every $i \in I$, let $\sigma''_i = \sigma_i \setminus \{v\}$. Write $I = \{i_1,\ldots,i_r\}$, where $i_1 < \cdots < i_r$, and let $\mathcal{A}'' = (\sigma''_{i_1},\ldots,\sigma''_{i_r})$.

For any $\sigma \in \mathrm{lk}(X, v)$, the simplex $\sigma \cup \{v\}$ belongs to X; hence, there exists some $i \in [m]$ such that $\sigma \cup \{v\} \subset \sigma_i$. Since $v \in \sigma \cup \{v\}$, we must have $i \in I$, and therefore $\sigma \subset \sigma''_i = \sigma_i \setminus \{v\}$. So, by the induction hypothesis, $\mathrm{lk}(X, v)$ is $d(\mathrm{lk}(X, v),\mathcal{A}'', <)$-collapsible.

Let $\sigma \in \mathrm{lk}(X, v)$. We will show that

$$M_{X,\mathcal{A},<}(\sigma \cup \{v\}) = M_{\mathrm{lk}(X,v),\mathcal{A}'',<}(\sigma) \cup \{v\}.$$

Let

$$\mathrm{mes}_{X,\mathcal{A},<}(\sigma \cup \{v\}) = (v_1,\ldots,v_n),$$

and

$$\mathrm{mes}_{\mathrm{lk}(X,v),\mathcal{A}'',<}(\sigma) = (u_1,\ldots,u_t).$$

For any $j \in [r]$, $\sigma \subset \sigma''_{i_j}$ if and only if $\sigma \cup \{v\} \subset \sigma_{i_j}$. Also, for $i \notin I$, $\sigma \cup \{v\} \not\subset \sigma_i$ (since $v \notin \sigma_i$). Therefore, $n = i_{t+1} - 1$.

The vertex v is the minimal vertex in $V \setminus \sigma_1$, therefore it is the minimal vertex in $(\sigma \cup \{v\}) \setminus \sigma_1$. Hence, we have $v_1 = v$. Now, let $i > 1$ such that $i \notin I$. Then, $v_1 = v$ is the vertex of minimal index in the sequence $(v_1,\ldots,v_{i-1})$ that is contained in $(\sigma \cup \{v\}) \setminus \sigma_i$. Therefore, $v_i = v$.

Finally, we will show that $v_{i_j} = u_j$ for all $j \in [t]$. We argue by induction on j. Let $j \in [t]$, and assume that $v_{i_\ell} = u_\ell$ for all $\ell < j$.

For any $k < i_j$, either $v_k = v$ (if $k \notin I$) or $v_k = u_\ell$ for some $\ell < j$ (if $k = i_\ell \in I$). Also, since $v \in \sigma_{i_j}$, we have $(\sigma \cup \{v\}) \setminus \sigma_{i_j} = \sigma \setminus \sigma''_{i_j}$. So, for any $k < i_j$, $v_k \in (\sigma \cup \{v\}) \setminus \sigma_{i_j}$ if and only if $k = i_\ell$ for some $\ell < j$ such that $u_\ell \in \sigma \setminus \sigma''_{i_j}$. Therefore, v_{i_j} is old at i_j

if and only if u_j is old at j, and if v_{i_j} and u_j are both old, then $v_{i_j} = u_j$. Otherwise, assume that v_{i_j} is new at i_j and u_j is new at j. Then, v_{i_j} is the minimal vertex in $(\sigma \cup \{v\}) \setminus \sigma_{i_j}$, and u_j is the minimal vertex in $\sigma \setminus \sigma''_{i_j} = (\sigma \cup \{v\}) \setminus \sigma_{i_j}$. Thus, $v_{i_j} = u_j$.

So, for any $\sigma \in \mathrm{lk}(X, v)$ we obtain

$$|M_{\mathrm{lk}(X,v),\mathcal{A}'',<}(\sigma)| = |M_{X,\mathcal{A},<}(\sigma \cup \{v\})| - 1.$$

Hence,

$$d(\mathrm{lk}(X, v), \mathcal{A}'', <) \leq d(X, \mathcal{A}, <) - 1.$$

So, $\mathrm{lk}(X, v)$ is $(d(X, \mathcal{A}, <) - 1)$-collapsible.

By Claim 3.1.2, Claim 3.1.3 and Lemma 2.3.14, X is $d(X, \mathcal{A}, <)$-collapsible.

For our applications, we will use the following simplified version of Theorem 3.1.1:

Theorem 1.1.3. *Let X be a simplicial complex on vertex set V. Let $S(X)$ be the collection of all sets $\{v_1, \ldots, v_k\} \subset V$ satisfying the following condition:*

There exist maximal faces $\sigma_1, \sigma_2, \ldots, \sigma_{k+1}$ of X such that:

- *$v_i \notin \sigma_i$ for all $i \in [k]$,*
- *$v_i \in \sigma_j$ for all $1 \leq i < j \leq k + 1$.*

Let $d'(X)$ be the maximum size of a set in $S(X)$. Then X is $d'(X)$-collapsible.

Proof. Let $<$ be some linear order on the vertex set V, and let $\mathcal{A} = (\sigma_1, \ldots, \sigma_m)$ be the sequence of maximal faces of X (ordered in any way).

Let $i \in [m]$ and let $\sigma \in X$ with $m_{X,\mathcal{A},<}(\sigma) = i$. Let $\mathrm{mes}_{X,\mathcal{A},<}(\sigma) = (v_1, \ldots, v_{i-1})$. Then $M_{X,\mathcal{A},<}(\sigma) = \{v_{i_1}, \ldots, v_{i_k}\}$ for some $i_1 < \cdots < i_k \in [i - 1]$ (these are exactly the indices i_j such that v_{i_j} is new at i_j). For each $j \in [k]$ we have $v_{i_j} \notin \sigma_{i_j}$. In addition, since v_{i_j} is new at i_j, we have $v_{i_\ell} \in \sigma_{i_j}$ for all $\ell < j$. Let $i_{k+1} = i$. Since $m_{X,\mathcal{A},<}(\sigma) = i = i_{k+1}$, we have $\sigma \subset \sigma_{i_{k+1}}$. In particular, $v_{i_\ell} \in \sigma_{i_{k+1}}$ for all $\ell < k + 1$.

Therefore, $M_{X,\mathcal{A},<}(\sigma) \in S(X)$. Thus, $d(X, \mathcal{A}, <) \leq d'(X)$, and by Theorem 3.1.1, X is $d'(X)$-collapsible. □

3.2 Collapsibility of independence complexes

Let $G = (V, E)$ be a graph. The independence complex $I(G)$ is the simplicial complex on vertex set V whose simplices are the independent sets in G.

Definition 3.2.1. Let $k(G)$ be the maximal size of a set $\{v_1, \ldots, v_k\} \subset V$ that satisfies:

- $\{v_i, v_j\} \notin E$ for all $i \neq j \in [k]$,
- There exist $u_1, \ldots, u_k \in V$ such that

- $\{v_i, u_i\} \in E$ for all $i \in [k]$,
- $\{v_i, u_j\} \notin E$ for all $1 \le i < j \le k$.

Proposition 3.2.2. $k(G) = d'(I(G))$.

Proof. Let $A = \{v_1, \dots, v_k\} \in S(I(G))$. Then, there exist maximal faces $\sigma_1, \dots, \sigma_{k+1}$ of $I(G)$ such that:

- $v_i \notin \sigma_i$ for all $i \in [k]$,
- $v_i \in \sigma_j$ for all $1 \le i < j \le k+1$.

Let $i \in [k]$. Since σ_i is a maximal independent set in G and $v_i \notin \sigma_i$, there exists some $u_i \in \sigma_i$ such that $\{v_i, u_i\} \in E$.

Let $1 \le i < j \le k$. Since v_i and u_j are both contained in the independent set σ_j, we have $\{v_i, u_j\} \notin E$. Furthermore, since $A \subset \sigma_{k+1}$, A is an independent set in G. That is, $\{v_i, v_j\} \notin E$ for all $i \neq j \in [k]$. So, A satisfies the conditions of Definition 3.2.1. Hence, $|A| \le k(G)$; therefore, $d'(I(G)) \le k(G)$.

Now, let $k = k(G)$, and let $v_1, \dots, v_k, u_1, \dots, u_k \in V$ such that

- $\{v_i, v_j\} \notin E$ for all $i \neq j \in [k]$,
- $\{v_i, u_i\} \in E$ for all $i \in [k]$,
- $\{v_i, u_j\} \notin E$ for all $1 \le i < j \le k$. □

Let $i \in [k]$, and let $V_i = \{v_j : 1 \le j < i\}$. Note that $V_i \cup \{u_i\}$ forms an independent set in G; therefore, it is a simplex in $I(G)$. Let σ_i be a maximal face of $I(G)$ containing $V_i \cup \{u_i\}$. Since $\{v_i, u_i\} \in E$, we have $v_i \notin \sigma_i$.

The set $\{v_1, \dots, v_k\}$ is also an independent set in G. Therefore, there is a maximal face $\sigma_{k+1} \in I(G)$ that contains it.

By the definition of $\sigma_1, \dots, \sigma_{k+1}$, we have $v_i \in \sigma_j$ for $1 \le i < j \le k+1$. Therefore, $\{v_1, \dots, v_k\} \in S(I(G))$; so, $k(G) = k \le d'(I(G))$.

Hence, $k(G) = d'(I(G))$, as wanted.

As an immediate consequence of Proposition 3.2.2 and Theorem 1.1.3, we obtain:

Proposition 3.2.3. *The complex $I(G)$ is $k(G)$-collapsible.*

3.3 Complexes of hypergraphs

In this section we prove our main results, Theorems 1.1.1 and 1.1.2.

Theorem 1.1.1. *Let $\mathcal{H}$ be a hypergraph of rank r. Then $\mathrm{Cov}_{\mathcal{H},p}$ is $\left(\binom{r+p}{r} - 1\right)$-collapsible.*

Proof. Let $\mathcal{H}$ be a hypergraph of rank r on vertex set $[n]$, and let

$$\{A_1, \ldots, A_k\} \in S(\mathrm{Cov}_{\mathcal{H},p}).$$

Then, there exist maximal faces $\mathcal{F}_1, \ldots, \mathcal{F}_{k+1} \in \mathrm{Cov}_{\mathcal{H},p}$ such that

- $A_i \notin \mathcal{F}_i$ for all $i \in [k]$,
- $A_i \in \mathcal{F}_j$ for all $1 \leq i < j \leq k+1$.

For any $i \in [k+1]$, there is some $C_i \subset [n]$ of size at most p that covers $\mathcal{F}_i$. Since $\mathcal{F}_i$ is maximal, then, for any $A \in \mathcal{H}$, $A \in \mathcal{F}_i$ if and only if $A \cap C_i \neq \emptyset$. Therefore, we obtain

- $A_i \cap C_i = \emptyset$ for all $i \in [k]$,
- $A_i \cap C_j \neq \emptyset$ for all $1 \leq i < j \leq k+1$. □

Hence, the pair of families

$$\{A_1, \ldots A_k, \emptyset\}$$

and

$$\{C_1, \ldots, C_k, C_{k+1}\}$$

satisfies the conditions of Lemma 1.1.4; thus, $k+1 \leq \binom{r+p}{r}$. Therefore,

$$d'(\mathrm{Cov}_{\mathcal{H},p}) \leq \binom{r+p}{r} - 1,$$

and by Theorem 1.1.3, $\mathrm{Cov}_{\mathcal{H},p}$ is $\left(\binom{r+p}{r} - 1\right)$-collapsible.

Theorem 1.1.2. *Let $\mathcal{H}$ be a hypergraph of rank r. Then $\mathrm{Int}_{\mathcal{H}}$ is $\frac{1}{2}\binom{2r}{r}$-collapsible.*

Proof. Let $\mathcal{H}$ be a hypergraph of rank r and let G be the graph on vertex set $\mathcal{H}$ whose edges are the pairs $\{A, B\} \subset \mathcal{H}$ such that $A \cap B = \emptyset$. Then $\mathrm{Int}_{\mathcal{H}} = I(G)$.

Let $k = k(G)$ and let $\{A_1, \ldots, A_k\} \subset \mathcal{H}$ that satisfies the conditions of Definition 3.2.1. That is,

- $A_i \cap A_j \neq \emptyset$ for all $i \neq j \in [k]$,
- There exist $B_1, \ldots, B_k \in \mathcal{H}$ such that
  - $A_i \cap B_i = \emptyset$ for all $i \in [k]$,
  - $A_i \cap B_j \neq \emptyset$ for all $1 \leq i < j \leq k$.

Then, the pair of families

$$\{A_1, \ldots, A_k, B_k, \ldots, B_1\}$$

and

$$\{B_1, \ldots, B_k, A_k, \ldots, A_1\}$$

satisfies the conditions of Lemma 1.1.4; therefore, $2k \leq \binom{2r}{r}$. Thus, by Proposition 3.2.3, $\mathrm{Int}_{\mathcal{H}} = I(G)$ is $\frac{1}{2}\binom{2r}{r}$-collapsible. $\square$

3.4 More complexes of hypergraphs

Let $\mathcal{H}$ be a hypergraph. A set C is a *t-transversal* of $\mathcal{H}$ if $|A \cap C| \geq t$ for all $A \in \mathcal{H}$. Let $\tau_t(\mathcal{H})$ be the minimal size of a t-transversal of $\mathcal{H}$. The hypergraph $\mathcal{H}$ is *pairwise t-intersecting* if $|A \cap B| \geq t$ for all $A, B \in \mathcal{H}$. Let

$$\mathrm{Cov}^t_{\mathcal{H},p} = \{\mathcal{F} \subset \mathcal{H} : \tau_t(\mathcal{F}) \leq p\}$$

and

$$\mathrm{Int}^t_{\mathcal{H}} = \{\mathcal{F} \subset \mathcal{H} : \mathcal{F} \text{ is pairwise } t\text{-intersecting}\}.$$

The following generalization of Lemma 1.1.4 was proved by Füredi in [Für84].

Lemma 3.4.1 (Füredi [Für84])**.** *Let $\{A_1, \ldots, A_k\}$ and $\{B_1, \ldots, B_k\}$ be families of sets such that:*

- $|A_i| \leq r$, $|B_i| \leq p$ *for all* $i \in [k]$,
- $|A_i \cap B_i| \leq t$ *for all* $i \in [k]$,
- $|A_i \cap B_j| > t$ *for all* $1 \leq i < j \leq k$.

Then

$$k \leq \binom{r+p-2t}{r-t}.$$

We obtain the following:

Theorem 3.4.2. *Let $\mathcal{H}$ be a hypergraph of rank r and let $t \leq \min\{r,p\} - 1$. Then $\mathrm{Cov}^{t+1}_{\mathcal{H},p}$ is $\left(\binom{r+p-2t}{r-t} - 1\right)$-collapsible.*

Theorem 3.4.3. *Let $\mathcal{H}$ be a hypergraph of rank r and let $t \leq r-1$. Then $\mathrm{Int}^{t+1}_{\mathcal{H}}$ is $\frac{1}{2}\binom{2(r-t)}{r-t}$-collapsible.*

Note that by setting $t = 0$ we recover Theorems 1.1.1 and 1.1.2. The proofs are essentially the same as the proofs of Theorems 1.1.1 and 1.1.2, except for the use of Lemma 3.4.1 instead of Lemma 1.1.4. The extremal examples are also similar: Let

$$\mathcal{H}_1 = \left\{A \cup [t] : A \in \binom{[r+p-t] \setminus [t]}{r-t}\right\}$$

and

$$\mathcal{H}_2 = \left\{A \cup [t] : A \in \binom{[2r-t] \setminus [t]}{r-t}\right\}.$$

The complex $\mathrm{Cov}^{t+1}_{\mathcal{H}_1}$ is the boundary of the $\left(\binom{r+p-2t}{r-t}-1\right)$-dimensional simplex, hence it is not $\left(\binom{r+p-2t}{r-t}-2\right)$-collapsible, and the complex $\mathrm{Int}^{t+1}_{\mathcal{H}_2}$ is the boundary of the $\frac{1}{2}\binom{2(r-t)}{r-t}$-dimensional cross-polytope, hence it is not $\left(\frac{1}{2}\binom{2(r-t)}{r-t}-1\right)$-collapsible.

Restricting ourselves to special classes of hypergraphs we may obtain better bounds on the collapsibility of their associated complexes. For example, we may look at r-partite r-uniform hypergraphs (that is, hypergraphs $\mathcal{H}$ on vertex set $V = V_1 \uplus V_2 \uplus \cdots \uplus V_r$ such that $|A \cap V_i| = 1$ for all $A \in \mathcal{H}$ and $i \in [r]$). In this case we have the following result:

Theorem 3.4.4. *Let $\mathcal{H}$ be an r-partite r-uniform hypergraph. Then $\mathrm{Int}_{\mathcal{H}}$ is 2^{r-1}-collapsible.*

The next example shows that the bound on the collapsibility of $\mathrm{Int}_{\mathcal{H}}$ in Theorem 3.4.4 is tight: Let $\mathcal{H}$ be the complete r-partite r-uniform hypergraph with all sides of size 2. It has 2^r edges, and any edge $A \in \mathcal{H}$ intersects all the edges of $\mathcal{H}$ except its complement. Therefore the complex $\mathrm{Int}_{\mathcal{H}}$ is the boundary of the 2^{r-1}-dimensional cross-polytope, so it is homeomorphic to a $(2^{r-1}-1)$-dimensional sphere. Hence, $\mathrm{Int}_{\mathcal{H}}$ is not $(2^{r-1}-1)$-collapsible.

For the proof we need the following Lemma, due to Lovász, Nešetřil and Pultr.

Lemma 3.4.5 (Lovász, Nešetřil, Pultr [LNP80, Prop. 5.3])**.** *Let $V = V_1 \uplus V_2 \uplus \cdots \uplus V_r$ be a finite set, and let $\{A_1, \ldots, A_k\}$ and $\{B_1, \ldots, B_k\}$ be families of subsets of V such that:*

- $|A_i \cap V_j| = 1$, $|B_i \cap V_j| = 1$ *for all* $i \in [k]$ *and* $j \in [r]$,
- $A_i \cap B_i = \emptyset$ *for all* $i \in [k]$,
- $A_i \cap B_j \neq \emptyset$ *for all* $1 \leq i < j \leq k$.

Then

$$k \leq 2^r.$$

A common generalization of Lemma 1.1.4 and Lemma 3.4.5 was proved by Alon in [Alo85].

The proof of Theorem 3.4.4 is the same as the proof of Theorem 1.1.2, except that we replace Lemma 1.1.4 by Lemma 3.4.5. A similar argument was also used by Aharoni and Berger ([AB09, Theorem 5.1]) in order to prove a related result about rainbow matchings in r-partite r-uniform hypergraphs.

3.5 More applications of minimal exclusion sequences

In this section we present further applications of Theorem 1.1.3. We will need the following simple lemma:

Lemma 3.5.1. *Let $\mathbb{F}$ be a field, and let V be a vector space over $\mathbb{F}$. Let $\mathcal{F}$ be a family of linear subspaces of V. Denote by $r(\mathcal{F})$ the maximal dimension of a subspace in $\mathcal{F}$.*

Let $v_1, \dots, v_k \in V$ and $U_1, \dots, U_{k+1} \in \mathcal{F}$ such that

- *$v_i \notin U_i$ for all $i \in [k]$,*
- *$v_i \in U_j$ for all $1 \leq i < j \leq k+1$.*

Then $k \leq r(\mathcal{F})$.

Proof. Let $i \in [k]$. We have $v_i \notin U_i$, but $v_i \in U_j$ for all $i+1 \leq j \leq k+1$. Hence, $\cap_{j=i}^{k+1} U_j \subsetneq \cap_{j=i+1}^{k+1} U_j$. So, we have a flag

$$U_{k+1} \supsetneq U_{k+1} \cap U_k \supsetneq U_{k+1} \cap U_k \cap U_{k-1} \supsetneq \cdots \supsetneq \cap_{i=1}^{k+1} U_i$$

of length $k+1$. In particular, we must have $\dim(U_{k+1}) \geq k$. Thus, $r(\mathcal{F}) \geq k$. □

3.5.1 Complexes from projective varieties

Let $\mathbb{F}$ be a field, and let V be a vector space over $\mathbb{F}$. Let $\mathcal{F}$ be a family of linear subspaces of V. Let $A \subset V$ be a finite set. We define the simplicial complex

$$X_{\mathcal{F}}[A] = \{\sigma \subset A : \sigma \subset U \text{ for some } U \in \mathcal{F}\}.$$

Proposition 3.5.2. *The complex $X_{\mathcal{F}}[A]$ is $r(\mathcal{F})$-collapsible.*

Proof. Let $\{v_1, \dots, v_k\} \in S(X_{\mathcal{F}}[A])$. Then, there exist maximal faces $\sigma_1, \dots, \sigma_{k+1}$ of $X_{\mathcal{F}}[A]$ such that $v_i \notin \sigma_i$ for all $i \in [k]$ and $v_i \in \sigma_j$ for $1 \leq i < j \leq k+1$. For every $i \in [k+1]$, there is some subspace $U_i \in \mathcal{F}$ such that $\sigma_i \subset U_i$.

We have $v_i \in U_j$ for all $1 \leq i < j \leq k+1$, and $v_i \notin U_i$ for $i \in [k]$ (otherwise, $\{v_i\} \cup \sigma_i \subset U_i$, but then $\{v_i\} \cup \sigma_i \in X_{\mathcal{F}}[A]$, a contradiction to the maximality of σ_i). So, by Lemma 3.5.1, we have $k \leq r(\mathcal{F})$. Thus, by Theorem 1.1.3, $C(X_{\mathcal{F}}[A]) \leq r(\mathcal{F})$. □

Remark. A different proof of the fact that $X_{\mathcal{F}}[A]$ is $r(\mathcal{F})$-Leray goes as follows:

First, since, for any $A' \subset A$, $X_{\mathcal{F}}[A][A'] = X_{\mathcal{F}}[A']$, it is enough to show that $\tilde{H}_k(X_{\mathcal{F}}[A]) = 0$ for all $k \geq r(\mathcal{F})$. Furthermore, we may assume that $0 \notin A$, since otherwise $X_{\mathcal{F}}[A]$ is a cone over 0, and in particular $\tilde{H}_k(X_{\mathcal{F}}[A]) = 0$ for all k.

Let

$$\tilde{\mathcal{F}} = \{\cap_{U \in \mathcal{F}'} U : \emptyset \neq \mathcal{F}' \subset \mathcal{F}\}$$

be the set of all subspaces of V obtained as intersections of subspaces in $\mathcal{F}$. Let X be the simplicial complex whose vertex set is $\tilde{\mathcal{F}}$, and whose simplices are the sets $\{U_1, \dots, U_k\} \subset \tilde{\mathcal{F}}$ forming a flag $U_1 \subsetneq U_2 \subsetneq \cdots \subsetneq U_k$.

For $u \in A$, let $X_u = X[\mathcal{F}_u]$, where $\mathcal{F}_u = \{U \in \tilde{\mathcal{F}} : u \in U\}$.

We have

$$\begin{aligned} N(\{X_u\}_{u\in A}) &= \{\sigma\subset A:\ \sigma\subset U \ \text{ for some } U\in\tilde{\mathcal{F}}\} \\ &= \{\sigma\subset A:\ \sigma\subset U \ \text{ for some } U\in\mathcal{F}\} = X_{\mathcal{F}}[A]. \end{aligned}$$

Moreover, for any $\sigma\in N(\{X_u\}_{u\in A})$, let

$$U_\sigma = \bigcap_{\substack{U\in\mathcal{F}:\\ \sigma\subset U}} U$$

be the minimal subspace in $\tilde{\mathcal{F}}$ containing σ. Then, $\cap_{u\in\sigma}X_u$ is a cone over the vertex U_σ. In particular, $\tilde{H}_k\left(\cap_{u\in\sigma}X_u\right)=0$ for all k. So, by the Nerve Theorem (Theorem 2.2.4), we have

$$\tilde{H}_k\left(X_{\mathcal{F}}[A]\right)\cong\tilde{H}_k\left(\cup_{u\in A}X_u\right)$$

for all k. Since the trivial subspace $\{0\}$ is not a vertex of $\cup_{u\in A}X_u$, and $\dim(U)\le r(\mathcal{F})$ for all $U\in\tilde{\mathcal{F}}$, we have $\dim(\cup_{u\in A}X_u)\le r(\mathcal{F})-1$. Hence,

$$\tilde{H}_k\left(X_{\mathcal{F}}[A]\right)\cong\tilde{H}_k\left(\cup_{u\in A}X_u\right)=0$$

for $k\ge r(\mathcal{F})$. □

Let $\mathbb{P}^n$ be the n-dimensional projective space over $\mathbb{F}$. That is, $\mathbb{P}^n$ is the set of the lines through the origin in $\mathbb{F}^{n+1}$, or equivalently, the set $(\mathbb{F}^{n+1}\setminus\{0\})/\sim$, where $x\sim y$ if $x=\lambda y$ for some $\lambda\in\mathbb{F}\setminus\{0\}$.

Let $p:\mathbb{F}^{n+1}\setminus\{0\}\to\mathbb{P}^n$ be the map that sends any vector $v\neq 0$ to the line through the origin containing v. A set $W\subset\mathbb{P}^n$ is called a projective subspace if $W=p(U)$ for some linear subspace $U\subset\mathbb{F}^n$. The dimension of W is defined by $\dim(W)=\dim(U)-1$. For a set $A\subset\mathbb{P}^n$, let $\operatorname{span}(A)$ be the minimal projective subspace containing A.

Let $f_1,\ldots,f_t\in\mathbb{F}[x_0,\ldots,x_n]$ be homogeneous polynomials. Let

$$V=V(f_1,\ldots,f_t)=\{x\in\mathbb{P}^n:\ f_i(x)=0 \text{ for all } i\in[t]\}\subset\mathbb{P}^n.$$

Let A be a finite subset of V. Define the simplicial complex

$$K_V[A]=\{\sigma\subset A:\operatorname{span}(\sigma)\subset V\}.$$

Proposition 3.5.3. *Let d be the maximal dimension of a projective subspace contained in V. Then, the complex $K_V[A]$ is $(d+1)$-collapsible.*

Proof. Let $\tilde{A}=p^{-1}(A)\subset\mathbb{F}^{n+1}$. Let $W_1,\ldots,W_k\subset\mathbb{P}^n$ be the maximal projective subspaces contained in V. For each $i\in[k]$, there is some subspace $U_i\subset\mathbb{F}^{n+1}$ such that $W_i=p(U_i)$ and $\dim(U_i)=\dim(W_i)+1\le d+1$. Let $\mathcal{F}=\{U_1,\ldots,U_k\}$.

For $\tau \subset \tilde{A}$ and $i \in [k]$, we have $\tau \subset U_i$ if and only if $p(\tau) \subset W_i$. So, $\tau \in X_{\mathcal{F}}[\tilde{A}]$ if and only if $p(\tau) \in K_V[A]$. Thus, we have

$$X_{\mathcal{F}}[\tilde{A}] = p^{-1}(K_V[A]).$$

Therefore, by (the easy direction of) Lemma 2.3.19 and Proposition 3.5.2, we obtain

$$C(K_V[A]) \leq C(X_{\mathcal{F}}[\tilde{A}]) \leq d + 1,$$

as wanted. $\square$

Example 3.5.4. Let $\mathbb{F} = \mathbb{F}_q$. Let n be even and let $f(x_1, \ldots, x_n) = x_1x_2 + x_3x_4 + \cdots + x_{n-1}x_n$. Let $V = V(f)$. It is well known that the maximum dimension of a subspace of $\mathbb{P}^{n-1}$ contained in V is $\frac{n}{2} - 1$ (see e.g. [VLW01, Theorem 26.6]). Therefore, by Proposition 3.5.3, for any $U \subset V$, the complex $K_V[U]$ is $\frac{n}{2}$-collapsible.

On the other hand, if U consists of the (equivalence classes of the) points $e_1 = (1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1)$, then the complex $K_V[U]$ is the boundary of a $\frac{n}{2}$-dimensional cross-polytope (its missing faces are the edges $\{e_1, e_2\}, \{e_2, e_3\}, \ldots, \{e_{n-1}, e_n\}$). So, $K_V[U]$ is homeomorphic to an $\left(\frac{n}{2} - 1\right)$-dimensional sphere. In particular, $K_V[U]$ is not $\left(\frac{n}{2} - 1\right)$-collapsible.

3.5.2 Matrices with bounded maximal rank

Let $\mathbb{F}$ be a field. Let $\mathcal{A}$ be a finite set of matrices in $\mathbb{F}^{m\times n}$. For $r \in \mathbb{N}$, we defined the complex

$$\mathrm{M}_{\mathcal{A},r} = \{\mathcal{B} \subset \mathcal{A} : \rho(\mathcal{B}) \leq r\},$$

where $\rho(\mathcal{B})$ is the maximal rank of a matrix in the span of $\mathcal{B}$.

Theorem 1.1.5. *Assume that $\mathbb{F}$ is infinite. Then, the complex $M_{\mathcal{A},r}$ is $r(r+1)$-collapsible.*

For the proof we will need the following result by Dieudonné (later extended by Flanders [Fla62] and Meshulam [Mes85]):

Theorem 3.5.5 (Dieudonné [Die48])**.** *Let $\mathbb{F}$ be a field, and let $r \in \mathbb{N}$. Let U be a linear subspace of $\mathbb{F}^{(r+1)\times(r+1)}$ satisfying $\rho(U) \leq r$. Then,* $\dim(U) \leq r(r+1)$.

We will also need the following lemma:

Lemma 3.5.6. *Let $\mathbb{F}$ be an infinite field, and let $r \in \mathbb{N}$. Let $B_1, \ldots, B_k \in \mathbb{F}^{m\times n}$ such that $r(B_i) > r$ for all $i \in [k]$. Then, there exist $P \in \mathbb{F}^{(r+1)\times m}$ and $Q \in \mathbb{F}^{n\times(r+1)}$ such that $r(PB_iQ) = r + 1$ for all $i \in [k]$.*

Proof. For each $i \in [k]$, let $p_i(P, Q) = \det(PB_iQ)$. We consider p_i as a polynomial in $(r + 1)(m + n)$ variables.

Note that, since $r(B_i) \geq r+1$, p_i is not identically zero: indeed, let $e_1, \ldots, e_{r+1}$ be a basis of $\mathbb{F}^{r+1}$, and let $v_1, \ldots, v_{r+1}$ be $r+1$ linearly independent columns of B_i. Let $\tilde{P} \in \mathbb{F}^{(r+1)\times m}$ be a linear transformation that, for each $j \in [r+1]$, maps the vector v_j to the basis vector e_j. Then, since the columns $\{\tilde{P}v_1, \ldots, \tilde{P}v_{r+1}\} = \{e_1, \ldots, e_{r+1}\}$ of $\tilde{P}B_i$ are linearly independent, $\tilde{P}B_i$ is of rank $r+1$. Hence, the rows $w_1, \ldots, w_{r+1}$ of $\tilde{P}B_i$ are also linearly independent. Let $\tilde{Q} \in \mathbb{F}^{(r+1)\times n}$ be a linear transformation that maps each w_j to the basis vector e_j. Then, the rows of $\tilde{P}B_i\tilde{Q}^t$ are exactly the vectors $e_1, \ldots, e_{r+1}$. So, $r(\tilde{P}B_i\tilde{Q}^t) = r+1$, and therefore $p_i(\tilde{P}, \tilde{Q}^t) = \det(\tilde{P}B_i\tilde{Q}^t) \neq 0$.

Now, since $\mathbb{F}$ is a field and none of the polynomials p_i is the zero polynomial, then the polynomial $\prod_{i=1}^k p_i$ is also not the zero polynomial. Therefore, since $\mathbb{F}$ is infinite, it is not identically zero. Hence, there exist $P \in \mathbb{F}^{(r+1)\times m}$ and $Q \in \mathbb{F}^{n\times(r+1)}$ such that $\prod_{i=1}^k p_i(P,Q) \neq 0$. So, $p_i(P,Q) = \det(PB_iQ) \neq 0$ for all $i \in [k]$. That is, $r(PB_iQ) = r+1$ for all $i \in [k]$, as wanted. □

Proof of Theorem 1.1.5. Let $\{A_1, \ldots, A_k\} \in S(\mathrm{M}_{\mathcal{A},r})$. Then, there exists maximal faces $\mathcal{B}_1, \ldots, \mathcal{B}_{k+1}$ of $\mathrm{M}_{\mathcal{A},r}$ such that $A_i \notin \mathcal{B}_i$ for all $i \in [k]$, and $A_i \in \mathcal{B}_j$ for all $1 \leq i < j \leq k+1$.

For $i \in [k+1]$, let $U_i = \operatorname{span}(\mathcal{B}_i)$. So, for all $1 \leq i < j \leq k+1$, we have $A_i \in U_j$. Moreover, for all $i \in [k]$, since $A_i \notin \mathcal{B}_i$, we must have $\rho(\mathcal{B}_i \cup \{A_i\}) > r$ (otherwise, we have $\rho(\mathcal{B}_i \cup \{A_i\}) \leq r$, and therefore $\mathcal{B}_i \cup \{A_i\} \in \mathrm{M}_{\mathcal{A},r}$, in contradiction to the maximality of $\mathcal{B}_i$). Therefore, there exists a matrix $B_i \in \operatorname{span}(\mathcal{B}_i \cup \{A_i\})$ such that $r(B_i) > r$. Moreover, since $\rho(\mathcal{B}_i) \leq r$, we have $A_i \notin U_i$ for all $i \in [k]$.

By Lemma 3.5.6, there exist $P \in \mathbb{F}^{(r+1)\times m}$ and $Q \in \mathbb{F}^{n\times(r+1)}$ such that $r(PB_iQ) = r+1$ for all $i \in [k]$.

For $i \in [k]$, let $A'_i = PA_iQ$, and for $i \in [k+1]$, let $U'_i = \{PBQ : B \in U_i\}$. For $1 \leq i < j \leq k+1$, we have $A_i \in U_j$, and thus $A'_i \in U'_j$. Note that, for any $B \in U_i$, we have $r(B) \leq r$, and therefore

$$r(PBQ) \leq r(B) \leq r.$$

Thus, $\rho(U'_i) \leq r$ for all $i \in [k+1]$. Since $r(PB_iQ) = r+1$, we must have $PB_iQ \notin U'_i$. Hence, since $PB_iQ \in \operatorname{span}(\{A'_i\}) + U'_i$, we must have $A'_i \notin U'_i$ for all $i \in [k]$.

By Theorem 3.5.5, $\dim(U'_i) \leq r(r+1)$ for all $i \in [k+1]$. So, by Lemma 3.5.1, we have $k \leq r(r+1)$. Therefore, by Theorem 1.1.3, $\mathrm{M}_{\mathcal{A},r}$ is $r(r+1)$-collapsible. □

3.5.3 Complexes of graphs with bounded matching number and some conjectures

Let $\mathcal{H}$ be a hypergraph. A *matching* in $\mathcal{H}$ is a set of edges that are pairwise disjoint. Let $\nu(\mathcal{H})$ be the maximal size of a matching in $\mathcal{H}$. For $\nu \in \mathbb{N}$ let

$$\mathrm{Mat}_{\mathcal{H},\nu} = \{\mathcal{F} \subset \mathcal{H} : \nu(\mathcal{H}) \leq \nu\}.$$

Note that, for $\nu = 1$, we obtain

$$\mathrm{Mat}_{\mathcal{H},1} = \mathrm{Int}_{\mathcal{H}}\,.$$

The case when $\mathcal{H}$ is a graph has been previously studied: Let K_n be the complete graph on n vertices, and let $K_{r,s}$ be the complete bipartite graph with parts of sizes r and s. The homotopy type of the complexes $\mathrm{Mat}_{K_n,\nu}$ and $\mathrm{Mat}_{K_{r,s},\nu}$ was determined by Linusson, Shareshian and Welker in [LSW08]:

Theorem 3.5.7 (Linusson, Shareshian, Welker [LSW08])**.** *The complex* $\mathrm{Mat}_{K_n,\nu}$ *is homotopy equivalent to a wedge of spheres of dimension* $3\nu - 1$*. The number of spheres in the wedge is*

$$\sum_{\{A_1,A_2,\ldots,A_{n-2\nu-1}\}\in\Pi} \left(\prod_{i=1}^{n-2\nu-1} (|A_i|-2)!! \right)^2,$$

where Π *is the set of all partitions of* $[n-1]$ *into* $n-2\nu-1$ *subsets* $A_1, A_2, \ldots, A_{n-2\nu-1}$ *of odd size.*

Theorem 3.5.8 (Linusson, Shareshian, Welker [LSW08])**.** *The complex* $\mathrm{Mat}_{K_{r,s},\nu}$ *is homotopy equivalent to a wedge of spheres of dimension* $2\nu - 1$*. The number of spheres in the wedge is*

$$\binom{r-1}{\nu}\binom{s-1}{\nu}.$$

In particular, these results imply that $C(\mathrm{Mat}_{K_n},\nu) \geq 3\nu$ and $C(\mathrm{Mat}_{K_{r,s},\nu}) \geq 2\nu$. The bipartite case was solved by Aharoni, Holzman and Jiang in [AHJ19], where it was shown that $C(\mathrm{Mat}_{K_{r,s},\nu}) = 2\nu$. In [HL20], Holmsen and Lee showed that $L(\mathrm{Mat}_{K_n,\nu}) = 3\nu$.

The complexes $\mathrm{M}_{\mathcal{A},r}$ are related to the complexes $\mathrm{Mat}_{G,\nu}$ by the following results (see also [Lov89]).

Theorem 3.5.9 (Edmonds [Edm67])**.** *Let* G *be a bipartite graph on vertex set* $A \uplus B$*, where* $A = \{a_1, a_2, \ldots, a_n\}$ *and* $B = \{b_1, b_2, \ldots, b_n\}$*.*

Let F_G *be the* $n \times n$ *matrix defined by*

$$F_G(i,j) = \begin{cases} x_{i,j} & \text{if } \{a_i, b_j\} \in G, \\ 0 & \text{otherwise,} \end{cases}$$

where $x_{i,j}$ *are variables. Then, the maximal rank of the matrix* F_G *(over all possible substitutions of the variables) is equal to* $\nu(G)$*.*

Theorem 3.5.10 (Tutte [Tut47])**.** *Let* G *be a graph on vertex set* $[n]$*. Let* T_G *be the*

$n \times n$ matrix defined by

$$T_G(i,j) = \begin{cases} x_{i,j} & \text{if } \{i,j\} \in G \text{ and } i < j, \\ -x_{i,j} & \text{if } \{i,j\} \in G \text{ and } i > j, \\ 0 & \text{otherwise,} \end{cases}$$

where $x_{i,j}$ are variables. Then, the maximal rank of the matrix T_G (over all possible substitutions of the variables) is equal to $2\nu(G)$.

As a consequence, we obtain:

Proposition 3.5.11. *Let Let $\{e_{i,j}\}_{i,j=1}^n$ be the matrices in the standard basis of $\mathbb{F}^{n\times n}$.*

1. *Let G be a bipartite graph on vertex set $A \cup B$, where $A = \{a_1, a_2, \dots, a_n\}$ and $B = \{b_1, b_2, \dots, b_n\}$. Let $\mathcal{A}_G = \{e_{i,j} : \{a_i, b_j\} \in G\}$. Then*

$$M_{\mathcal{A}_G,k} = \mathrm{Mat}_{G,k}\,.$$

2. *Let G be a graph on vertex set $[n]$. Let $\mathcal{A}_G = \{e_{ij} - e_{ji} : \{i,j\} \in G\}$. Then*

$$M_{\mathcal{A}_G,2k} = \mathrm{Mat}_{G,k}\,.$$

In [BK19], Briggs and Kim obtained the following generalization of Aharoni, Holzman and Jiang's result:

Theorem 3.5.12 (Briggs, Kim [BK19])**.** *Let $r \geq 1$. Let $\mathcal{A} \subset \mathbb{F}^{n\times m}$ be a finite family of rank one matrices. Then, $M_{\mathcal{A},r}$ is $2r$-collapsible.*

In fact, Theorem 3.5.12 is a special case of [BK19, Theorem 8], that bounds the collapsibility of complexes associated to fractional matchings in a family of matroids.

Based on Proposition 3.5.11 and the results in [HL20], we conjecture the following:

Conjecture 3.5.13. *Let $r \geq 2$ be an even integer. Let $\mathcal{A}$ be a finite family of rank two skew-symmetric matrices in $\mathbb{F}^{n\times n}$. Then, $M_{\mathcal{A},r}$ is $\frac{3}{2}r$-collapsible.*

Note that, since $C(\mathrm{M}_{\mathcal{A}_{K_n},r}) = C(\mathrm{Mat}_{K_n}, \frac{r}{2}) \geq \frac{3r}{2}$, the bound in the conjecture is tight. We also conjecture:

Conjecture 3.5.14. *Let $r \geq 1$. Let $\mathcal{A}$ be a finite family of rank two symmetric matrices in $\mathbb{F}^{n\times n}$. Then, $M_{\mathcal{A},r}$ is r-collapsible.*

We believe that the condition on the size of the field $\mathbb{F}$ in the statement of Theorem 1.1.5 is a byproduct of our proof and is not actually necessary. Moreover, we don't expect the bound $r(r+1)$ to be tight. In fact, the following may be true:

Conjecture 3.5.15. *Let $\mathcal{A}$ be a finite family of matrices in $\mathbb{F}^{m\times n}$, and let $r \geq 1$. Then, $M_{\mathcal{A},r}$ is $2r$-collapsible.*

That is, the bound from Theorem 3.5.12 for rank one matrices may hold also for general families of matrices.

Chapter 4

Complexes of graphs with bounded independence number

This chapter is organized as follows. In Section 4.1 we introduce some basic definitions and facts about graphs that we will use throughout the chapter. In Section 4.2 we present several tools for bounding the collapsibility numbers of a general simplicial complex. Section 4.3 contains the proof of Theorem 1.2.5, dealing with the case of chordal graphs. Section 4.4 focuses on the class of graphs with bounded maximum degree. It contains the proofs of Theorems 1.2.6, 1.2.7, 1.2.8, and 1.2.9. Section 4.5 deals with the Leray numbers of the complexes $I_n(G)$. In particular, it presents extremal examples determining the tightness of our main results (Theorems 1.2.7, 1.2.8 and 1.2.9), and examples of 3-regular graphs for which the complexes $I_n(G)$ do not satisfy the bound in Question 1.2.4 (for various values of n).

This chapter is based on joint work with Minki Kim.

4.1 Preliminaries on graphs

Let $G = (V, E)$ be a simple graph. For a set $U \subset V$, the subgraph of G *induced* by U is the graph

$$G[U] = (U, \{e \in E : e \subset U\}) .$$

A set $U \subset V$ is called a *clique* in G if the induced subgraph $G[U]$ is the complete graph on vertex set U.

For any vertex $v \in V$, we define the *deletion* of v in G to be the induced subgraph $G \setminus v = G[V \setminus \{v\}]$.

For each $v \in V$, we define the *open neighborhood* of v in G as the vertex subset

$$N_G(v) = \{u \in V : u \text{ is adjacent to } v\},$$

and we define the *closed neighborhood* of v in G as

$$N_G[v] = \{v\} \cup N_G(v).$$

For a set $A \subset V$, let

$$N_G(A) = \bigcup_{u \in A} N_G(u).$$

A vertex $v \in V$ is called a *simplicial vertex* if $N_G[v]$ is a clique. The *degree* of v in G is the number $\deg_G(v) = |N_G(v)|$.

We say G is *k-colorable* (or k-partite) if we can partition the vertex set V into k parts so that each part is independent in G. The following is a classical result in graph theory that states a relation between the maximum degree and the k-colorability of G.

Theorem 4.1.1 (Brooks [Bro41])**.** *Let G be a connected graph with maximum degree k. Then G is k-colorable unless G is the complete graph K_{k+1} or an odd cycle.*

The complete bipartite graph $K_{1,3}$ is called a *claw*. A graph is said to be *claw-free* if it does not have a claw as an induced subgraph.

We say a graph is *chordal* if it does not contain a cycle of length at least 4 as an induced subgraph. Chordal graphs satisfy the following special property:

Theorem 4.1.2 (Lekkerkerker, Boland [LB63])**.** *Every chordal graph contains a simplicial vertex.*

4.2 Upper bounds for collapsibility numbers

In this section we present our main technical tools for proving d-collapsibility of a simplicial complex.

Lemma 4.2.1. *Let X be a simplicial complex, and let $\sigma = \{v_1, \dots, v_k\} \in X$. For every $0 \le i \le k$, define $\sigma_i = \{v_j : 1 \le j \le i\}$. Let $d \ge k$. If for all $0 \le i \le k-1$,*

$$C(\mathrm{lk}(X \setminus v_{i+1}, \sigma_i)) \le d - i,$$

and

$$C(\mathrm{lk}(X, \sigma_k)) \le d - k,$$

then $C(X) \le d$.

Proof. We will show that, for any $i \in \{0, \dots, k\}$,

$$C(\mathrm{lk}(X, \sigma_i)) \le d - i.$$

We argue by backwards induction on i. For $i = k$, $C(\mathrm{lk}(X, \sigma_k)) \le d - k$ by assumption.

Let $i < k$. By Lemma 2.3.14, we have

$$C(\mathrm{lk}(X, \sigma_i)) \le \max\{C(\mathrm{lk}(X \setminus v_{i+1}, \sigma_i)), C(\mathrm{lk}(X, \sigma_{i+1})) + 1\}.$$

But $C(\mathrm{lk}(X \setminus v_{i+1}, \sigma_i)) \le d - i$ by assumption, and $C(\mathrm{lk}(X, \sigma_{i+1})) \le d - i - 1$ by the induction hypothesis. Therefore,

$$C(\mathrm{lk}(X, \sigma_i)) \le d - i.$$

Setting $i = 0$, we obtain (since $\sigma_0 = \emptyset$),

$$C(X) = C(\mathrm{lk}(X, \sigma_0)) \le d - 0 = d.$$

As a consequence of Lemma 4.2.1, we obtain:

Proposition 4.2.2. *Let X be a simplicial complex on vertex set V. If all the missing faces of X are of dimension at most d, then*

$$C(X) \le \left\lfloor \frac{d|V|}{d+1} \right\rfloor.$$

Moreover, equality $C(X) = \frac{d|V|}{d+1}$ is obtained if and only if X is the join of $r = \frac{|V|}{d+1}$ disjoint copies of the boundary of a d-dimensional simplex (or equivalently, if the set of missing faces of X consists of r disjoint sets of size $d+1$).

Proof. We argue by induction on $|V|$. If $|V| = 0$, then X is 0-collapsible, and the inequality holds. Assume $|V| > 0$. If X is a complete complex, then it is 0-collapsible, and the inequality holds. Otherwise, let $\sigma = \{v_1, \ldots, v_{k+1}\} \subset V$ be a missing face of X. Since all the missing faces of X are of dimension at most d, we have $k \le d$. For each $0 \le i \le k$, let $\sigma_i = \{v_j : 1 \le j \le i\} \in X$. Let V_i be the vertex set of $\mathrm{lk}(X \setminus v_{i+1}, \sigma_i)$. Note that for every $0 \le i \le k$,

$$V_i \subset V \setminus \sigma_{i+1}.$$

Therefore, by the induction hypothesis,

$$C(\mathrm{lk}(X \setminus v_{i+1}, \sigma_i)) \le \frac{d}{d+1}|V_i| \le \frac{d}{d+1}(|V| - i - 1).$$

Since $i \le k \le d$, we obtain

$$C(\mathrm{lk}(X \setminus v_{i+1}, \sigma_i)) \le \frac{d}{d+1}|V| - \frac{i}{i+1}(i+1) = \frac{d}{d+1}|V| - i.$$

Also, since σ is a missing face, we have

$$\mathrm{lk}(X, \sigma_k) = \mathrm{lk}(X \setminus v_{k+1}, \sigma_k),$$

and in particular $C(\mathrm{lk}(X,\sigma_k)) \leq \frac{d}{d+1}|V| - k$. Therefore, by Lemma 4.2.1, we obtain

$$C(X) \leq \frac{d}{d+1}|V|.$$

Since $C(X)$ is an integer, we obtain $C(X) \leq \left\lfloor \frac{d|V|}{d+1} \right\rfloor$.

Now, assume $C(X) = \frac{d}{d+1}|V|$. Note that, since $C(X)$ is an integer and $\gcd(d, d+1) = 1$, then $d+1$ must divide $|V|$.

Then, there exists some $0 \leq i \leq k$ such that

$$C(\mathrm{lk}(X \setminus v_{i+1}, \sigma_i)) = \frac{d}{d+1}(|V| - i - 1)$$

(otherwise, by the same argument as above, we could prove that $C(X) < \frac{d}{d+1}|V|$). Since $d+1$ divides $|V|$, it must also divide $i+1$. Hence, we must have $i = k = d$. By the induction hypothesis, the missing faces of

$$\mathrm{lk}(X, \sigma_d) = \mathrm{lk}(X \setminus v_{d+1}, \sigma_d)$$

form a set of $r-1$ disjoint sets of size $d+1$. Therefore, the set of missing faces of X consists of r disjoint sets of size $d+1$ plus, possibly, some additional faces of the form $\tau \cup \{v_{d+1}\}$, where $\tau \in V \setminus \sigma$. But the choice of the order $v_1, \ldots, v_{d+1}$ on the vertices of σ was arbitrary. Thus, repeating the same argument with a different order (e.g. $v'_i = v_i$ for $i \leq d-1$, $v'_d = v_{d+1}$, $v'_{d+1} = v_d$), we obtain that the set of missing faces of X consists exactly of r disjoint sets of size $d+1$. □

Remark. An analogous bound in terms of Leray numbers was proved in [Ada14, Prop. 5.4].

Lemma 4.2.3. *Let X be a complex on vertex set V, and let $B \subset V$. Let $<$ be a linear order on the vertices of B. Let $\mathcal{P} = \mathcal{P}(X, B)$ be the family of partitions (B_1, B_2) of B satisfying:*

- *$B_2 \in X$.*
- *For any $v \in B_2$, the complex*

  $$\mathrm{lk}(X[V \setminus \{u \in B_1 : u < v\}], \{u \in B_2 : u < v\})$$

  is not a cone over v.

If for every $(B_1, B_2) \in \mathcal{P}$,

$$C(\mathrm{lk}(X[V \setminus B_1], B_2)) \leq d - |B_2|,$$

then $C(X) \leq d$.

Proof. We argue by induction on $|B|$. If $|B| = 0$ there is nothing to prove. So, assume $|B| > 0$, and let v be the minimal vertex in B (with respect to the order $<$). Let $X' = X \setminus v$, and let $V' = V \setminus \{v\}$ be its vertex set. Let $B' = B \setminus \{v\}$, and let $(B'_1, B'_2) \in \mathcal{P}(X', B')$. Define $B_1 = B'_1 \cup \{v\}$ and $B_2 = B'_2$. Then, $B_2 \in X \setminus v \subset X$, and for any $u \in B_2$, the complex

$$\begin{aligned}\mathrm{lk}(X[V \setminus \{w \in B_1 : w < u\}], \{w \in B_2 : w < u\})\\ = \mathrm{lk}(X'[V' \setminus \{w \in B'_1 : w < u\}], \{w \in B'_2 : w < u\})\end{aligned}$$

is not a cone over u (since $(B'_1, B'_2) \in \mathcal{P}(X', B')$). Therefore $(B_1, B_2) \in \mathcal{P}(X, B)$. So,

$$C(\mathrm{lk}(X'[V' \setminus B'_1], B'_2)) = C(\mathrm{lk}(X[V \setminus B_1], B_2)) \leq d - |B_2| = d - |B'_2|.$$

Hence, by the induction hypothesis, $C(X \setminus v) = C(X') \leq d$.

If X is a cone over v then, by Lemma 2.3.9, $C(X) = C(X \setminus v) \leq d$, as wanted. Otherwise, let $X'' = \mathrm{lk}(X, v)$, and let $V'' \subset V \setminus \{v\}$ be its vertex set. Let $B'' = B \cap V''$, and let $(B''_1, B''_2) \in \mathcal{P}(X'', B'')$. Let $B_2 = B''_2 \cup \{v\}$ and $B_1 = B \setminus B_2$.

Since $B''_2 \in X'' = \mathrm{lk}(X, v)$, we have $B_2 = B''_2 \cup \{v\} \in X$. Let $u \in B_2$. If $u = v$, then

$$\mathrm{lk}(X[V \setminus \{w \in B_1 : w < u\}], \{w \in B_2 : w < u\}) = X$$

is not a cone over $u = v$. If $u > v$, then

$$\begin{aligned}\mathrm{lk}(X[V \setminus \{w \in B_1 : w < u\}], \{w \in B_2 : w < u\})\\ = \mathrm{lk}(X''[V'' \setminus \{w \in B''_1 : w < u\}], \{w \in B''_2 : w < u\})\end{aligned}$$

is not a cone over u (since $(B''_1, B''_2) \in \mathcal{P}(X'', B'')$). Therefore, $(B_1, B_2) \in \mathcal{P}(X, B)$. So,

$$C(\mathrm{lk}(X''[V'' \setminus B''_1], B''_2)) = C(\mathrm{lk}(X[V \setminus B_1], B_2)) \leq d - |B_2| = (d-1) - |B''_2|.$$

Thus, by the induction hypothesis, $C(\mathrm{lk}(X, v)) = C(X'') \leq d - 1$. Hence, by Lemma 2.3.14, $C(X) \leq d$. ☐

4.3 Chordal graphs

In this section we prove Theorem 1.2.5, which bounds the collapsibility of $I_n(G)$ in the case that G is a chordal graph. The proof relies on the next result.

Lemma 4.3.1. *Let $G = (V, E)$ be a graph, and let $v \in V$ be a simplicial vertex in G. Then, for any $n \geq 2$,*

$$C(I_n(G)) \leq \max\{C(I_n(G \setminus v)), C(I_{n-1}(G[V \setminus N_G[v]])) + 1\}.$$

Proof. Let $W \subset V \setminus \{v\}$. Then, W belongs to $\mathrm{lk}(I_n(G), v)$ if and only if $W \setminus N_G(v) \in I_{n-1}(G)$. Indeed, assume that $W \setminus N_G(v) \notin I_{n-1}(G)$; that is, $G[W \setminus N_G(v)]$ contains an independent set A of size $n-1$. Then, $A \cup \{v\}$ is an independent set of size n in G, and therefore $W \notin \mathrm{lk}(I_n(G), v)$. For the opposite direction, suppose $W \notin \mathrm{lk}(I_n(G), v)$. Then, $W \cup \{v\}$ contains an independent set A of size n in G. Since $N_G[v]$ is a clique in G, A contains at most one vertex from $N_G[v]$. Thus, $A \setminus N_G[v] \subset W \setminus N_G(v)$ is an independent set of size at least $n-1$. So, $W \setminus N_G(v) \notin I_{n-1}(G)$.

It follows that $\mathrm{lk}(I_n(G), v) = 2^{N_G(v)} * I_{n-1}(G[V \setminus N_G[v]])$. By Lemma 2.3.8, we have

$$C(\mathrm{lk}(I_n(G), v)) = C(I_{n-1}(G[V \setminus N_G[v]])).$$

So, by Lemma 2.3.14, we obtain

$$\begin{aligned} C(I_n(G)) &\leq \max\{C(I_n(G \setminus v)), C(\mathrm{lk}(I_n(G), v)) + 1\} \\ &= \max\{C(I_n(G \setminus v)), C(I_{n-1}(G[V \setminus N_G[v]])) + 1\}. \end{aligned}$$

Theorem 1.2.5. *Let $G = (V, E)$ be a chordal graph and $n \geq 1$ an integer. Then,*

$$C(I_n(G)) \leq n - 1.$$

Moreover, if $\alpha(G) \geq n$, then $C(I_n(G)) = n - 1$.

Proof. We argue by induction on $|V|$. For $|V| = 0$ the statement is obvious. Suppose $|V| > 0$. For $n = 1$, $C(I_1(G)) = C(\{\emptyset\}) = 0$, so the claim holds. Let $n \geq 2$. Since G is a chordal graph, there exists a simplicial vertex v in G. By the induction hypothesis,

$$C(I_n(G - v)) \leq n - 1$$

and

$$C(I_{n-1}(G[V \setminus N_G[v]])) \leq n - 2.$$

Hence, by Lemma 4.3.1,

$$C(I_n(G)) \leq \max\{C(I_n(G \setminus v)), C(I_{n-1}(G[V \setminus N_G[v]])) + 1\} \leq n - 1.$$

Now, let G be a graph with $\alpha(G) \geq n$, and let A be an independent set of size n in G. Then $I_n(G)[A]$ is the boundary of an $(n-1)$-dimensional simplex, and in particular $\tilde{H}_{n-2}(I_n(G)[A]; \mathbb{F}) \neq 0$. Hence, $C(I_n(G)) \geq L(I_n(G)) \geq n - 1$. So, any chordal graph G with $\alpha(G) \geq n$ satisfies $C(I_n(G)) = n - 1$.

4.4 Graphs with bounded maximum degree

In this section we prove our main results about graphs with bounded maximum degree, Theorems 1.2.6, 1.2.7, 1.2.8, and 1.2.9.

We begin with the following related problem: Let $\mathcal{X}(k)$ be the class of all k-colorable graphs. In [ABKK19] it was observed that $f_{\mathcal{X}(k)}(n) = k(n-1)+1$. The following proposition (combined with Proposition 1.2.2) offers an alternative proof for this result.

Proposition 4.4.1. *Let G be a k-colorable graph and $n \geq 1$ an integer. Then,*

$$C(I_n(G)) \leq k(n-1).$$

Proof. Take a proper vertex-coloring of G with k colors. Note that each color class forms an independent set in G. Let $\sigma \in I_n(G)$. Since σ contains no independent set of size n in G, it contains at most $n-1$ vertices from each color class. It follows that $|\sigma| \leq k(n-1)$. Hence, by Lemma 2.3.5,

$$C(I_n(G)) \leq \dim(I_n(G)) + 1 \leq k(n-1).$$

Next, we present the proof of Theorem 1.2.6. We deal with the case $\Delta = 2$ separately:

Theorem 4.4.2. *Let $G = (V, E)$ be a graph with maximum degree at most 2 and $n \geq 1$ an integer. Then $I_n(G)$ is $2(n-1)$-collapsible.*

Recall that a graph with maximum degree bounded by 2 is a disjoint union of cycles and paths. In other to apply an inductive argument, we state the following more general claim:

Proposition 4.4.3. *Let $G = (V, E)$ be a graph with maximum degree at most 2. Let A be an independent set in G of size at most $n-1$ that is contained in the union of all the components of G that are paths. Then,*

$$C(\operatorname{lk}(I_n(G), A)) \leq 2(n-1) - |A|.$$

Proof. We argue by induction on the number of cycles c in G.

If $c = 0$, then G is a disjoint union of paths. In particular, it is a chordal graph, and by Theorem 1.2.5, $C(I_n(G)) \leq n-1$. By Lemma 2.3.15, we obtain

$$C(\operatorname{lk}(I_n(G), A)) \leq C(I_n(G)) \leq n-1 \leq 2(n-1) - |A|.$$

Let $c \geq 1$, and assume that the claim holds for all graphs with less than c cycles. Let $C = \{v_1, \ldots, v_k\}$ be the vertex set of a cycle in G (such that $\{v_i, v_{i+1}\} \in E$ for all $i \in [k]$, where the indices are taken modulo k). Let

$$r = \min\left\{ \left\lfloor \frac{k}{2} \right\rfloor, n - |A| - 1 \right\},$$

and let

$$U = \{v_{2i-1} : 1 \leq i \leq r\}.$$

So, U is an independent set in G of size r.

For each $0 \leq i \leq r$, let $U_i = \{v_{2j-1} : 1 \leq j \leq i\}$. Let $0 \leq i \leq r-1$. The graph $G \setminus v_{2i+1}$ has $c-1$ cycles, and the set $A \cup U_i$ is an independent set contained in components of $G \setminus v_{2i+1}$ that are paths. Therefore, by the induction hypothesis,

$$C(\mathrm{lk}(I_n(G \setminus v_{2i+1}), A \cup U_i)) \leq 2(n-1) - |A| - i.$$

Next, we divide into two cases. First, assume $r = n - |A| - 1 < \left\lfloor \frac{k}{2} \right\rfloor$. Then $2r + 1 \leq k$, and, by the same argument as before, we obtain

$$C(\mathrm{lk}(I_n(G \setminus v_{2r+1}), A \cup U_r)) \leq 2(n-1) - |A| - r.$$

Since $r = n - |A| - 1$, the set $A \cup U_r \cup \{v_{2r+1}\}$ is an independent set of size n in G; therefore, $v_{2r+1} \notin \mathrm{lk}(I_n(G), A \cup U_r)$. Hence,

$$\mathrm{lk}(I_n(G), A \cup U_r) = \mathrm{lk}(I_n(G \setminus v_{2r+1}), A \cup U_r).$$

So,

$$C(\mathrm{lk}(I_n(G), A \cup U_r)) \leq 2(n-1) - |A| - r.$$

Now, assume $r = \left\lfloor \frac{k}{2} \right\rfloor$. Then, U_r is a maximum independent set in $G[C]$, and we have

$$\mathrm{lk}(I_n(G), A \cup U_r) = 2^{C \setminus U_r} * \mathrm{lk}(I_{n-r}(G[V \setminus C]), A).$$

Therefore, by Lemma 2.3.8, we obtain

$$\begin{aligned} C(\mathrm{lk}(I_n(G), A \cup U_r)) = C(\mathrm{lk}(I_{n-r}(G[V \setminus C]), A)) &\leq 2(n-r-1) - |A| \\ &= 2(n-1) - |A| - 2r \leq 2(n-1) - |A| - r, \end{aligned}$$

where the first inequality follows by the induction hypothesis (since the number of cycles in $G[V \setminus C]$ is $c-1$).

In both cases we obtained

$$C(\mathrm{lk}(I_n(G), A \cup U_r)) \leq 2(n-1) - |A| - r.$$

So, by Lemma 4.2.1, we obtain

$$C(\mathrm{lk}(I_n(G), A)) \leq 2(n-1) - |A|,$$

as wanted.

Theorem 4.4.2 follows from Proposition 4.4.3 by setting $A = \emptyset$.

Now we can prove the general case of Theorem 1.2.6:

Theorem 1.2.6. *Let $G = (V, E)$ be a graph with maximum degree at most Δ and $n \geq 1$ an integer. Then,*

$$C(I_n(G)) \leq \Delta(n-1).$$

Proof. We argue by induction on n. For $n = 1$ the claim is trivial. Assume $n \geq 2$.

If $\Delta = 1$ then the edges of G are pairwise disjoint. In particular, G is a chordal graph; therefore, the claim follows from Theorem 1.2.5. If $\Delta = 2$, the claim follows from Theorem 4.4.2. Assume $\Delta \geq 3$, and let G be a graph with maximum degree at most Δ. We will show that $C(I_n(G)) \leq \Delta(n-1)$. Let $c(G)$ be the number of connected components of G that are isomorphic to the complete graph $K_{\Delta+1}$. We argue by induction on $c(G)$.

If $c(G) = 0$, then by Brooks' Theorem (Theorem 4.1.1) G is Δ-colorable. Then, by Proposition 4.4.1, $I_n(G)$ is $\Delta(n-1)$-collapsible, as wanted.

Otherwise, assume there exists a component of G that is isomorphic to $K_{\Delta+1}$, and let v be a vertex in that component. Note that v is a simplicial vertex in G. Since $c(G \setminus v) = c(G) - 1$, we obtain by the induction hypothesis

$$C(I_n(G \setminus v)) \leq \Delta(n-1).$$

Also, by the (first) induction hypothesis, we have

$$C(I_{n-1}(G[V \setminus N_G[v]])) \leq \Delta(n-2) \leq \Delta(n-1) - 1.$$

So, by Lemma 4.3.1, we obtain

$$C(I_n(G)) \leq \max\{C(I_n(G \setminus v)), C(I_{n-1}(G[V \setminus N_G[v]])) + 1\} \leq \Delta(n-1),$$

as wanted. □

4.4.1 The $n \leq 3$ case and claw-free graphs

Next, we prove Theorems 1.2.7 and 1.2.8, which give tight upper bounds on the collapsibility of $I_n(G)$ for graphs G with bounded maximum degree, for $n \leq 3$. We also prove Proposition 4.4.5, bounding the collapsibility of certain subcomplexes of $I_n(G)$, in the case where G is a bounded degree claw-free graph.

Theorem 1.2.7. *Let $G = (V, E)$ be a graph with maximum degree at most Δ. Then,*

$$C(I_2(G)) \leq \left\lceil \frac{\Delta + 1}{2} \right\rceil.$$

Proof. We argue by induction on $|V|$. For $|V| = 0$ the bound holds trivially. Assume $|V| > 0$, and let $v \in V$. By Lemma 2.3.14, we have

$$C(I_2(G)) \leq \max\{C(I_2(G \setminus v)), C(\mathrm{lk}(I_2(G), v)) + 1\}. \tag{4.1}$$

Note that $\mathrm{lk}(I_2(G), v)$ is a flag complex on vertex set $N_G(v)$. Thus, by Proposition 4.2.2, we have

$$C(\mathrm{lk}(I_2(G), v)) \leq \left\lfloor \frac{|N_G(v)|}{2} \right\rfloor \leq \left\lfloor \frac{\Delta}{2} \right\rfloor = \left\lceil \frac{\Delta+1}{2} \right\rceil - 1.$$

Also, by the induction hypothesis,

$$C(I_2(G \setminus v)) \leq \left\lceil \frac{\Delta+1}{2} \right\rceil.$$

Hence, by (4.1), we obtain

$$C(I_2(G)) \leq \left\lceil \frac{\Delta+1}{2} \right\rceil.$$

Lemma 4.4.4. *Let $G = (V, E)$ be a graph and $n \geq 2$ an integer. Let A be an independent set of size $n-1$ in G, such that any vertex in $V \setminus A$ is adjacent to at most two vertices in A. Let*

$$B = \bigcup_{\{u,v\} \in \binom{A}{2}} N_G(u) \cap N_G(v).$$

Assume that $A \cup B$ does not contain an independent set of size n (that is, $A \cup B \in I_n(G)$). Then, $\mathrm{lk}(I_n(G), A \cup B)$ is a flag complex.

Proof. Let $X = \mathrm{lk}(I_n(G), A \cup B)$, and let τ be a missing face of X. Then, there exists an independent set I of G of size n, such that $\tau \subset I \subset \tau \cup A \cup B$. We may choose I such that $|A \cap I|$ is maximal. Each vertex in $A \setminus I$ is adjacent to at least two vertices in $I \setminus A$: otherwise, assume there exists $a \in A \setminus I$ that is adjacent to at most one vertex in $I \setminus A$. We divide into two cases:

- If a is not adjacent to any vertex in $I \setminus A$, let $\tau' = \tau \setminus \{u\}$ for any vertex $u \in \tau$.
- If a is adjacent to a single vertex $u \in I \setminus A$, observe that u should be contained in τ. If not, we can take an independent set $I' = I \setminus \{u\} \cup \{a\}$ of size n in G such that $\tau \subset I' \subset \tau \cup A \cup B$. Since $|A \cap I'| = |A \cap I| + 1$, this contradicts the maximality assumption of $|A \cap I|$. Hence, $u \in \tau$. Now, let $\tau' = \tau \setminus \{u\}$.

In both cases, $I \setminus \{u\} \cup \{a\}$ is an independent set of size n satisfying $\tau' \subset I \setminus \{u\} \cup \{a\} \subset \tau' \cup A \cup B$. It follows that $\tau' \notin X$, which is a contradiction to τ being a missing face.

Let $|\tau| = k$ and $|A \cap I| = t$. Then, $|A \setminus I| = n-t-1$; so, there are at least $2(n-t-1)$ edges between A and $I \setminus A$.

By assumption, each vertex $v \in I \setminus (A \cup \tau)$ is adjacent to at most 2 vertices in A. Therefore, since $|I \setminus (A \cup \tau)| = n-t-k$, there are at least $2(n-t-1) - 2(n-t-k) = 2k-2$

edges between A and τ. But, since $\tau \subset V \setminus B$, each vertex in τ is adjacent to at most one vertex in A. Therefore, we must have $2k-2 \leq k$; that is, $|\tau| = k \leq 2$. Thus, X is a flag complex. $\square$

Proposition 4.4.5. *Let $G=(V,E)$ be a claw-free graph with maximum degree at most Δ, and let $n \geq 1$ be an integer. Let A be an independent set of size $n-1$ in G. Then,*

$$C(\mathrm{lk}(I_n(G), A)) \leq \left\lfloor \frac{(n-1)\Delta}{2} \right\rfloor.$$

Proof. For $n=1$ the claim holds trivially. Assume $n \geq 2$.

Let $v \in V \setminus (A \cup N_G(A))$. Then, $A \cup \{v\}$ is an independent set of size n in G; hence, $v \notin \mathrm{lk}(I_n(G), A)$. So, we may assume without loss of generality that $V = N_G(A) \cup A$. Let

$$B = \bigcup_{\{u,v\} \in \binom{A}{2}} N_G(u) \cap N_G(v)$$

and $U = N_G(A) \setminus B$. Since G is claw-free, each vertex is adjacent to at most 2 vertices in A. Hence, we have

$$|N_G(A)| = \sum_{v \in A} |N_G(v)| - \sum_{\{u,v\} \in \binom{A}{2}} |N_G(u) \cap N_G(v)| = \sum_{v \in A} |N_G(v)| - |B|.$$

So, since the maximum degree in G is at most Δ, we obtain

$$|U| \leq (n-1)\Delta - 2|B|.$$

Let $X = \mathrm{lk}(I_n(G), A)$. We will use Lemma 4.2.3 in order to show that $C(X) \leq \left\lfloor \frac{(n-1)\Delta}{2} \right\rfloor$:

Let (B_1, B_2) be a partition of B such that $B_2 \in X = \mathrm{lk}(I_n(G), A)$. Let $G' = G[V \setminus B_1]$, and let

$$Y = \mathrm{lk}(X[V \setminus B_1], B_2) = \mathrm{lk}(I_n(G)[V \setminus B_1], A \cup B_2) = \mathrm{lk}(I_n(G'), A \cup B_2).$$

Note that

$$B_2 = \bigcup_{\{u,v\} \in \binom{A}{2}} N_{G'}(u) \cap N_{G'}(v)$$

Also, since G' is claw-free and A is independent in G', then every vertex in $V \setminus B_1$ is adjacent to at most 2 vertices in A. Therefore, by Lemma 4.4.4, Y is a flag complex.

The vertex set of Y is contained in $U = N_G(A) \setminus B$. Thus, by Proposition 4.2.2, we obtain

$$C(Y) \leq \left\lfloor \frac{|U|}{2} \right\rfloor \leq \left\lfloor \frac{(n-1)\Delta - 2|B|}{2} \right\rfloor \leq \left\lfloor \frac{(n-1)\Delta}{2} \right\rfloor - |B_2|.$$

Therefore, by Lemma 4.2.3,

$$C(\mathrm{lk}(I_n(G),A)) \le \left\lfloor \frac{(n-1)\Delta}{2} \right\rfloor.$$

Now we are ready to prove Theorem 1.2.9.

Theorem 1.2.9. *Let G be a claw-free graph with maximum degree at most Δ, and let $n \ge 1$ be an integer. Then,*

$$f_G(n) \le \left\lfloor \left(\frac{\Delta}{2}+1\right)(n-1) \right\rfloor + 1.$$

Proof. We argue by induction on n. The case $n=1$ is trivial. Now, assume $n>1$. Let $t = \left\lfloor \left(\frac{\Delta}{2}+1\right)(n-1) \right\rfloor + 1$ and let $J_1,\dots,J_t$ be independent sets of size n in G. Since $t \ge \left\lfloor \left(\frac{\Delta}{2}+1\right)(n-2) \right\rfloor + 1$, then, by the induction hypothesis, there exists a rainbow independent set A of size $n-1$. Without loss of generality, we may assume that $A = \{v_1,\dots,v_{n-1}\}$, where $v_i \in J_i$ for all $i \in [n-1]$.

Let $X = \mathrm{lk}(I_n(G),A)$. By Proposition 4.4.5, X is $\left\lfloor \frac{\Delta}{2}(n-1) \right\rfloor$-collapsible.

The family $\{J_i\}_{n \le i \le t}$ consists of $\left\lfloor \frac{\Delta}{2}(n-1) \right\rfloor + 1$ sets not belonging to X. Thus, by Theorem 1.2.1, there exists a set $R = \{v_n,\dots,v_t\}$, where $v_i \in J_i$ for all $n \le i \le t$, such that $R \notin X$. Therefore, the set $A \cup R$ contains a set I of size n that is independent in G. I is a rainbow independent set of size n in G, as wanted. ☐

Proposition 4.4.6. *Let $G=(V,E)$ be a graph with maximum degree at most Δ. Let $A = \{a_1,a_2\}$ be an independent set of size 2 in G. Assume that there exists an independent set in G of the form $\{a_1,w,w'\}$, where $w,w' \in N_G(a_2)$, or there exists an independent set of the form $\{a_2,v,v'\}$, where $v,v' \in N_G(a_1)$. Then,*

$$C(\mathrm{lk}(I_3(G),A)) \le \begin{cases} \Delta & \text{if } \Delta \text{ is even,} \\ \Delta-1 & \text{if } \Delta \text{ is odd.} \end{cases}$$

Proof. Let $v \in V \setminus (N_G(A) \cup A)$. Then $A \cup \{v\}$ is an independent set of size 3 in G; hence, $v \notin \mathrm{lk}(I_3(G),A)$. So, we may assume without loss of generality that $V = N_G(A) \cup A$.

Let $B = N_G(a_1) \cap N_G(a_2)$ and $U = N_G(A) \setminus B$. Since the maximum degree of a vertex in G is at most Δ, we have

$$|N_G(A)| = |N_G(a_1)| + |N_G(a_2)| - |N_G(a_1) \cap N_G(a_2)| \le 2\Delta - |B|.$$

So, $|U| \le 2\Delta - 2|B|$.

Let $X = \mathrm{lk}(I_3(G),A)$. We will use Lemma 4.2.3 in order to bound the collapsibility number of X:

Write $B = \{u_1,\dots,u_k\}$. Let $\mathcal{P} = \mathcal{P}(\mathrm{lk}(I_3(G),A),B)$ be the family of partitions (B_1,B_2) of B satisfying:

- $B_2 \in X = \mathrm{lk}(I_3(G), A)$.

- For any $u_i \in B_2$, the complex

$$\begin{aligned}&\mathrm{lk}(X[V \setminus \{u_j \in B_1 : j < i\}], \{u_j \in B_2 : j < i\}) \\ &\qquad\qquad = \mathrm{lk}(I_3(G)[V \setminus \{u_j \in B_1 : j < i\}], A \cup \{u_j \in B_2 : j < i\})\end{aligned}$$

  is not a cone over u_i.

Let $(B_1, B_2) \in \mathcal{P}$. Let $G' = G[V \setminus B_1]$, and let

$$Y = \mathrm{lk}(X[V \setminus B_1], B_2) = \mathrm{lk}(I_3(G)[V \setminus B_1], A \cup B_2) = \mathrm{lk}(I_3(G'), A \cup B_2).$$

Note that $B_2 = N_{G'}(a_1) \cap N_{G'}(a_2)$. Also, since A is of size 2, then every vertex in $V \setminus B_1$ is adjacent to at most 2 vertices in A. Therefore, by Lemma 4.4.4, Y is a flag complex.

The vertex set of Y is contained in $U = N_G(A) \setminus B$. So, by Proposition 4.2.2, we obtain

$$C(Y) \leq \frac{|U|}{2} \leq \frac{2\Delta - 2|B|}{2} = \Delta - |B| \leq \Delta - |B_2|. \tag{4.2}$$

Therefore, by Lemma 4.2.3,

$$C(\mathrm{lk}(I_3(G), A)) \leq \Delta.$$

Now, assume Δ is odd. Again, let $(B_1, B_2) \in \mathcal{P}$, and let

$$Y = \mathrm{lk}(I_3(G)[V \setminus B_1], A \cup B_2).$$

If $B_2 \neq B$ then, by (4.2),

$$C(Y) \leq \Delta - |B| \leq \Delta - 1 - |B_2|.$$

Now, assume $B_2 = B$. By the equality case of Proposition 4.2.2, we have $C(Y) \leq \Delta - 1 - |B|$ unless Y contains exactly $2\Delta - 2|B|$ vertices, and its set of missing faces consists of $\Delta - |B| = \Delta - k$ pairwise disjoint sets of size 2. We will show that this case cannot in fact hold:

Assume for contradiction that the equality case holds. Then, Y is a simplicial complex on vertex set $U = U_1 \cup U_2$, where $U_1 = N_G(a_1) \setminus N_G(a_2)$ and $U_2 = N_G(a_2) \setminus N_G(a_1)$, and $|U_1| = |U_2| = \Delta - k$ (see Figure 4.1).

Claim 4.4.7. *Let J be an independent set of size 3 in G. Then J is of one of the following forms:*

- $J = \{a_1, v, w\}$, *where* $v, w \in U_2$,

- $J = \{a_2, v, w\}$, *where* $v, w \in U_1$, *or*

- $J = \{u_i, v, w\}$ *for some* $i \in [k]$, *where* $v, w \in U$.

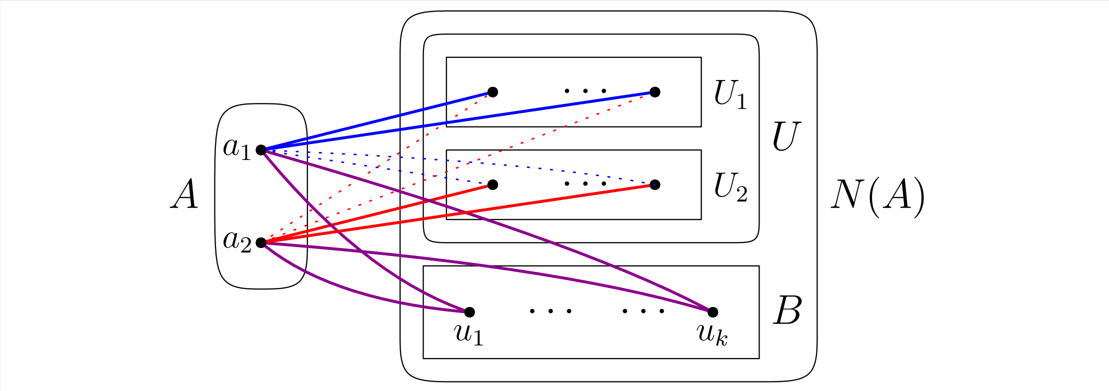

Figure 4.1: The sets A, B, U. The vertices in B are adjacent to both a_1 and a_2. The vertices in U_1 are adjacent to a_1 but not to a_2, while the vertices in U_2 are adjacent to a_2 but not to a_1.

Proof. Since $B_2 = B$ and $(B_1, B_2) \in \mathcal{P}$, we have $B \in \mathrm{lk}(I_3(G), A)$. Thus, any independent set J of size 3 in G contains at least one vertex from U. Also, since Y is a flag complex, at least one vertex in J must belong to $A \cup B$ (otherwise J is a missing face of size 3 in Y).

Note that since $U \subset N_G(A)$, each independent set of size 3 contains at most one of the vertices a_1 or a_2.

Assume that $a_1 \in J$. Then, since all the vertices in $B \cup U_1$ are adjacent to a_1, the two vertices in $J \setminus \{a_1\}$ must belong to U_2, as wanted. Similarly, if $a_2 \in J$, then the two vertices in $J \setminus \{a_2\}$ must belong to U_1.

Now, assume that $a_1, a_2 \notin J$. Then, there exists some $i \in [k]$ such that $u_i \in J$. For all $j \in [k] \setminus \{i\}$, $u_j \notin J$, otherwise the unique vertex v in $J \setminus \{u_i, u_j\}$ does not belong to Y, a contradiction to the assumption that the vertex set of Y is the whole set U. So, the two vertices in $J \setminus \{u_i\}$ must belong to U, as wanted. $\square$

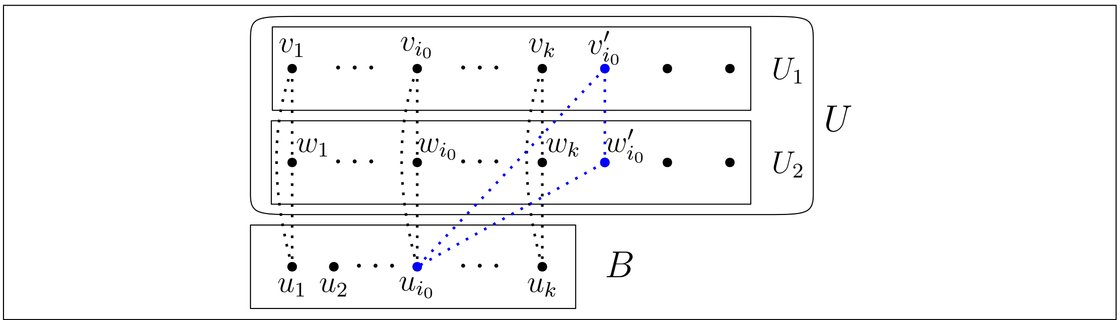

Figure 4.2: The pairs $\{v_i, w_i\}$ $(i = 1, \dots, k)$, and the additional pair $\{v'_{i_0}, w'_{i_0}\}$. For each $i \in [k]$, $\{u_i, v_i, w_i\}$ is independent in G. The set $\{u_{i_0}, v'_{i_0}, w'_{i_0}, \}$ is also independent. On the other hand, $\{u_j, v_i, w_i\}$ is *not* independent for $j < i$.

Claim 4.4.8. *There exist distinct vertices $v_1, \dots, v_k \in U_1$ and $w_1, \dots, w_k \in U_2$ such that:*

- *For all $i \in [k]$, $\{u_i, v_i, w_i\}$ is an independent set in G.*

- *For all $1 \leq j < i \leq k$, $\{u_j, v_i, w_i\}$ is not independent in G.*

Proof. We define the vertices $v_1, \ldots, v_k, w_1, \ldots, w_k$ recursively, as follows. Let $i \in [k]$, and assume that we already defined $v_1, \ldots, v_{i-1}$ and $w_1, \ldots, w_{i-1}$. Since $(B_1, B_2) = (\emptyset, B) \in \mathcal{P}$, then the complex

$$Y' = \mathrm{lk}(I_3(G), A \cup \{u_j \in B : j < i\})$$

is not a cone over u_i. Therefore, there exists a missing face τ of Y' containing u_i. Since τ is a missing face of Y', there exists an independent set J of size 3 in G containing τ. By Claim 4.4.7, J is of the form $J = \{u_i, v_i, w_i\}$, for some $v_i, w_i \in U$.

Note that actually $J = \tau$. Otherwise, assume without loss of generality that $\tau = \{u_i, v_i\}$. Then $w_i \notin Y'$. But then $w_i \notin Y$, a contradiction to the assumption that the vertex set of Y is the whole set U.

If both v_i and w_i belong to U_1, or both of them belong to U_2, then $\{v_i, w_i\} \notin Y'$, a contradiction to $\{u_i, v_i, w_i\}$ being a missing face. So, we may assume that $v_i \in U_1$ and $w_i \in U_2$. Moreover, for all $j < i$, $\{u_j, v_i, w_i\}$ is not independent in G, otherwise $\{v_i, w_i\} \notin Y'$, a contradiction to $\{u_i, v_i, w_i\}$ being a missing face.

The pairs $\{\{v_i, w_i\}\}_{i \in [k]}$ are missing faces of the complex Y. Hence, they must be pairwise disjoint. Thus, the vertices $v_1, \ldots, v_k, w_1, \ldots, w_k$ are all distinct. $\square$

Claim 4.4.9. *There exist some $i_0 \in [k]$ and vertices $v'_{i_0} \in U_1 \setminus \{v_1, \ldots, v_k\}$, $w'_{i_0} \in U_2 \setminus \{w_1, \ldots, w_k\}$ such that $\{u_{i_0}, v'_{i_0}, w'_{i_0}\}$ is independent in G.*

Proof. Recall that, by assumption, the missing faces of Y consist of $\Delta - k$ pairwise disjoint sets of size 2. In particular, each vertex $v \in U$ belongs to exactly one missing face of Y.

Assume for contradiction that the only missing faces of Y of the form $\{v, w\}$, where $v \in U_1$ and $w \in U_2$, are the pairs $\{v_i, w_i\}$, $i \in [k]$, from Claim 4.4.8.

Then, the $\Delta - 2k$ remaining missing faces must be of the form $\{v, w\}$, where $v, w \in U_1$ or $v, w \in U_2$. In particular, the set $U_1 \setminus \{v_1, \ldots, v_k\}$ must be of even size (otherwise, there exists a vertex $v \in U_1 \setminus \{v_1, \ldots, v_k\}$ that does not belong to any missing face of Y, a contradiction). But

$$|U_1 \setminus \{v_1 \ldots, v_k\}| = \Delta - 2k$$

is odd, since Δ is odd.

Therefore, there exists some additional missing face of the form $\{v, w\}$, where $v \in U_1$, $w \in U_2$. That is, there is some $i_0 \in [k]$ such that $\{u_{i_0}, v, w\}$ is independent in G. So, we can choose $v'_{i_0} = v$ and $w'_{i_0} = w$ (see Figure 4.2). $\square$

Claim 4.4.10. $\Delta \geq 2k + 3$.

Proof. By assumption, there exists in G either an independent set of the form $\{a_1, w, w'\}$, where $w, w' \in N_G(a_2)$, or an independent set of the form $\{a_2, v, v'\}$, where $v, v' \in$

$N_G(a_1)$. Assume without loss of generality that the first case holds: there exists an independent set in G of the form $\{a_1, w, w'\}$, where $w, w' \in N_G(a_2)$. Then, the set $\{w, w'\}$ is a missing face in Y. Since the missing faces of Y are all disjoint, the vertices $w_1, \ldots, w_k, w'_{i_0}, w, w' \in U_2$ must be all distinct. Therefore,

$$\Delta - k = |U_2| \geq k + 3.$$

Hence, $\Delta \geq 2k + 3$. $\square$

Let

$$S = \{j \in [k] \setminus \{i_0\} : \{v_{i_0}, u_j\} \notin E \text{ or } \{w_{i_0}, u_j\} \notin E\}.$$

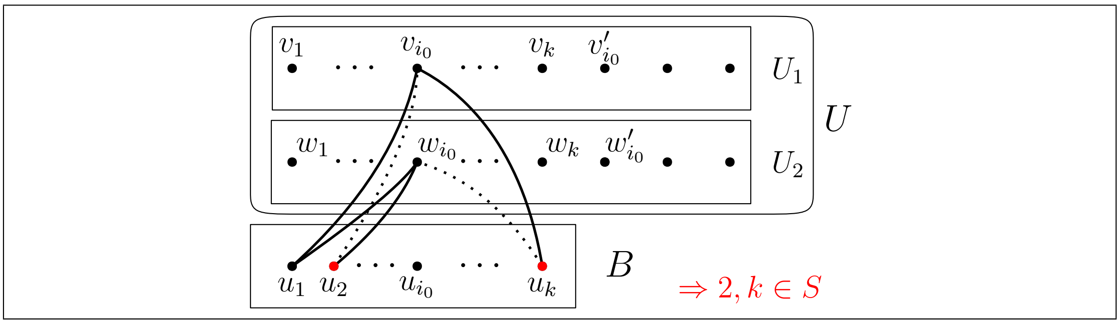

Figure 4.3: The set S consists of the indices $j \in [k] \setminus \{i_0\}$ such that u_j is adjacent in G to at most one of the vertices v_{i_0} or w_{i_0}. For example, in the picture presented here, 2 and k belong to S, but 1 does not.

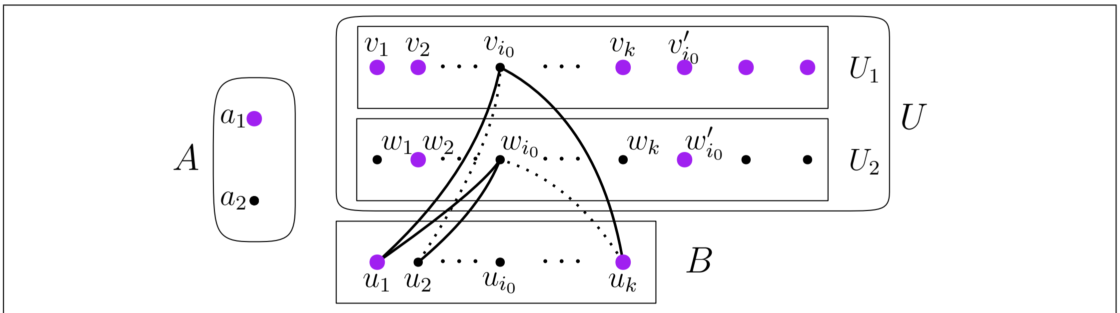

Figure 4.4: The bold purple vertices are the neighbours of v_{i_0} in G. The vertex v_{i_0} is adjacent to a_1, to w'_{i_0} and to all the vertices in U_1 other than itself. In addition, for each $i \in [k]$, it is adjacent to exactly one of the two vertices u_i or w_i. In particular, for indices $j \neq i_0$ that *do not* belong to S, v_{i_0} is adjacent to u_j (and not adjacent to w_j); see Claim 4.4.11.

Claim 4.4.11. *There exists a set N_1 consisting of exactly one vertex from each pair $\{w_j, u_j\}$, for all $j \in S$, such that*

$$N_G(v_{i_0}) = \{a_1\} \cup \{w'_{i_0}\} \cup (U_1 \setminus \{v_{i_0}\}) \cup \{u_j : j \in [k] \setminus (S \cup \{i_0\})\} \cup N_1.$$

In particular, $|N_G(v_{i_0})| = \Delta$.

Similarly, there exists a set N_2 consisting of exactly one vertex from each pair $\{v_j, u_j\}$, for all $j \in S$, such that

$$N_G(w_{i_0}) = \{a_2\} \cup \{v'_{i_0}\} \cup (U_2 \setminus \{w_{i_0}\}) \cup \{u_j : j \in [k] \setminus (S \cup \{i_0\})\} \cup N_2.$$

And, in particular, $|N_G(w_{i_0})| = \Delta$.

Proof. We prove the claim for v_{i_0}. The proof for w_{i_0} is identical.

First, since $v_{i_0} \in U_1$, then a_1 is adjacent to v_{i_0}. Also, for every $v_{i_0} \neq v \in U_1$, v is adjacent to v_{i_0}, since otherwise the set $\{a_2, v_{i_0}, v\}$ is independent in G, but then the set $\{v, v_{i_0}\}$ is a missing face of Y that intersects the missing face $\{v_{i_0}, w_{i_0}\}$, a contradiction to the assumption that the missing faces are pairwise disjoint.

The vertex w'_{i_0} must also be adjacent to v_{i_0}, otherwise $\{u_{i_0}, v_{i_0}, w'_{i_0}\}$ is an independent set in G. But then, $\{v_{i_0}, w'_{i_0}\}$ is a missing face of Y intersecting the missing face $\{v_{i_0}, w_{i_0}\}$, again a contradiction.

By the definition of S, v_{i_0} is adjacent to u_j for all $j \in [k] \setminus (S \cup \{i_0\})$.

Finally, let $j \in S$. If $\{v_{i_0}, u_j\} \notin E$ and $\{v_{i_0}, w_j\} \notin E$, then $\{v_{i_0}, u_j, w_j\}$ is independent in G; therefore, $\{v_{i_0}, w_j\}$ is a missing face of Y, a contradiction. So, v_{i_0} is adjacent to either u_j or w_j. Let $S' = \{j \in S : \{u_j, v_{i_0}\} \in E\}$. Let

$$N_1 = \{u_j : j \in S'\} \cup \{w_j : j \in S \setminus S'\}.$$

Then, $N_1 \subset N_G(v_{i_0})$. Let

$$N = \{a_1\} \cup \{w'_{i_0}\} \cup (U_1 \setminus \{v_{i_0}\}) \cup \{u_j : j \in [k] \setminus (S \cup \{i_0\})\} \cup N_1.$$

We showed that $N \subset N_G(v_{i_0})$. Note that

$$|N| = 1 + 1 + (\Delta - k - 1) + (k - |S| - 1) + |S| = \Delta.$$

Since the maximal degree of a vertex in G is at most Δ, then we must have $N_G(v_{i_0}) = N$, as wanted. ☐

Claim 4.4.12. *For all $j \in [k] \setminus \{i_0\}$, u_{i_0} is adjacent in G to at least one of the vertices v_j or w_j.*

Proof. Let $j \neq i_0$. Assume for contradiction that u_{i_0} is not adjacent to any of the two vertices v_j and w_j. Then $\{u_{i_0}, v_j, w_j\}$ is independent in G. So, by Claim 4.4.8, we must have $i_0 > j$. Moreover, either $\{v_{i_0}, u_j\} \in E$ or $\{w_{i_0}, u_j\} \in E$ (otherwise $\{u_j, v_{i_0}, w_{i_0}\}$ is independent in G, a contradiction to Claim 4.4.8). Assume without loss of generality that $\{v_{i_0}, u_j\} \in E$. The vertex v_{i_0} must be also adjacent to w_j, since otherwise the set $\{u_{i_0}, v_{i_0}, w_j\}$ is independent in G. But then $\{v_{i_0}, w_j\}$ is a missing face of Y, a contradiction to the assumption that the missing faces are pairwise disjoint.

But, by Claim 4.4.11, the set of neighbors of v_{i_0} in G, $N_G(v_{i_0})$, contains at most one of the vertices u_j or w_j, a contradiction.

So, u_{i_0} must be adjacent in G to at least one of the vertices v_j or w_j. $\square$

Claim 4.4.13. *There is some vertex*

$$w \in U \setminus (\{v_j, w_j : j \in S\} \cup \{v_{i_0}, w_{i_0}, v'_{i_0}, w'_{i_0}\})$$

such that $\{u_{i_0}, w\} \notin E$.

Proof. Let $U' = U \setminus (\{v_j, w_j : j \in S\} \cup \{v_{i_0}, w_{i_0}, v'_{i_0}, w'_{i_0}\})$. The vertex u_{i_0} is adjacent in G to both a_1 and a_2 (since $u_{i_0} \in B = N_G(a_1) \cap N_G(a_2)$). Also, by Claim 4.4.12, it is adjacent to at least $|S|$ vertices from the set $\{v_j, w_j : j \in S\}$. By the definition of S, for each $j \in S$, u_j is not adjacent to one of the vertices v_{i_0} or w_{i_0}. Thus u_{i_0} must be adjacent in G to u_j (otherwise, one of the sets $\{u_j, u_{i_0}, v_{i_0}\}$ or $\{u_j, u_{i_0}, w_{i_0}\}$ is independent in G, in contradiction to Claim 4.4.7).

So, u_{i_0} is adjacent to at least $2|S| + 2$ vertices outside of U'. Since the degree of u_{i_0} is at most Δ, u_{i_0} is adjacent to at most $\Delta - 2 - 2|S|$ vertices in U'.

But $|U'| = |U| - 2|S| - 4 = 2\Delta - 2k - 2|S| - 4$. So, u_{i_0} is not adjacent to at least $\Delta - 2k - 2$ vertices in U'. By Claim 4.4.10, $\Delta \geq 2k + 3$. Therefore, u_{i_0} is not adjacent to at least one vertex $w \in U'$. $\square$

Assume without loss of generality that the vertex w from Claim 4.4.13 belongs to U_2. If $\{v_{i_0}, w\} \notin E$, then $\{u_{i_0}, v_{i_0}, w\}$ is independent in G. But then, $\{v_{i_0}, w\}$ is a missing face of Y intersecting $\{v_{i_0}, w_{i_0}\}$, a contradiction to the assumption that all the missing faces are disjoint. So, $w \in N_G(v_{i_0})$. But this is a contradiction to Claim 4.4.11.

Therefore, $C(Y) \leq (\Delta - 1) - |B|$; so, by Lemma 4.2.3, $\mathrm{lk}(I_3(G), A)$ is $(\Delta - 1)$-collapsible.

Proposition 4.4.14. *Let* $\Delta \geq 2$. *Let* $G = (V, E)$ *be a graph with maximum degree at most* Δ, *and let* $a_1 \in V$. *Then,*

$$C(\mathrm{lk}(I_3(G), a_1)) \leq \begin{cases} \Delta + 1 & \text{if } \Delta \text{ is even,} \\ \Delta & \text{if } \Delta \text{ is odd.} \end{cases}$$

Proof. Let $d = \Delta + 2$ if Δ is even, and $d = \Delta + 1$ if Δ is odd. Let V' be the vertex set of $\mathrm{lk}(I_3(G), a_1)$. We argue by induction on $|V'|$. If $|V'| \leq \Delta$, then by Proposition 4.2.2,

$$C(\mathrm{lk}(I_3(G), a_1)) \leq \frac{2|V'|}{3} \leq \frac{2\Delta}{3} \leq d - 1,$$

as wanted. Otherwise, let $|V'| > \Delta$. We will show that there exists a vertex $a_2 \notin N_G(a_1)$ such that $C(\mathrm{lk}(I_3(G), \{a_1, a_2\})) \leq d - 2$. We divide into three different cases:

Case 1: There exists an independent set in G of the form $\{u, v, a_2\}$, where $u, v \in N_G(a_1)$ and $a_2 \notin N_G(a_1)$. Then, by Proposition 4.4.6, we have

$$C(\mathrm{lk}(I_3(G), \{a_1, a_2\})) \leq d - 2.$$

Case 2: There exists a triple $\{u, v, a_2\} \subset V'$ such that $u, v, a_2 \notin N_G(a_1)$, $\{u, v\} \notin E$, $\{u, a_2\} \in E$ and $\{v, a_2\} \in E$. Then, $\{a_1, u, v\}$ is an independent set in G, and $u, v \in N_G(a_2)$. Thus, by Proposition 4.4.6,

$$C(\mathrm{lk}(I_3(G), \{a_1, a_2\})) \leq d - 2.$$

Case 3: Assume none of the two first cases holds. Since $|V'| > \Delta$, there exists a vertex $a_2 \in \mathrm{lk}(I_3(G), a_1)$ such that $a_2 \notin N_G(a_1)$ (otherwise $\deg_G(a_1) = |N_G(a_1)| > \Delta$, a contradiction).

Note that the set $N_G(a_2) \backslash N_G(a_1)$ is contained in the vertex set of $\mathrm{lk}(I_3(G), \{a_1, a_2\})$. We will show that there are no missing faces of $\mathrm{lk}(I_3(G), \{a_1, a_2\})$ contaning vertices from $N_G(a_2) \setminus N_G(a_1)$:

Assume for contradiction that there exists a missing face τ of the complex $\mathrm{lk}(I_3(G), \{a_1, a_2\})$ that contains a vertex $w \in N_G(a_2) \setminus N_G(a_1)$. First, assume that $\tau = \{u, v, w\}$ is an independent set of size 3. Then, both u and v must belong to $N_G(a_1)$. Otherwise, assume without loss of generality that $v \notin N_G(a_1)$. Then $\{w, v, a_1\}$ is an independent set in G, and therefore $\{v, w\} \notin \mathrm{lk}(I_3(G), \{a_1, a_2\})$, a contradiction to τ being a missing face. But then, the existence of the independent set $\{u, v, w\}$ is a contradiction to the assumption that Case 1 does not hold.

Now, assume $\tau = \{v, w\}$ is of size 2. Then there exists an independent set J of size 3 such that $\tau \subset J \subset \tau \cup \{a_1, a_2\}$. Since $w \in N_G(a_2)$, we must have $J = \{a_1, v, w\}$. In particular $v \notin N_G(a_1)$. So, we must have $v \in N_G(a_2)$. But then, the triple $\{a_2, v, w\}$ satisfies $a_2, v, w \notin N_G(a_1)$, $\{v, w\} \notin E$, $\{a_2, v\} \in E$ and $\{a_2, w\} \in E$. This is a contradiction to the assumption that Case 2 does not hold.

Therefore, there are no missing faces of $\mathrm{lk}(I_3(G), \{a_1, a_2\})$ containing vertices in $N_G(a_2) \setminus N_G(a_1)$. Let $U = N_G(a_1) \cup \{a_1, a_2\}$. Then, we have

$$\mathrm{lk}(I_3(G), \{a_1, a_2\}) = 2^{N_G(a_2) \backslash N_G(a_1)} * \mathrm{lk}(I_3(G[U]), \{a_1, a_2\}).$$

So, by Lemma 2.3.8, we have

$$C(\mathrm{lk}(I_3(G), \{a_1, a_2\})) = C(\mathrm{lk}(I_3(G[U]), \{a_1, a_2\})).$$

By Proposition 4.2.2, we obtain

$$C(\mathrm{lk}(I_3(G), \{a_1, a_2\})) \leq \frac{2|N_G(a_1)|}{3} \leq \frac{2\Delta}{3}.$$

Note that $\frac{2\Delta}{3} \leq \Delta$, and $\frac{2\Delta}{3} \leq \Delta - 1$ for $\Delta \geq 3$. Hence, we obtain

$$C(\mathrm{lk}(I_3(G), \{a_1, a_2\})) \leq \frac{2\Delta}{3} \leq d - 2$$

for all $\Delta \geq 2$.

For any of the three cases we have $C(\mathrm{lk}(I_3(G \setminus a_2), a_1)) \leq d - 1$ by the induction hypothesis. Also, we showed that $C(\mathrm{lk}(I_3(G), \{a_1, a_2\})) \leq d - 2$ in all three cases. So, by Lemma 2.3.14,

$$\begin{aligned} C(\mathrm{lk}(I_3(G), a_1)) &\leq \max\{C(\mathrm{lk}(I_3(G \setminus a_2), a_1)), C(\mathrm{lk}(I_3(G), \{a_1, a_2\})) + 1\} \\ &\leq d - 1, \end{aligned}$$

as wanted. □

Theorem 1.2.8. *Let $G = (V, E)$ be a graph with maximum degree Δ. Then,*

$$C(I_3(G)) \leq \begin{cases} \Delta + 2 & \text{if } \Delta \text{ is even,} \\ \Delta + 1 & \text{if } \Delta \text{ is odd.} \end{cases}$$

Proof. For $\Delta = 1$ the claim holds by Theorem 1.2.6. Assume $\Delta \geq 2$.

Let $d = \Delta + 2$ if Δ is even, and $d = \Delta + 1$ if Δ is odd. We argue by induction on $|V|$. If $|V| = 0$ the claim holds trivially. Otherwise, let $a_1 \in V$. By the induction hypothesis, $C(I_3(G \setminus a_1)) \leq d$. Also, by Proposition 4.4.14, $C(\mathrm{lk}(I_3(G), a_1)) \leq d - 1$. So, by Lemma 2.3.14,

$$C(I_3(G)) \leq \max\{C(I_3(G \setminus a_1)), C(\mathrm{lk}(I_3(G), a_1)) + 1\} \leq d,$$

as wanted. □

4.5 Lower bounds on Leray numbers

In this section we present some examples establishing the sharpness of our different bounds on the collapsibility of $I_n(G)$. Also, we present a family of counterexamples to the conjectural bound presented in Question 1.2.4, in the case of graphs with maximum degree at most 3.

4.5.1 Extremal examples

Let n be an integer, and k be an even integer. Let $G_{k,n}$ be the graph obtained from a cycle of length $\left(\frac{k}{2} + 1\right) n$ by adding all edges connecting any two vertices of distance at most $\frac{k}{2}$ in the cycle. Note that $G_{k,n}$ is a k-regular graph, i.e. every vertex has degree exactly k. Moreover, $G_{k,n}$ is claw-free.

In [ABKK19] it is shown that $f_{G_{k,n}}(n) \geq \left(\frac{k}{2}+1\right)(n-1)+1$. In particular, this shows the tightness of Theorem 1.2.9, in the case that k is even. Moreover, by Proposition 1.2.2, we obtain

$$C(I_n(G_{k,n})) \geq f_{G_{k,n}}(n) - 1 \geq \left(\frac{k}{2}+1\right)(n-1).$$

This shows that the bound in Question 1.2.4, whenever it holds, is tight. A different way to show this is as follows.

Proposition 4.5.1.

$$\tilde{H}_i(I_n(G_{k,n});\mathbb{F}) = \begin{cases} \mathbb{F} & \textit{if } i = \left(\frac{k}{2}+1\right)(n-1)-1, \\ 0 & \textit{otherwise.} \end{cases}$$

In particular, $L(I_n(G_{k,n})) \geq \left(\frac{k}{2}+1\right)(n-1)$.

Proof. Let $t = \frac{k}{2}+1$. It is easy to check that there are precisely t independent sets of size n in $G_{k,n}$, and they are pairwise disjoint. Therefore, $I_n(G_{k,n})$ can be described as the join of t disjoint copies of the boundary of an $(n-1)$-dimensional simplex. Since the boundary of an $(n-1)$-dimensional simplex is an $(n-2)$-dimensional sphere, we obtain by Theorem 2.2.3:

$$\tilde{H}_i(I_n(G_{k,n});\mathbb{F}) = \begin{cases} \mathbb{F} & \text{if } i = t(n-1)-1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $L(I_n(G)) \geq t(n-1) = \left(\frac{k}{2}+1\right)(n-1)$. $\square$

Therefore, we obtain

$$C(I_n(G_{k,n})) \geq L(I_n(G_{k,n})) \geq \left(\frac{k}{2}+1\right)(n-1).$$

On the other hand, $I_n(G_{k,n})$ is a $\left(\left(\frac{k}{2}+1\right)(n-1)-1\right)$-dimensional complex, and therefore it is $\left(\frac{k}{2}+1\right)(n-1)$-collapsible. So,

$$C(I_n(G_{k,n})) = \left(\frac{k}{2}+1\right)(n-1).$$

Proposition 4.5.1 also shows that the bound in Proposition 4.4.1 is tight, since $G_{2k-2,n}$ is a k-partite graph with $C(I_n(G_{2k-2,n})) = k(n-1)$. Another such extremal example is the complete k-partite graph $K_{n,\ldots,n}$. In this case, it easy to see that $I_n(K_{n,\ldots,n}) \cong I_n(G_{2k-2,n})$.

4.5.2 A negative answer to Question 1.2.4

Let $G = (V, E)$ be the dodecahedral graph. It will be convenient to represent G as a generalized Petersen graph (see [Wat69]), as follows:

$$V = \{a_1, \ldots, a_{10}, b_1, \ldots, b_{10}\}$$

and

$$E = \{\{a_i, b_i\},\ \{a_i, a_{i+1}\},\ \{b_i, b_{i+2}\} : i = 1, 2, \ldots, 10\},$$

where the indices are taken modulo 10.

Every vertex in G is adjacent to exactly 3 vertices; that is, G is 3-regular. The maximum independent sets in G are the sets

$$I_i = \{a_i, a_{i+2}, a_{i+5}, a_{i+7}, b_{i-2}, b_{i-1}, b_{i+3}, b_{i+4}\}$$

for $i = 1, \ldots, 5$ (also here, the indices are to be taken modulo 10). In particular, $\alpha(G) = 8$.

Proposition 4.5.2. *Let $G = (V, E)$ be the dodecahedral graph. Then,*

$$\tilde{H}_i(I_8(G); \mathbb{F}) = \begin{cases} \mathbb{F}^4 & \text{if } i = 15, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $L(I_8(G)) \geq 16$.

Proof. Let $\mathcal{F} = \{V \setminus I_1, V \setminus I_2, \ldots, V \setminus I_5\}$. The family $\mathcal{F}$ is the set of maximal faces of $I_8(G)^V$. So, by the Nerve Theorem (Corollary 2.2.5),

$$\tilde{H}_i(N(\mathcal{F}); \mathbb{F}) \cong \tilde{H}_i(I_8(G)^V; \mathbb{F})$$

for all $i \geq -1$. So, by Alexander duality (Corollary 2.2.11),

$$\tilde{H}_i(N(\mathcal{F}); \mathbb{F}) = \tilde{H}_{|V|-i-3}(I_8(G); \mathbb{F}) = \tilde{H}_{17-i}(I_8(G); \mathbb{F}) \tag{4.3}$$

for all $-1 \leq i \leq |V| - 2 = 18$. We have

$$N(\mathcal{F}) = \left\{ A \subset [5] : \bigcap_{i \in A} V \setminus I_i \neq \emptyset \right\} = \left\{ A \subset [5] : \bigcup_{i \in A} I_i \neq V \right\}.$$

It is easy to check that $N(\mathcal{F})$ is the complete 2-dimensional complex on 5 vertices. So,

$$\tilde{H}_i(N(\mathcal{F}); \mathbb{F}) = \begin{cases} \mathbb{F}^4 & \text{if } i = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, by (4.3),

$$\tilde{H}_i(I_8(G);\mathbb{F}) = \begin{cases} \mathbb{F}^4 & \text{if } i = 15, \\ 0 & \text{otherwise,} \end{cases}$$

as wanted. $\square$

We obtain $C(I_8(G)) \geq L(I_8(G)) \geq 16 > 2 \cdot (8-1) = 14$. Therefore, $I_8(G)$ does not satisfy the bound in Question 1.2.4. However, this is not a counterexample for Conjecture 1.2.3. Indeed, it is not hard to check that $f_G(8) = 11$.

4.5.3 Leray number of the disjoint union of graphs

The following result will help us in constructing more examples of complexes that do not satisfy the bound in Question 1.2.4:

Theorem 4.5.3. *Let G be the disjoint union of the graphs $G_1, \dots, G_m$. For $i \in [m]$, let $t_i = \alpha(G_i)$ and let $\ell_i = L(I_{t_i}(G_i))$. Let $t = \sum_{i=1}^m t_i = \alpha(G)$ and $\ell = L(I_t(G))$. Then,*

$$\ell = \sum_{i=1}^m \ell_i + m - 1.$$

The proof relies on the following result.

Proposition 4.5.4. *Let G be the disjoint union of the graphs $G_1, \dots, G_m$. For $i \in [m]$, let $t_i = \alpha(G_i)$. Let $t = \sum_{i=1}^m t_i = \alpha(G)$. Then, $\tilde{H}_k(I_t(G);\mathbb{F}) = 0$ if and only if for every choice of integers $k_1, \dots, k_m$ satisfying $\sum_{i=1}^m k_i = k - 2m + 2$, $\tilde{H}_{k_i}(I_{t_i}(G_i);\mathbb{F}) = 0$ for all $i \in [m]$.*

Proof. For all $i \in [m]$, let V_i be the vertex set of G_i, and let $V = \bigcup_{i=1}^m V_i$ be the vertex set of G. Let $N_i = |V_i|$ for all $i \in [m]$, and $N = |V| = \sum_{i=1}^m N_i$.

A set $U \subset V$ contains an independent set of size t in G if and only if $U \cap V_i$ contains an independent set of size t_i in G_i for all $i \in [m]$. That is, $U \notin I_t(G)$ if and only if $U \cap V_i \notin I_{t_i}(G_i)$ for all $i \in [m]$. Equivalently, a set $W \subset V$ belongs to $I_t(G)^V$ if and only if $W \cap V_i \in I_{t_i}(G_i)^V$ for all $i \in [m]$. Thus, we have

$$I_t(G)^V = I_{t_1}(G_1)^V * \cdots * I_{t_m}(G_m)^V.$$

Note that for every $i \in [m]$, $V_i \notin I_{t_i}(G_i)$ (since G_i contains an independent set of size $t_i = \alpha(G_i)$). Similarly, $V \notin I_t(G)$. So, by Alexander duality (Corollary 2.2.11), we have

$$\tilde{H}_j(I_{t_i}(G_i)^V;\mathbb{F}) = \tilde{H}_{N_i-j-3}(I_{t_i}(G_i);\mathbb{F})$$

for all $i \in [m]$ and $-1 \leq j \leq |V_i| - 2$, and

$$\tilde{H}_j(I_t(G)^V;\mathbb{F}) = \tilde{H}_{N-j-3}(I_t(G);\mathbb{F})$$

for all $-1 \leq j \leq |V| - 2$.

Therefore, by Theorem 2.2.3, we obtain

$$\begin{aligned}
&\tilde{H}_{N-j-3}(I_t(G);\mathbb{F}) = \tilde{H}_j(I_t(G)^V;\mathbb{F}) \\
&= \bigoplus_{j_1+\cdots+j_m=j-m+1} \tilde{H}_{j_1}(I_{t_1}(G_1)^V;\mathbb{F}) \otimes \cdots \otimes \tilde{H}_{j_m}(I_{t_m}(G_m)^V;\mathbb{F}) \\
&= \bigoplus_{j_1+\cdots+j_m=j-m+1} \tilde{H}_{N_1-j_1-3}(I_{t_1}(G_1);\mathbb{F}) \otimes \cdots \otimes \tilde{H}_{N_m-j_m-3}(I_{t_m}(G_m);\mathbb{F}).
\end{aligned}$$

Setting $k = N - j - 3$ and $k_i = N_i - j_i - 3$ for all $i \in [m]$, we obtain

$$\tilde{H}_k(I_t(G);\mathbb{F}) = \bigoplus_{k_1+\cdots+k_m=k-2m+2} \tilde{H}_{k_1}(I_{t_1}(G_1);\mathbb{F}) \otimes \cdots \otimes \tilde{H}_{k_m}(I_{t_m}(G_m);\mathbb{F}).$$

In particular, $\tilde{H}_k(I_t(G);\mathbb{F}) = 0$ if and only if for every choice of $k_1, \ldots, k_m$ satisfying $\sum_{i=1}^m k_i = k - 2m + 2$, $\tilde{H}_{k_i}(I_{t_i}(G_i);\mathbb{F}) = 0$ for all $i \in [m]$.

Proof of Theorem 4.5.3. For all $i \in [m]$, let V_i be the vertex set of G_i, and let $V = \bigcup_{i=1}^m V_i$ be the vertex set of G.

Since $L(I_t(G)) = \ell$, there exists a set $U \subset V$ such that

$$\tilde{H}_{\ell-1}(I_t(G[U]);\mathbb{F}) \neq 0.$$

Let $G' = G[U]$ and $G'_i = G_i[U \cap V_i]$ for all $i \in [m]$. Note that $I_t(G')$ is not the complete complex, since it has non-trivial homology; hence, $\alpha(G') = t$. Since G' is the disjoint union of the graphs $G'_1, \ldots, G'_m$, we must have $\alpha(G'_i) = t_i$ for all $i \in [m]$. By Proposition 4.5.4, there exists $k_1, \ldots, k_m$ satisfying $\sum_{i=1}^m k_i = \ell - 2m + 1$ such that

$$\tilde{H}_{k_i}(I_{t_i}(G'_i);\mathbb{F}) \neq 0.$$

In particular, $\ell_i = L(I_{t_i}(G_i)) \geq k_i + 1$. Summing over all $i \in [m]$, we obtain

$$\sum_{i=1}^m \ell_i \geq \sum_{i=1}^m k_i + m = \ell - m + 1.$$

Now, let $i \in [m]$. Since $\ell_i = L(I_{t_i}(G_i))$, there exists a set $U_i \subset V_i$ such that

$$\tilde{H}_{\ell_i-1}(I_{t_i}(G_i[U_i]);\mathbb{F}) \neq 0.$$

Let $G'_i = G_i[U_i]$. Note that $I_{t_i}(G'_i)$ is not the complete complex, since it has non-trivial homology. Therefore, $\alpha(G'_i) = t_i$. Let $U = U_1 \cup \cdots \cup U_m$, and let $G' = G[U]$. Then, G' is the disjoint union of $G'_1, \ldots, G'_m$. By Proposition 4.5.4, we have

$$\tilde{H}_{\sum_{i=1}^m (\ell_i-1)+2m-2}(I_t(G');\mathbb{F}) = \tilde{H}_{\sum_{i=1}^m \ell_i+m-2}(I_t(G');\mathbb{F}) \neq 0.$$

Thus, $\ell = L(I_t(G)) \geq \sum_{i=1}^m \ell_i + m - 1$.

Corollary 4.5.5. *Let G_k be the union of k disjoint copies of the dodecahedral graph. Then,*

$$L(I_{8k}(G_k)) \geq 17k - 1.$$

Proof. Let $H_1, \dots, H_k$ be k disjoint copies of the dodecahedral graph. Then, by Propositions 4.5.2 and 4.5.3, we obtain

$$\begin{aligned} L(I_{8k}(G_k)) &= L(I_{8k}(H_1 \cup H_2 \cup \cdots \cup H_k)) \\ &= \sum_{i=1}^{k} L(I_8(H_i)) + k - 1 \geq 16k + k - 1 = 17k - 1. \end{aligned}$$

Note that the graphs G_k are 3-regular, and

$$\frac{L(I_{8k}(G_k))}{8k-1} \geq \frac{17k-1}{8k-1} > 2\frac{1}{8} > 2.$$

Thus, the complexes $I_{8k}(G_k)$ do not satisfy the bound in Question 1.2.4.

Note that the graphs G_k are *not* counterexamples for Conjecture 1.2.3. This can be shown by the following observation.

Proposition 4.5.6. *Let G be the disjoint union of two graphs G_1 and G_2 with $\alpha(G_1) = t_1$ and $\alpha(G_2) = t_2$. Then,*

$$f_G(t_1 + t_2) \leq \max\{f_{G_1}(t_1), f_{G_2}(t_2) + t_1\}.$$

Proof. Let V_1 and V_2 denote the vertex sets of G_1 and G_2 respectively. Let $t = \max\{f_{G_1}(t_1), f_{G_2}(t_2) + t_1\}$.

Let $\mathcal{A} = \{A_1, \dots, A_t\}$ be a family of independent sets of size $t_1 + t_2$ in G. Note that any independent set of size $t_1 + t_2 = \alpha(G)$ in G has t_1 vertices in V_1 and t_2 vertices in V_2.

Thus, $A_1 \cap V_1, A_2 \cap V_1, \dots, A_t \cap V_1$ is a family of $t \geq f_{G_1}(t_1)$ independent sets of size t_1 in G_1. Hence, it contains a rainbow independent set R_1 of size t_1. Without loss of generality, we may assume that $R_1 = \{a_{t-t_1+1}, \dots, a_t\}$, where $a_i \in A_i$ for all $i \in \{t - t_1 + 1, \dots, t\}$.

The family $A_1 \cap V_2, A_2 \cap V_2, \dots, A_{t-t_1} \cap V_2$ is a family of $t - t_1 \geq f_{G_2}(t_2)$ independent sets of size t_2 in G_2; therefore, it contains a rainbow independent set R_2 of size t_2.

Then, the set $R_1 \cup R_2$ is a rainbow independent set of size $t_1 + t_2$ in G with respect to $\mathcal{A}$, as wanted. □

Applying Proposition 4.5.6 repeatedly, we obtain that $f_{G_k}(8k) \leq 8k + 3 < 16k - 1$.

Chapter 5

Leray numbers of tolerance complexes

This chapter is organized as follows. In Section 5.1 we present some auxiliary topological results that we will use later. In Section 5.2 we prove our main result, Theorem 1.3.5. In Section 5.3 we prove Theorem 1.3.6 about the Leray number of the 1-tolerance complex of a 2-collapsible complex. In Section 5.4 we describe Montejano and Oliveros' example of a d-representable complex whose 1-tolerance complex is not $\left(\left\lfloor \left(\frac{d+3}{2}\right)^2 \right\rfloor - 2\right)$-Leray.

This chapter is based on joint work with Minki Kim.

5.1 Some topological preliminaries

In this section we prove some auxiliary results that we will later need. Let $\mathbb{F}$ be a field.

Lemma 5.1.1. *Let X be a simplicial complex on vertex set V, and $Y \subset X$ a subcomplex. Assume that there is some $\sigma \in X$ and subcomplexes $W \subset Z \subset X[V \setminus \sigma]$ such that*

$$X \setminus Y = \{\eta \cup \sigma : \eta \in Z \setminus W\}.$$

Then,

$$H_k\left(X, Y; \mathbb{F}\right) \cong H_{k-|\sigma|}\left(Z, W; \mathbb{F}\right)$$

for all k.

Proof. For all k, let $\phi_k : C_k(X, Y; \mathbb{F}) \to C_{k-|\sigma|}(Z, W; \mathbb{F})$ be defined by

$$\phi_k(\eta \cup \sigma) = \eta$$

and extended linearly. Note that the maps ϕ_k are linear isomorphisms. Denote by ∂_k the boundary operator of $C_k(X, Y; \mathbb{F})$ and by ∂'_k the boundary operator of $C_k(Z, W; \mathbb{F})$. We are left to show that ϕ is a chain map. That is, for any $\eta \in Z(k) \setminus W(k)$, we want to show that

$$\phi_{k+|\sigma|-1}(\partial_{k+|\sigma|}(\eta \cup \sigma)) = \partial'_k(\phi_{k+|\sigma|}(\eta \cup \sigma)).$$

Let $\eta = \{u_0, \ldots, u_k\}$. For any $i \in \{0, \ldots, k\}$, let $\eta_i = \{u_0, \ldots, u_{i-1}, u_{i+1}, \ldots, u_k\}$. Then, since any subset of $\eta \cup \sigma$ belonging to $X \setminus Y$ must contain σ, we have

$$\partial_{k+|\sigma|}(\eta \cup \sigma) = \sum_{\substack{i \in \{0,\ldots,k\}: \\ \eta_i \cup \sigma \notin Y}} (-1)^i \eta_i \cup \sigma = \sum_{\substack{i \in \{0,\ldots,k\}: \\ \eta_i \notin W}} (-1)^i \eta_i \cup \sigma.$$

Hence,

$$\phi_{k+|\sigma|-1}(\partial_{k+|\sigma|}(\sigma \cup \eta)) = \sum_{\substack{i \in \{0,\ldots,k\}: \\ \eta_i \notin W}} (-1)^i \eta_i = \partial'_k(\eta) = \partial'_k(\phi_{k+|\sigma|}(\eta \cup \sigma)).$$

So $C_k(X, Y; \mathbb{F})$ and $C_{k-|\sigma|}(Z, W; \mathbb{F})$ are isomorphic as chain complexes, and in particular have isomorphic homology groups. □

Lemma 5.1.2. *Let $X_1, \ldots, X_m$ be simplicial complexes, and let $X = \cup_{i=1}^m X_i$. If for all $I \subset [m]$ of size at least 2, the complex $\cap_{i \in I} X_i$ is non-empty and acyclic, then*

$$\tilde{H}_k(X; \mathbb{F}) \cong \bigoplus_{i=1}^m \tilde{H}_k(X_i; \mathbb{F}).$$

for all $k \geq -1$.

Proof. We argue by induction on m. For $m = 1$ the claim is trivial. Assume $m > 1$. Since $\cap_{i \in I} X_i$ is non-empty and acyclic for every $I \subset [m-1]$, we obtain, by the induction hypothesis,

$$\tilde{H}_k\left(\cup_{i=1}^{m-1} X_i; \mathbb{F}\right) \cong \bigoplus_{i=1}^{m-1} \tilde{H}_k(X_i; \mathbb{F})$$

for all $k \geq -1$.

Since $X = \left(\cup_{i=1}^{m-1} X_i\right) \cup X_m$, we have by Mayer-Vietoris (Theorem 2.2.1) a long exact sequence

$$\cdots \to \tilde{H}_k\left(\cup_{i=1}^{m-1}(X_i \cap X_m); \mathbb{F}\right) \to \bigoplus_{i=1}^m \tilde{H}_k(X_i; \mathbb{F}) \to$$
$$\to \tilde{H}_k(X; \mathbb{F}) \to \tilde{H}_{k-1}\left(\cup_{i=1}^{m-1}(X_i \cap X_m); \mathbb{F}\right) \to \cdots$$

Hence, it is enough to show that

$$\tilde{H}_k\left(\cup_{i=1}^{m-1}(X_i \cap X_m); \mathbb{F}\right) = 0$$

for all $k \geq -1$.

By the assumptions of this lemma, the nerve $N = N(\{X_i \cap X_m\}_{i=1}^{m-1})$ is the complete complex on vertex set $[m-1]$. Moreover, for all $I \subset [m-1]$, the complex

$$\cap_{i\in I}(X_i \cap X_m) = \cap_{i\in I\cup\{m\}} X_i$$

is acyclic. Therefore, by the Nerve Theorem (Theorem 2.2.4), we obtain

$$\tilde{H}_k \left(\cup_{i=1}^{m-1}(X_i \cap X_m); \mathbb{F}\right) \cong \tilde{H}_k (N; \mathbb{F}) = 0$$

for all $k \geq -1$. Thus,

$$\tilde{H}_k (X; \mathbb{F}) \cong \bigoplus_{i=1}^{m} \tilde{H}_k (X_i; \mathbb{F})$$

for all $k \geq -1$, as wanted. □

Remark. We can give a shorter proof of Lemma 5.1.2 by applying the stronger version of the Nerve Theorem, Theorem 2.A.2: By the assumption of the Lemma, the nerve $N = N(\{X_1, \ldots, X_m\})$ is the complete complex on vertex set $[m]$, and in particular is acyclic. By Theorem 2.A.2, we have a long exact sequence

$$\cdots \to \tilde{H}_{k+1} (N; \mathbb{F}) \to \bigoplus_{i=1}^{m} \tilde{H}_k (X_i; \mathbb{F}) \to \tilde{H}_k (X; \mathbb{F}) \to \tilde{H}_k (N; \mathbb{F}) \to \cdots$$

Therefore, we obtain $\tilde{H}_k (X; \mathbb{F}) \cong \bigoplus_{i=1}^{m} \tilde{H}_k (X_i; \mathbb{F})$ for all $k \geq -1$.

5.2 Proof of Theorem 1.3.5

In this section we prove our main result, Theorem 1.3.5.

Note that the construction of the tolerance complexes depends on the vertex set of the original complex. Let K be a complex on vertex set V. Let $U \subset V$ and $\sigma \in K$. For the construction of tolerance complexes, we will consider the vertex set of the induced subcomplex $K[U]$ to be the set U, the vertex set of $\mathrm{cost}(K, \sigma)$ to be V, and the vertex set of $\mathrm{lk}(K, \sigma)$ to be $V \setminus \sigma$.

Lemma 5.2.1. *Let K be a simplicial complex on vertex set V, and let $\sigma \in K$. Then,*

$$\begin{aligned}&\mathcal{T}_t (K) \setminus \mathcal{T}_t (\mathrm{cost}(K, \sigma)) \\ &\quad = \left\{ \sigma \cup \eta : \ \eta \in \mathcal{T}_t (\mathrm{lk}(K, \sigma)) \setminus \left(\bigcup_{\substack{\sigma' \subset \sigma: \\ 1 \leq |\sigma'| \leq t}} \mathcal{T}_{t-|\sigma'|} \left(\mathrm{lk}(K[V \setminus \sigma'], \sigma \setminus \sigma')\right) \right) \right\}.\end{aligned}$$

Proof. Let $\tau \in \mathcal{T}_t (K) \setminus \mathcal{T}_t (\mathrm{cost}(K, \sigma))$. Since $\tau \in \mathcal{T}_t (K)$, we can write $\tau = \tau' \cup \tau''$, where $\tau' \in K$ and $|\tau''| \leq t$. Moreover, we must have $\tau' \supset \sigma$. Otherwise, $\tau' \in \mathrm{cost}(K, \sigma)$, a contradiction to $\tau \notin \mathcal{T}_t (\mathrm{cost}(K, \sigma))$.

Let $\eta = \tau \setminus \sigma$. Then, we can write $\eta = (\tau' \setminus \sigma) \cup \tau''$. Since $\tau' \setminus \sigma \in \mathrm{lk}(K, \sigma)$, we obtain $\eta \in \mathcal{T}_t\left(\mathrm{lk}(K, \sigma)\right)$. We claim that

$$\eta \notin \bigcup_{\substack{\sigma' \subset \sigma:\\ 1 \leq |\sigma'| \leq t}} \mathcal{T}_{t-|\sigma'|}\left(\mathrm{lk}(K[V \setminus \sigma'], \sigma \setminus \sigma')\right).$$

Assume for contradiction that $\eta \in \mathcal{T}_{t-|\sigma'|}\left(\mathrm{lk}(K[V \setminus \sigma'], \sigma \setminus \sigma')\right)$ for some $\sigma' \subset \sigma$, $1 \leq |\sigma'| \leq t$. Then, we can write

$$\eta = \eta_1 \cup \eta_2,$$

where $\eta_1 \cap \sigma = \emptyset$, $\eta_1 \cup (\sigma \setminus \sigma') \in K$ and $|\eta_2| \leq t - |\sigma'|$. Hence, we obtain

$$\tau = \sigma \cup \eta = (\eta_1 \cup (\sigma \setminus \sigma')) \cup (\sigma' \cup \eta_2).$$

Since $\sigma \not\subset \eta_1 \cup (\sigma \setminus \sigma')$ and $|\sigma' \cup \eta_2| \leq t$, we have $\tau \in \mathcal{T}_t\left(\mathrm{cost}(K, \sigma)\right)$, which is a contradiction to the assumption $\tau \in \mathcal{T}_t(K) \setminus \mathcal{T}_t\left(\mathrm{cost}(K, \sigma)\right)$.

For the opposite direction, let $\tau = \sigma \cup \eta$, where

$$\eta \in \mathcal{T}_t\left(\mathrm{lk}(K, \sigma)\right) \setminus \left(\bigcup_{\substack{\sigma' \subset \sigma:\\ 1 \leq |\sigma'| \leq t}} \mathcal{T}_{t-|\sigma'|}\left(\mathrm{lk}(K[V \setminus \sigma'], \sigma \setminus \sigma')\right) \right).$$

We claim that $\tau \in \mathcal{T}_t(K) \setminus \mathcal{T}_t\left(\mathrm{cost}(K, \sigma)\right)$. Since $\eta \in \mathcal{T}_t\left(\mathrm{lk}(K, \sigma)\right)$, we can write $\eta = \eta_1 \cup \eta_2$, where $\eta_1 \cap \sigma = \emptyset$, $\eta_1 \cup \sigma \in K$ and $|\eta_2| \leq t$. Therefore, $\tau = (\eta_1 \cup \sigma) \cup \eta_2 \in \mathcal{T}_t(K)$. We are left to show that $\tau \notin \mathcal{T}_t\left(\mathrm{cost}(K, \sigma)\right)$. Assume for contradiction that $\tau \in \mathcal{T}_t\left(\mathrm{cost}(K, \sigma)\right)$. Then, we can write $\tau = \tau_1 \cup \tau_2$, where $\tau_1 \in K$, $\sigma \not\subset \tau_1$ and $|\tau_2| \leq t$. Let $\sigma' = \tau_2 \cap \sigma$. Since $\sigma \not\subset \tau_1$ and $\sigma \subset \tau$, we must have $\sigma' \neq \emptyset$. Then,

$$\eta = \tau \setminus \sigma = (\tau_1 \setminus (\sigma \setminus \sigma')) \cup (\tau_2 \setminus \sigma').$$

Since $\tau_1 \setminus (\sigma \setminus \sigma') \in \mathrm{lk}(K[V \setminus \sigma'], \sigma \setminus \sigma')$ and $|\tau_2 \setminus \sigma'| \leq t - |\sigma'|$, we have $\eta \in \mathcal{T}_{t-|\sigma'|}\left(\mathrm{lk}(K[V \setminus \sigma'], \sigma \setminus \sigma')\right)$. But this is a contradiction to the assumption on η. This completes the proof.

By Lemma 5.2.1 and Lemma 5.1.1, we obtain:

Corollary 5.2.2. *Let K be a simplicial complex, and let $\sigma \in K$. Then, for all k, we have*

$$\begin{aligned}
&H_k\left(\mathcal{T}_t(K), \mathcal{T}_t\left(\mathrm{cost}(K, \sigma)\right); \mathbb{F}\right)\\
&\cong H_{k-|\sigma|}\left(\mathcal{T}_t\left(\mathrm{lk}(K, \sigma)\right), \mathcal{T}_t\left(\mathrm{lk}(K, \sigma)\right) \cap \left(\bigcup_{\substack{\sigma' \subset \sigma:\\ 1 \leq |\sigma'| \leq t}} \mathcal{T}_{t-|\sigma'|}\left(\mathrm{lk}(K[V \setminus \sigma'], \sigma \setminus \sigma')\right) \right); \mathbb{F} \right).
\end{aligned}$$

Proposition 5.2.3. *Let K be a simplicial complex, and let $\sigma \in K$ such that σ is contained in a unique maximal simplex $\sigma \cup U \in K$, where $U \neq \emptyset$. Then, for all k,*

$$H_k\left(\mathcal{T}_t\left(K\right), \mathcal{T}_t\left(\operatorname{cost}(K,\sigma)\right); \mathbb{F}\right) \cong \bigoplus_{\substack{W \subset V \setminus (\sigma \cup U):\\ |W|=t}} \tilde{H}_{k-|\sigma|-1}\left(\bigcup_{\substack{\sigma' \subset \sigma:\\ 1 \leq |\sigma'| \leq t}} \mathcal{T}_{t-|\sigma'|}\left(\operatorname{lk}(K, \sigma \setminus \sigma')[U \cup W]\right); \mathbb{F}\right).$$

Proof. Let

$$Y = \bigcup_{\substack{\sigma' \subset \sigma:\\ 1 \leq |\sigma'| \leq t}} \mathcal{T}_{t-|\sigma'|}\left(\operatorname{lk}(K[V \setminus \sigma'], \sigma \setminus \sigma')\right).$$

By Corollary 5.2.2, we have

$$H_k\left(\mathcal{T}_t\left(K\right), \mathcal{T}_t\left(\operatorname{cost}(K,\sigma)\right); \mathbb{F}\right) \cong H_{k-|\sigma|}\left(\mathcal{T}_t\left(\operatorname{lk}(K,\sigma)\right), \mathcal{T}_t\left(\operatorname{lk}(K,\sigma)\right) \cap Y; \mathbb{F}\right).$$

By Theorem 2.2.6, we have a long exact sequence

$$\cdots \to \tilde{H}_{k-|\sigma|}\left(\mathcal{T}_t\left(\operatorname{lk}(K,\sigma)\right); \mathbb{F}\right) \to H_{k-|\sigma|}\left(\mathcal{T}_t\left(\operatorname{lk}(K,\sigma)\right), \mathcal{T}_t\left(\operatorname{lk}(K,\sigma)\right) \cap Y; \mathbb{F}\right) \to$$
$$\to \tilde{H}_{k-|\sigma|-1}\left(\mathcal{T}_t\left(\operatorname{lk}(K,\sigma)\right) \cap Y; \mathbb{F}\right) \to \tilde{H}_{k-|\sigma|-1}\left(\mathcal{T}_t\left(\operatorname{lk}(K,\sigma)\right); \mathbb{F}\right) \to \cdots$$

Note that $\operatorname{lk}(K,\sigma) = 2^U$; therefore,

$$\mathcal{T}_t\left(\operatorname{lk}(K,\sigma)\right) = 2^U * \{\tau \subset V \setminus (U \cup \sigma) :\, |\tau| \leq t\}.$$

In particular, since $U \neq \emptyset$, $\mathcal{T}_t\left(\operatorname{lk}(K,\sigma)\right)$ is contractible. Hence,

$$H_{k-|\sigma|}\left(\mathcal{T}_t\left(\operatorname{lk}(K,\sigma)\right), \mathcal{T}_t\left(\operatorname{lk}(K,\sigma)\right) \cap Y; \mathbb{F}\right) \cong \tilde{H}_{k-|\sigma|-1}\left(\mathcal{T}_t\left(\operatorname{lk}(K,\sigma)\right) \cap Y; \mathbb{F}\right).$$

We can write

$$\mathcal{T}_t\left(\operatorname{lk}(K,\sigma)\right) \cap Y = \bigcup_{\substack{W \subset V \setminus (\sigma \cup U):\\ |W|=t}} 2^{U \cup W} \cap Y = \bigcup_{\substack{W \subset V \setminus (\sigma \cup U):\\ |W|=t}} Y_W,$$

where

$$Y_W = Y[U \cup W] = \bigcup_{\substack{\sigma' \subset \sigma:\\ 1 \leq |\sigma'| \leq t}} \mathcal{T}_{t-|\sigma'|}\left(\operatorname{lk}(K, \sigma \setminus \sigma')[U \cup W]\right).$$

Let $m > 1$, and let $W_1, \ldots, W_m \subset V \setminus (\sigma \cup U)$ be distinct sets, such that $|W_i| = t$ for all $i \in [m]$. Then,

$$\bigcap_{i=1}^m Y_{W_i} = \bigcup_{\substack{\sigma' \subset \sigma:\\ 1 \leq |\sigma'| \leq t}} \mathcal{T}_{t-|\sigma'|}\left(\operatorname{lk}(K, \sigma \setminus \sigma')[U \cup (\cap_{i=1}^m W_i)]\right).$$

Since $|\cap_{i=1}^m W_i| \le t-1$, we have, for any $v \in \sigma$,

$$U \cup (\cap_{i=1}^m W_i) \in \mathcal{T}_{t-1}\left(\mathrm{lk}(K, \sigma \setminus \{v\})[U \cup (\cap_{i=1}^m W_i)]\right).$$

In particular,

$$U \cup (\cap_{i=1}^m W_i) \in \bigcap_{i=1}^m Y_{W_i},$$

and hence, we conclude

$$\bigcap_{i=1}^m Y_{W_i} = 2^{U \cup (\cap_{i=1}^m W_i)}.$$

Since $U \neq \emptyset$, the intersection $\bigcap_{i=1}^m Y_{W_i}$ is non-empty and acyclic. Therefore, by Lemma 5.1.2,

$$\begin{aligned}\tilde{H}_{k-|\sigma|-1}\left(\mathcal{T}_t\left(\mathrm{lk}(K,\sigma)\right) \cap Y; \mathbb{F}\right) &\cong \bigoplus_{\substack{W \subset V \setminus (\sigma \cup U): \\ |W|=t}} \tilde{H}_{k-|\sigma|-1}\left(Y_W; \mathbb{F}\right) \\ &\cong \bigoplus_{\substack{W \subset V \setminus (\sigma \cup U): \\ |W|=t}} \tilde{H}_{k-|\sigma|-1}\left(\bigcup_{\substack{\sigma' \subset \sigma: \\ 1 \le |\sigma'| \le t}} \mathcal{T}_{t-|\sigma'|}\left(\mathrm{lk}(K, \sigma \setminus \sigma')[U \cup W]\right); \mathbb{F}\right),\end{aligned}$$

as wanted. $\square$

Recall that $h(t,d)$ is defined as follows: $h(0,d) = d$ for all $d \ge 0$, and for $t > 0$,

$$h(t,d) = \left(\sum_{s=1}^{\min\{t,d\}} \binom{d}{s}(h(t-s,d)+1)\right) + d.$$

Lemma 5.2.4. *For $d = 1$, we have*

$$h(t,1) = 2t+1.$$

For $t = 1$, we have

$$h(1,d) = d^2 + 2d.$$

For fixed t, we have

$$h(t,d) = O(d^{t+1}).$$

Proof. First, we show that $h(t,1) = 2t+1$. We argue by induction on t. For $t = 0$ we have $h(0,1) = 1 = 2t+1$. Now, assume $t > 0$. Then, by the definition of $h(t,d)$ and the induction hypothesis, we obtain

$$h(t,1) = h(t-1,1) + 1 + 1 = 2(t-1) + 3 = 2t+1.$$

Next, we show that $h(1, d) = d^2 + 2d$. Indeed,

$$h(1, d) = d(h(0, d) + 1) + d = d^2 + 2d,$$

as wanted.

Finally, we show that, for fixed t, $h(t, d) = O(d^{t+1})$. We argue by induction on t. For $t = 0$ we have $h(0, d) = d = O(d)$. Let $t > 1$. We will show that there is some constant C_t such that, for large enough d, $h(t, d) \leq C_t d^{t+1}$. By the definition of $h(t, d)$ and the induction hypothesis, we have,

$$\begin{aligned} h(t, d) &= \left(\sum_{s=1}^{t} \binom{d}{s} (h(t - s, d) + 1) \right) + d \\ &\leq \left(\sum_{s=1}^{t} \frac{d^s}{s!} (C_{t-s} d^{t-s+1} + 1) \right) + d \\ &= \left(\sum_{s=1}^{t} \frac{C_{t-s}}{s!} \right) d^{t+1} + \left(\sum_{s=1}^{t} \frac{d^s}{s!} + d \right) \\ &\leq C_t d^{t+1} \end{aligned}$$

for $C_t > \sum_{s=1}^{t} \frac{C_{t-s}}{s!}$ and large enough d. So, for fixed t, we have $h(t, d) = O(d^{t+1})$. $\square$

Theorem 1.3.5. *Let K be a d-collapsible complex on vertex set V and let $t \geq 0$. Then, $\mathcal{T}_t(K)$ is $h(t, d)$-Leray.*

Proof. We will show that $\tilde{H}_k(\mathcal{T}_t(K); \mathbb{F}) = 0$ for $k \geq h(t, d)$. This is sufficient to prove the statement of the theorem, since $\mathcal{T}_t(K)[W] = \mathcal{T}_t(K[W])$ and, by Lemma 2.3.6, $K[W]$ is d-collapsible for every $W \subset V$.

We argue by induction on t. If $t = 0$ the statement obviously holds, since every d-collapsible complex is d-Leray.

Let $t \geq 1$. We argue by induction on the size of K, that is, the number of simplices in K. If $\dim(K) < d$, then $\dim(\mathcal{T}_t(K)) < d + t < h(t, d)$, so the statement holds. Otherwise, by Lemma 2.3.18, there is some $\sigma \in K$ such that $|\sigma| = d$, σ is contained in a unique maximal face $\tau \neq \sigma$ of K, and $\text{cost}(K, \sigma)$ is d-collapsible.

Let $U = \tau \setminus \sigma$. By Theorem 2.2.6, the following sequence is exact:

$$\cdots \to \tilde{H}_k(\mathcal{T}_t(\text{cost}(K, \sigma)); \mathbb{F}) \to \tilde{H}_k((\mathcal{T}_t(K)); \mathbb{F}) \to H_k(\mathcal{T}_t(K), \mathcal{T}_t(\text{cost}(K, \sigma)); \mathbb{F}) \to \cdots$$

By the induction hypothesis, $\tilde{H}_k(\mathcal{T}_t(\text{cost}(K, \sigma)); \mathbb{F}) = 0$ for $k \geq h(t, d)$. Therefore, it is sufficient to show that $H_k(\mathcal{T}_t(K), \mathcal{T}_t(\text{cost}(K, \sigma)); \mathbb{F}) = 0$ for $k \geq h(t, d)$.

By Proposition 5.2.3, it is sufficient to show that, for every $W \subset V \setminus (\sigma \cup U)$ of size

t, the homology group

$$\tilde{H}_k\left(\bigcup_{\substack{\sigma'\subset\sigma:\\ 1\le|\sigma'|\le t}} \mathcal{T}_{t-|\sigma'|}\left(\mathrm{lk}(K,\sigma\setminus\sigma')[U\cup W]\right);\mathbb{F}\right)$$

is trivial for $k \geq h(t,d)-d-1$. Note that, for any $\sigma' \subset \sigma$, by Lemma 2.3.15 and Lemma 2.3.6, the complex $\mathrm{lk}(K,\sigma\setminus\sigma')[U\cup W]$ is also d-collapsible. Hence, by Theorem 2.2.13 and the induction hypothesis, the above homology group is trivial for

$$k \geq \left(\sum_{\substack{\sigma'\subset\sigma:\\ 1\le|\sigma'|\le t}} h(t-|\sigma'|,d)+1\right)-1 = \left(\sum_{s=1}^{\min\{t,d\}}\binom{d}{s}(h(t-s,d)+1)\right)-1$$
$$= h(t,d)-d-1,$$

as wanted. □

5.3 Improved bound for $d=2$, $t=1$

By Theorem 1.3.5 and Lemma 5.2.4, for any d-collapsible complex K, the 1-tolerance complex $\mathcal{T}_1(K)$ is (d^2+2d)-Leray. This is of the same order of magnitude, but larger, than the conjectural bound $\eta(d+1,2)-1=\left\lfloor\left(\frac{d+3}{2}\right)^2\right\rfloor-1$. In this section we prove Theorem 1.3.6, which gives a tight bound for the Leray number of $\mathcal{T}_1(K)$, in the special case that K is 2-collapsible.

For the proof we will need the following Lemma:

Lemma 5.3.1. *Let K be a 2-collapsible complex on vertex set V. Let $\sigma=\{u,v\}\in K$ such that σ is contained in a unique maximal face $\sigma\cup U$, where $U\neq\emptyset$. Let $w\in V\setminus(U\cup\sigma)$. Then,*

$$\tilde{H}_k\left(\mathrm{lk}(K,v)[U\cup\{w\}]\cup\mathrm{lk}(K,u)[U\cup\{w\}];\mathbb{F}\right)=0$$

for $k\geq 2$.

Proof. Let $A=\mathrm{lk}(K,v)[U\cup\{w\}]$ and $B=\mathrm{lk}(K,u)[U\cup\{w\}]$. By Mayer-Vietoris (Theorem 2.2.1), we have a long exact sequence

$$\cdots\to\tilde{H}_k(A;\mathbb{F})\bigoplus\tilde{H}_k(B;\mathbb{F})\to\tilde{H}_k(A\cup B;\mathbb{F})\to\tilde{H}_{k-1}(A\cap B;\mathbb{F})\to\cdots$$

Since K is 2-collapsible, then, by Lemma 2.3.6 and Lemma 2.3.15, A and B are also 2-collapsible. In particular, $\tilde{H}_k(A;\mathbb{F})=\tilde{H}_k(B;\mathbb{F})=0$ for $k\geq 2$. Therefore, it is enough to show that

$$\tilde{H}_k(A\cap B)=0$$

for $k \geq 1$. If $w \notin A \cap B$, then

$$A \cap B = 2^U,$$

and the claim holds. Otherwise, assume $w \in A \cap B$. By Theorem 2.2.2, we have a long exact sequence

$$\cdots \to \tilde{H}_k\left((A \cap B) \setminus w; \mathbb{F}\right) \to \tilde{H}_k\left(A \cap B; \mathbb{F}\right) \to \tilde{H}_{k-1}\left(\mathrm{lk}(A \cap B, w); \mathbb{F}\right) \to \cdots .$$

Note that $(A \cap B) \setminus w = 2^U$; hence, $\tilde{H}_k\left((A \cap B) \setminus w; \mathbb{F}\right) = 0$ for all k. Thus, it is enough to show that

$$\tilde{H}_k\left(\mathrm{lk}(A \cap B, w); \mathbb{F}\right) = \tilde{H}_k\left(\mathrm{lk}(K, \{v, w\})[U] \cap \mathrm{lk}(K, \{u, w\})[U]; \mathbb{F}\right) = 0$$

for $k \geq 0$. Let

$$Z = \mathrm{lk}(K, \{v, w\})[U] \cap \mathrm{lk}(K, \{u, w\})[U].$$

We will show that Z is in fact a complete complex.

Note that a set $\tau \subset U$ is a missing face of Z if and only if it is either a missing face of $\mathrm{lk}(K, \{v, w\})[U]$ or a missing face of $\mathrm{lk}(K, \{u, w\})[U]$. Moreover, $\tau \subset U$ is a missing face of $\mathrm{lk}(K, \{v, w\})[U]$ if and only if there is some $\eta \subset \{v, w\}$ such that $\tau \cup \eta$ is a missing face of K. Similarly, τ is a missing face of $\mathrm{lk}(K, \{u, w\})$ if and only if there is some $\eta \subset \{u, w\}$ such that $\tau \cup \eta$ is a missing face of K.

Assume for contradiction that Z contains a missing face $\tau \subset U$ of size at least two.

Recall that, since K is 2-collapsible, all the missing faces of K are of size at most 3. Then, since $U \in \mathrm{lk}(K, \{u, v\})$, τ must be of the form $\tau = \{x, y\}$, where $\{x, y, w\}$ is a missing face of K.

Now, we look at the induced subcomplex $L = K[\{u, v, w, x, y\}]$. By Lemma 2.3.6, L is 2-collapsible. The missing faces of L are exactly the two sets $\{u, v, w\}$ and $\{x, y, w\}$. It is easy to check (for example by applying Theorem 2.2.12) that $\tilde{H}_2\left(L; \mathbb{F}\right) \neq 0$. Therefore, L is not 2-Leray. This is a contradiction to L being 2-collapsible. So, Z is a complete complex, and therefore $\tilde{H}_k\left(Z; \mathbb{F}\right) = 0$ for all $k \geq 0$. □

Theorem 1.3.6. *Let K be a 2-collapsible complex. Then, $\mathcal{T}_1(K)$ is 5-Leray.*

Proof. The proof is exactly the same as the $t = 1$ case of the proof of Theorem 1.3.5, except that we replace the use of the Kalai-Meshulam bound (Theorem 2.2.13) by Lemma 5.3.1. □

5.4 Examples of 1-tolerance complexes

Let $d \geq 2$. The following example was presented in [MO11, Theorem 3.2]: Let $1 \leq n \leq d - 1$, and write $\mathbb{R}^d = \mathbb{R}^n \times \mathbb{R}^{d-n}$. Let $x_1, \ldots, x_{n+1} \subset \mathbb{R}^n$ be affinely independent. Let $y_1, \ldots, y_{d-n+1} \subset \mathbb{R}^{d-n}$ be affinely independent, and let $\Delta = \mathrm{conv}(\{y_1, \ldots, y_{d-n+1}\})$.

For $i \in [n+1]$, let

$$A_i = \operatorname{conv}\left(\{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}\}\right) \times \Delta.$$

For $i \in [n+1]$ and $j \in [d-n+1]$, let

$$B_{i,j} = \operatorname{conv}(\{(x_p, y_q) : \, p \in [n+1], q \in [d-n+1], \, (p,q) \neq (i,j)\})$$

Let $\mathcal{C} = \{A_i\}_{i=1}^{n+1} \cup \{B_{i,j}\}_{(i,j)\in[n+1]\times[d-n+1]}$. Note that $|\mathcal{C}| = (n+1)(d-n+2)$.

Let $K = N(\mathcal{C})$. For $i \in [n+1]$, we will denote the vertex of K corresponding to A_i by v_i, and for $(i,j) \in [n+1] \times [d-n+1]$, we will denote the vertex of K corresponding to $B_{i,j}$ by $u_{i,j}$.

Lemma 5.4.1. *The missing faces of K are the sets $\{v_1, \ldots, v_{n+1}\}$ and*

$$\{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{n+1}, u_{i,1}, \ldots, u_{i,d-n+1}\}$$

for $i \in [n+1]$.

Proof. First, note that $\cap_{i=1}^{n+1} A_i = \emptyset$. Therefore, $\{v_1, \ldots, v_{n+1}\} \notin K$.

For any $i \in [n+1]$, we have $\cap_{k\neq i} A_k = \{x_i\} \times \Delta$. Let $(x_i, y) \in \{x_i\} \times \Delta = \cap_{k \neq i} A_k$.

Assume that

$$(x_i, y) = \sum_{(p,q)\in[n+1]\times[d-n+1]} \alpha_{p,q}(x_p, y_q),$$

where $\alpha_{p,q} \geq 0$ for all p, q and $\sum_{(p,q)\in[n+1]\times[d-n+1]} \alpha_{p,q} = 1$. Then, since $x_1, \ldots, x_{n+1}$ are affinely independent, we must have $\alpha_{p,q} = 0$ for $p \neq i$. So, we have

$$(x_i, y) = \sum_{j=1}^{d-n+1} \alpha_{i,j}(x_i, y_j).$$

In particular, $y = \sum_{j=1}^{d-n+1} \alpha_{i,j} y_i$. Since $y_1, \ldots, y_{d-n+1}$ are affinely independent, this is the unique way to write (x_i, y) as a convex combination of the points $\{(x_p, y_q)\}_{(p,q)\in[n+1]\times[d-n+1]}$.

We will show that $(x_i, y) \notin \cap_{j=1}^{d-n+1} B_{i,j}$. Indeed, if $(x_i, y) \in B_{i,j}$ for some $j \in [d-n+1]$, we must have $\alpha_{i,j} = 0$. Hence, if $(x_i, y) \in \cap_{j=1}^{d-n+1} B_{i,j}$, we obtain $\alpha_{i,j} = 0$ for all j, a contradiction to $\sum_{j=1}^{d-n+1} \alpha_{i,j} = 1$.

Therefore, $(\cap_{k\neq i} A_k) \cap \left(\cap_{j=1}^{d-n+1} B_{i,j}\right) = \emptyset$. That is,

$$\{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{n+1}, u_{i,1}, \ldots, u_{i,d-n+1}\} \notin N(\mathcal{C}) = K.$$

On the other hand, let B be a set that does not contain any of the sets $\{v_1, \ldots, v_{n+1}\}$ or $\{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{n+1}, u_{i,1}, \ldots, u_{i,d-n+1}\}$ for $i \in [n+1]$. Then, there must be $(i,j) \in [n+1] \times [d-n+1]$ such that $v_i, u_{i,j} \notin B$. Since

$$(x_i, y_j) \in (\cap_{k\neq i} A_k) \cap \left(\cap_{(p,q)\neq(i,j)} B_{p,q}\right),$$

we have $B \in N(\mathcal{C}) = K$. Therefore, the missing faces of K are the sets $\{v_1, \ldots, v_{n+1}\}$ and

$$\{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{n+1}, u_{i,1}, \ldots, u_{i,d-n+1}\}$$

for $i \in [n+1]$. □

Note that the set $A = \{v_i\}_{i=1}^{n+1} \cup \{u_{i,j}\}_{(i,j)\in[n+1]\times[d-n+1]}$ is not a simplex in $\mathcal{T}_1(K)$, since even after removing any vertex from it, it contains a missing face. On the other hand, for any $i \in [n+1]$, the set $A \setminus \{v_i\}$ belongs to $\mathcal{T}_1(K)$, since, for any $j \in [d-n+1]$, $A \setminus \{v_i, u_{i,j}\} \in K$. Similarly, for any $(i,j) \in [n+1] \times [d-n+1]$, the set $A \setminus \{u_{i,j}\}$ belongs to $\mathcal{T}_1(K)$. Therefore $\mathcal{T}_1(K)$ is the boundary of a simplex on $(n+1)(d-n+2)$ vertices. That is, it is a $((n+1)(d-n+2)-2)$-dimensional sphere. Hence, it is not $((n+1)(d-n+2)-2)$-Leray.

Since

$$\max\{(n+1)(d-n+2) : 1 \leq n \leq d-1\} = \left\lfloor \left(\frac{d+3}{2}\right)^2 \right\rfloor,$$

we obtain for suitable n a d-representable complex such that its 1-tolerance complex is not $\left(\left\lfloor \left(\frac{d+3}{2}\right)^2 \right\rfloor - 2\right)$-Leray.

For $d = 2$, we have the following additional example of a 2-representable complex whose 1-tolerance complex is not 4-Leray:

Let $T \subset \mathbb{R}^2$ be a triangle. Let v_1, v_2, v_3 be its vertices and e_1, e_2, e_3 be its edges (where for each $i \in [3]$, v_i is the vertex disjoint from e_i). For each $i \in [3]$, let p_i be the midpoint of the edge e_i.

For each $i \in [3]$, let H_i be a line parallel to e_i separating v_i from the quadrilateral spanned by the four vertices $\{v_j, p_j\}_{j \neq i}$, and let H_i^+ be the half-plane defined by H_i that contains e_i.

Let $\mathcal{C} = \{e_1, e_2, e_3, H_1^+, H_2^+, H_3^+\}$ and let $K = N(\mathcal{C})$.

For $i \in [3]$, let w_i be the vertex of K that corresponds to e_i and let u_i be the vertex that corresponds to H_i^+ It is easy to check that the missing faces of K are the sets

$$\{w_1, w_2, w_3\}, \{w_1, w_2, u_3\}, \{w_1, u_2, w_3\}, \{u_1, w_2, w_3\}.$$

Now, note that the set $A = \{w_1, w_2, w_3, u_1, u_2, u_3\}$ does not belong to $\mathcal{T}_1(K)$, since even after removing any vertex from it, it contains a missing face. On the other hand, for any $i \in [3]$, the set $A \setminus \{w_i\}$ belongs to $\mathcal{T}_1(K)$, since $A \setminus \{w_i, u_i\} \in K$. Similarly, the set $A \setminus \{u_i\}$ belongs also to $\mathcal{T}_1(K)$. That is, $\mathcal{T}_1(K)$ is the boundary of a simplex on 6 vertices. In particular, it is a 4-dimensional sphere, and therefore it is not 4-Leray.

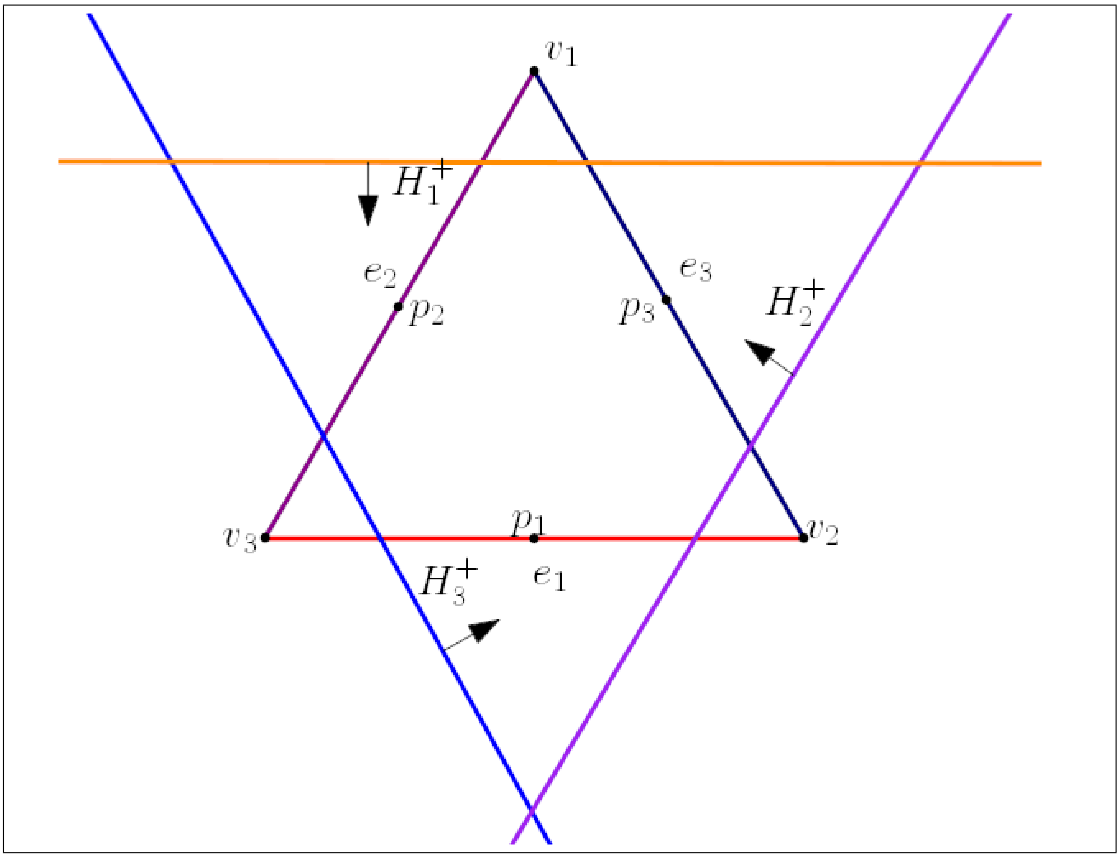

Figure 5.1: A family of convex sets in the plane $\mathcal{C} = \{e_1, e_2, e_3, H_1^+, H_2^+, H_3^+\}$ such that $\mathcal{T}_1(N(\mathcal{C}))$ is not 4-Leray.

Chapter 6

Representability and boxicity of simplicial complexes

This chapter is organized as follows. In Section 6.1 we prove some simple results about the missing faces and the representability of intersections of complexes. Section 6.2 contains the proof of Theorem 1.4.3. In Section 6.3 we prove Theorem 1.4.4. In Section 6.4 we prove our main result, Theorem 1.4.2. In Section 6.5 we present some related open problems.

6.1 Intersection of simplicial complexes

In this section we prove some basic results about the missing faces and the representability of intersections of complexes.

Proposition 6.1.1. *Let $X_1, \ldots, X_k$ be simplicial complexes on vertex set V, and $X = \cap_{i=1}^k X_i$. For each $i \in [k]$, let $\mathcal{M}_i$ be the set of missing faces of X_i, and let $\mathcal{M}$ be the set of missing faces of X. Then, $\mathcal{M}$ is the set of inclusion minimal elements of $\cup_{i=1}^k \mathcal{M}_i$. As a consequence, we obtain*

$$h(X) \leq \max_{i\in[k]} h(X_i).$$

Proof. Let $\tau \in \mathcal{M}$. Since $\tau \notin X$, then there exists some $j \in [k]$ such that $\tau \notin X_j$. Let $\sigma \subsetneq \tau$. Since τ is a missing face of X, we have $\sigma \in X = \cap_{i=1}^k X_i$. In particular, $\sigma \in X_j$. Hence, τ is a missing face of X_j. That is, $\tau \in \mathcal{M}_j \subset \cup_{i=1}^k \mathcal{M}_i$. Moreover, τ does not contain any other face of $\cup_{i=1}^k \mathcal{M}_i$. Otherwise, there exists some $r \in [k]$ and $\sigma \in \mathcal{M}_r$ such that $\sigma \subsetneq \tau$. Since $\sigma \notin X_r$, then $\sigma \notin X$. But this is a contradiction to τ being a missing face of X.

Now, let τ be an inclusion minimal element of $\cup_{i=1}^k \mathcal{M}_i$. Then $\tau \in \mathcal{M}_j$ for some $j \in [k]$. In particular, $\tau \notin X_j$, and therefore $\tau \notin X$. Now, let $\sigma \subsetneq \tau$. Assume for contradiction that $\sigma \notin X$. Then, there exists some $r \in [k]$ such that $\sigma \notin X_r$. So, there exists some $\eta \in \mathcal{M}_r$ such that $\eta \subset \sigma \subsetneq \tau$. This is a contradiction to τ being inclusion minimal in $\cup_{i=1}^k \mathcal{M}_i$. So, $\sigma \in X$. Therefore, τ is a missing face of X.

Since $\mathcal{M} \subset \cup_{i=1}^k \mathcal{M}_i$, we obtain $h(X) \le \max_{i\in[k]} h(X_i)$. $\square$

Lemma 6.1.2. *Let $X_1,\dots,X_k$ be simplicial complexes on vertex set V. If X_i is d_i-representable for each $i \in [k]$, then $\cap_{i=1}^k X_i$ is $\left(\sum_{i=1}^k d_i\right)$-representable.*

Proof. For $i \in [k]$, let $\{C_v^i\}_{v\in V}$ be a representation of X_i in $\mathbb{R}^{d_i}$. For $v \in V$, let

$$C_v = C_v^1 \times C_v^2 \times \cdots \times C_v^k.$$

We will show that $\mathcal{C} = \{C_v\}_{v\in V}$ is a representation of $\cap_{i=1}^k X_i$ in $\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_k} \cong \mathbb{R}^{d_1+\cdots+d_k}$.

Note that the sets C_v are convex, and for any $\sigma \subset V$,

$$\bigcap_{v\in\sigma} C_v = \left(\bigcap_{v\in\sigma} C_v^1\right) \times \cdots \times \left(\bigcap_{v\in\sigma} C_v^k\right). \tag{6.1}$$

Let $\sigma \subset V$. If $\sigma \in \cap_{i=1}^k X_i$, then $\sigma \in X_i$ for all $i \in [k]$. Hence, $\cap_{v\in\sigma} C_v^i \neq \emptyset$ for all $i \in [k]$. So, by Equation (6.1), $\cap_{v\in\sigma} C_v \neq \emptyset$. If $\sigma \notin \cap_{i=1}^k X_i$, then there exists some $i \in [k]$ such that $\sigma \notin X_i$. Therefore, $\cap_{v\in\sigma} C_v^i = \emptyset$. Thus, by Equation (6.1), $\cap_{v\in\sigma} C_v = \emptyset$. Hence, $\mathcal{C}$ is a representation of $\cap_{i=1}^k X_i$ in $\mathbb{R}^{d_1+\cdots+d_k}$. $\square$

6.2 Lower bounds on d-boxicity

In this section we prove Theorem 1.4.3. For the proof we will need the following simple lemma, which is a generalization of [Wit80, Lemma 3]:

Lemma 6.2.1. *Let A, B be two finite sets, such that $|A| = |B| = d+1$, and $|A\cap B| < d$. Let $V = A \cup B$. Let X be a simplicial complex on vertex set V that has A and B as missing faces, and such that for any other missing face τ of X, $\tau \cup A = V$ and $\tau \cup B = V$. Then, there exists some $k \ge d$ such that $\tilde{H}_k(X;\mathbb{F}) \neq 0$.*

Proof. Let $\mathcal{M}$ be the set of missing faces of X. Let $\Gamma(X)$ be the simplicial complex

$$\Gamma(X) = \left\{ \mathcal{N} \subset \mathcal{M} : \bigcup_{\tau\in\mathcal{N}} \tau \neq V \right\}.$$

By assumption, $A \cup B = V$, and for any missing face $\tau \in \mathcal{M} \setminus \{A, B\}$, $A \cup \tau = V$ and $B \cup \tau = V$. Therefore, both A and B are isolated vertices of the complex $\Gamma(X)$. In particular, $\Gamma(X)$ is disconnected. That is,

$$\tilde{H}_0(\Gamma(X);\mathbb{F}) \neq 0.$$

By Theorem 2.2.12, we have

$$\tilde{H}_{|V|-3}(X;\mathbb{F}) = \tilde{H}_0(\Gamma(X);\mathbb{F}) \neq 0.$$

Since $|A \cap B| < d$, we have

$$|V| - 3 = |A| + |B| - |A \cap B| - 3 \geq 2(d+1) - (d-1) - 3 = d.$$

Hence, we have $\tilde{H}_k(X;\mathbb{F}) \neq 0$ for some $k \geq d$. $\square$

Theorem 1.4.3. *Let X be a complex whose set of missing faces is a partial Steiner $(d, d+1, n)$-system $\mathcal{M}$. Then, X cannot be written as the intersection of less than $|\mathcal{M}|$ d-Leray complexes. On the other hand, the d-boxicity of X is at most $|\mathcal{M}|$. As a consequence,*

$$box_d(X) = |\mathcal{M}|.$$

Proof. Assume we can write X as

$$X = \cap_{i=1}^{s} X_i,$$

where, for all $i \in [s]$, X_i is a d-Leray complex. For each $i \in [s]$, let $\mathcal{M}_i$ be the set of missing faces of X_i.

By Proposition 6.1.1, $\mathcal{M}$ is the set of inclusion minimal elements in $\cup_{i=1}^{s} \mathcal{M}_i$. Since all the elements of $\mathcal{M}$ are of size $d+1$, and all the elements of $\mathcal{M}_i$ are of size at most $d+1$ (since, by Theorem 2.4.2, the missing faces of a d-Leray complex are of dimension at most d), we must in fact have

$$\mathcal{M} = \cup_{i=1}^{s} \mathcal{M}_i.$$

(Otherwise, assume there exists some $\tau \in \cup_{i=1}^{s} \mathcal{M}_i \setminus \mathcal{M}$. Then, there is some $\eta \in \mathcal{M}$ such that $\eta \subsetneq \tau$. But since all the elements of $\mathcal{M}$ are of size $d+1$, we obtain $|\tau| > d+1$, a contradiction).

Assume for contradiction that $s < |\mathcal{M}|$. Then, by the pigeonhole principle, there exist two distinct sets $\tau_1, \tau_2 \in \mathcal{M}$ such that τ_1 and τ_2 are both missing faces of X_i for some $i \in [s]$. Let τ_1 and τ_2 be such a pair with intersection $\tau_1 \cap \tau_2$ of maximal size.

Let us look at the induced subcomplex

$$Y = X_i[\tau_1 \cup \tau_2].$$

We will show that Y satisfies the conditions of Lemma 6.2.1: Note that τ_1 and τ_2 are missing faces of Y, the vertex set of Y is $\tau_1 \cup \tau_2$ and $|\tau_1| = |\tau_2| = d+1$. Moreover, since $\mathcal{M}$ is a partial Steiner $(d, d+1, n)$-system, we have $|\tau_1 \cap \tau_2| < d$. It is left to show that any other missing face τ of Y (if such a missing face exists) satisfies $\tau \cup \tau_1 = \tau_1 \cup \tau_2$ and $\tau \cup \tau_2 = \tau_1 \cup \tau_2$:

Let $\tau \neq \tau_1, \tau_2$ be a missing face of Y. That is, τ is a missing face of X_i that is contained in $\tau_1 \cup \tau_2$. Let $k = |\tau_1 \cap \tau_2|$, $t = |\tau_1 \cap \tau_2 \cap \tau|$, $t_1 = |\tau \setminus \tau_2|$ and $t_2 = |\tau \setminus \tau_1|$.

Since $\tau \in \mathcal{M}_i \subset \mathcal{M}$, we obtain, by the maximality of $|\tau_1 \cap \tau_2|$,

$$t_1 + t = |\tau \cap \tau_1| \leq k$$

and

$$t_2 + t = |\tau \cap \tau_2| \leq k.$$

We obtain

$$d + 1 = |\tau| = t_1 + t_2 + t \leq 2k - t.$$

That is,

$$t \leq 2k - d - 1.$$

Hence,

$$|\tau \setminus (\tau_1 \cap \tau_2)| = t_1 + t_2 = d + 1 - t \geq d + 1 - 2k + d + 1 = 2(d + 1 - k).$$

So, $\tau \setminus (\tau_1 \cap \tau_2)$ is a subset of size $t_1 + t_2 \geq 2(d + 1 - k)$ of the set $(\tau_1 \cup \tau_2) \setminus (\tau_1 \cap \tau_2)$. But $|(\tau_1 \cup \tau_2) \setminus (\tau_1 \cap \tau_2)| = 2(d + 1 - k)$. Therefore, $\tau \setminus (\tau_1 \cap \tau_2) = (\tau_1 \cup \tau_2) \setminus (\tau_1 \cap \tau_2)$. Hence, we have

$$\tau \cup \tau_1 = (\tau \setminus (\tau_1 \cap \tau_2)) \cup \tau_1 = ((\tau_1 \cup \tau_2) \setminus (\tau_1 \cap \tau_2)) \cup \tau_1 = \tau_1 \cup \tau_2,$$

and similarly

$$\tau \cup \tau_2 = (\tau \setminus (\tau_1 \cap \tau_2)) \cup \tau_2 = ((\tau_1 \cup \tau_2) \setminus (\tau_1 \cap \tau_2)) \cup \tau_2 = \tau_1 \cup \tau_2.$$

So, by Lemma 6.2.1, $\tilde{H}_r(Y; \mathbb{F}) \neq 0$ for some $r \geq d$. But this is a contradiction to the fact that X_i is d-Leray.

Since any d-representable complex is d-Leray, we obtain:

$$\text{box}_d(X) \geq |\mathcal{M}|.$$

On the other hand, it is easy to show that $\text{box}_d(X) \leq |\mathcal{M}|$: Let V be the vertex set of X. For each $\tau \in \mathcal{M}$, let X_τ be the simplicial complex on vertex set V whose only missing face is τ. It is easy to check that the complex X_τ is d-representable (for example, we may assign to each vertex in τ one of the facets of a simplex P in $\mathbb{R}^d$, and assign to all of the vertices in $V \setminus \tau$ the simplex P itself). Since $X = \cap_{\tau \in \mathcal{M}} X_\tau$, we obtain $\text{box}_d(X) \leq |\mathcal{M}|$. □

6.3 Upper bounds on representability

In this section we prove Theorem 1.4.4. We will need the following simple lemma:

Lemma 6.3.1. *Let $P \subset \mathbb{R}^d$ be a convex polytope. Let $F_1, \ldots, F_m$ be faces of P, and* *let $p_1, \ldots, p_k$ be points in P such that $p_i \notin F_j$ for all $i \in [k]$ and $j \in [m]$. Then, there exists a convex polytope $P' \subset P$ such that $P' \cap F_j = \emptyset$ for all $j \in [m]$, and $p_i \in P'$ for all $i \in [k]$.*

Proof. Let $P' = \operatorname{conv}(\{p_1, \ldots, p_k\})$. Let $j \in [m]$, and let H be a hyperplane supporting F_j. That is, $H \cap P = F_j$, and P is contained in one of the closed half-spaces H^+ defined by H.

Now, since the points $p_1, \ldots, p_k$ belong to $P \setminus F_j$, they must all lie in the interior of H^+. Therefore, their convex hull P' is also contained in the interior of H^+. Since F_j lies on the boundary H of H^+, we have $P' \cap F_j = \emptyset$, as wanted. □

Theorem 6.3.2. *Let X be a simplicial complex on vertex set V. Let $U \subset V$ such that $U \notin X$ and for any missing face τ of X, $|\tau \setminus U| \leq 1$. Then, X is $(|U|-1)$-representable.*

Proof. Let $d = |U| - 1$. Let P be a simplex in $\mathbb{R}^d$. Assign to each vertex $u \in U$ a facet F_u of P. For $\sigma \subset U$, let

$$F_\sigma = \cap_{u \in \sigma} F_u$$

(where we understand that $F_\emptyset = P$). Note that, unless $\sigma = U$, F_σ is a non-empty face of the simplex P. For $\sigma \subsetneq U$, let p_σ be a point in the relative interior of F_σ. Then, for any $\eta \subset U$ and $\sigma \subsetneq U$, $p_\sigma \in F_\eta$ if and only if $\eta \subset \sigma$.

Now we build a representation $\{F'_v\}_{v \in V}$ of X in $\mathbb{R}^d$, as follows:

We divide into two cases:

1. Let $u \in U$. Let $\eta \subset U$ and $\sigma \subsetneq U$ such that $u \in \sigma \cap \eta$, $\eta \notin X$ and $\sigma \in X$. Note that F_η is a face of F_u, and $p_\sigma \in F_u$. Also, since X is a simplicial complex, we must have $\eta \not\subset \sigma$, and therefore $p_\sigma \notin F_\eta$. Hence, by Lemma 6.3.1, there exists a convex polytope $F'_u \subset F_u$ such that $F'_u \cap F_\eta = \emptyset$ for all $\eta \subset U$ such that $u \in \eta$ and $\eta \notin X$, and $p_\sigma \in F'_u$ for all $\sigma \subsetneq U$ such that $u \in \sigma$ and $\sigma \in X$.
2. Let $v \in V \setminus U$. Let $\eta \subset U$ and $\sigma \subsetneq U$ such that $\eta \cup \{v\} \notin X$ and $\sigma \cup \{v\} \in X$. Since X is a simplicial complex, we must have $\eta \not\subset \sigma$; hence, $p_\sigma \notin F_\eta$. Therefore, by Lemma 6.3.1, there exists a convex polytope $F'_v \subset P$ such that $F'_v \cap F_\eta = \emptyset$ for all $\eta \subset U$ such that $\eta \cup \{v\} \notin X$ and $p_\sigma \in F'_v$ for all $\sigma \subsetneq U$ such that $\sigma \cup \{v\} \in X$.

We will show that the family $\{F'_v\}_{v \in V}$ is a representation of X.

First, let $\sigma \in X$. Let $\sigma_1 = \sigma \cap U$. Since $\sigma_1 \in X$ and $U \notin X$, we have $\sigma_1 \subsetneq U$. So, for any $u \in \sigma_1$, we have

$$p_{\sigma_1} \in F'_u.$$

Moreover, for any $v \in \sigma \setminus \sigma_1$, since $\sigma_1 \cup \{v\} \subset \sigma \in X$, we have

$$p_{\sigma_1} \in F'_v.$$

Hence,

$$p_{\sigma_1} \in \cap_{v \in \sigma} F'_v.$$

In particular, $\cap_{v\in\sigma} F'_v \neq \emptyset$.

Now, let $\sigma \subset V$ such that $\sigma \notin X$. Then, there exists some missing face τ of X such that $\tau \subset \sigma$. By assumption, we have $|\tau \setminus U| \leq 1$. We divide into two cases:

1. Assume $\tau \subset U$. Then, on the one hand, we have

$$\cap_{u\in\tau} F'_u \subset \cap_{u\in\tau} F_u = F_\tau.$$

On the other hand, for all $u \in \tau$, by the definition of F'_u, we have

$$F'_u \cap F_\tau = \emptyset.$$

Hence,

$$\cap_{u\in\tau} F'_u = \emptyset.$$

2. Assume that $|\tau \setminus U| = 1$. Let w be the unique vertex in $\tau \setminus U$. Then,

$$\cap_{u\in\tau\setminus\{w\}} F'_u \subset \cap_{u\in\tau\setminus\{w\}} F_u = F_{\tau\setminus\{w\}}.$$

But, since $(\tau \setminus \{w\}) \cup \{w\} = \tau \notin X$, we obtain, by the definition of F'_w,

$$F'_w \cap F_{\tau\setminus\{w\}} = \emptyset.$$

Hence,

$$\cap_{v\in\tau} F'_v = F'_w \cap \left(\cap_{u\in\tau\setminus\{w\}} F'_u\right) \subset F'_w \cap F_{\tau\setminus\{w\}} = \emptyset.$$

In both cases we obtain $\cap_{v\in\tau} F'_v = \emptyset$, and therefore

$$\cap_{v\in\sigma} F'_v \subset \cap_{v\in\tau} F'_v = \emptyset.$$

So, $\{F'_v\}_{v\in V}$ is a representation of X in $\mathbb{R}^d = \mathbb{R}^{|U|-1}$, as wanted. □

The proof of Theorem 6.3.2 is based on ideas developed by Wegner in his thesis [Weg67] (as presented in [Eck93, Tan13]). Indeed, we can think of Theorem 6.3.2 as an extension of the following result of Wegner:

Theorem 6.3.3 (Wegner [Weg67])**.** *Let X be a simplicial complex with n vertices. Then X is $(n-1)$-representable. Moreover, if X is not the complete $(n-2)$-dimensional complex, then it is $(n-2)$-representable.*

Proof. If X is the complete complex, then it is trivially 0-representable. Otherwise, let $U = V$. Since $V \notin X$ and $|\tau \setminus V| = 0 \leq 1$ for any missing face τ of X, then by Theorem 6.3.2, X is $(n-1)$-representable. If X is not the complete $(n-2)$-dimensional complex, then there exists some $U \subset V$ of size $n-1$ such that $U \notin X$. Since $|V \setminus U| \leq 1$, then $|\tau \setminus U| \leq 1$ for any missing face τ of X. Hence, by Theorem 6.3.2, X is $(n-2)$-representable. □

Theorem 1.4.4. *Let X be a simplicial complex on vertex set V. Let $V_1, \dots, V_k$ be subsets of V satisfying $V_i \notin X$ for all $i \in [k]$, such that for any missing face τ of X there exists some $i \in [k]$ satisfying $|\tau \setminus V_i| \le 1$. Then, X can be written as an intersection*

$$X = \cap_{i=1}^k X_i,$$

where, for all $i \in [k]$, X_i is a $(|V_i| - 1)$-representable complex. In particular, X is $\left(\sum_{i=1}^k (|V_i| - 1)\right)$-representable.

Proof. For $i \in [k]$, let $\mathcal{M}_i$ be the set consisting of all the missing faces τ of X such that $|\tau \setminus V_i| \le 1$. Let

$$X_i = \{\sigma \subset V : \tau \not\subset \sigma \text{ for all } \tau \in \mathcal{M}_i\}.$$

Note that $X = \cap_{i=1}^k X_i$. Indeed, if $\sigma \in X$, then σ does not contain any missing face of X; in particular, for all $i \in [k]$, σ does not contain any $\tau \in \mathcal{M}_i$. Therefore, $\sigma \in \cap_{i=1}^k X_i$. On the other hand, if $\sigma \notin X$, then $\tau \subset \sigma$ for some missing face τ of X. By the assumption of the theorem, there exists some $i \in k$ such that $\tau \in \mathcal{M}_i$. So, $\sigma \notin X_i$, and therefore $\sigma \notin \cap_{i=1}^k X_i$.

Let $i \in [k]$. The set of missing faces of X_i is exactly $\mathcal{M}_i$. Moreover, since $V_i \notin X$, there is some missing face τ of X such that $\tau \subset V_i$. Since $|\tau \setminus V_i| = 0 \le 1$, we have $\tau \in \mathcal{M}_i$; therefore, $V_i \notin X_i$. So, by Theorem 6.3.2, X_i is $(|V_i| - 1)$-representable.

Finally, by Lemma 6.1.2, X is $\left(\sum_{i=1}^k (|V_i| - 1)\right)$-representable. ☐

Remark. In [HW14, Theorem 1.2], an upper bound similar to the one in Theorem 1.4.4 is proved for the Leray number of a simplicial complex. Since $L(X) \le \text{rep}(X)$ for any complex X, we can see Theorem 1.4.4 as a generalization of that result.

6.4 Boxicity of complexes without large missing faces

In this section we prove our main result, Theorem 1.4.2.

First, we will need the following simple results about Steiner systems:

Lemma 6.4.1. *Let $\mathcal{F} \subset 2^V$ be a partial $(d, d+1, n)$-Steiner system. Then*

$$|\mathcal{F}| \le \left\lfloor \frac{1}{d+1} \binom{n}{d} \right\rfloor.$$

Moreover, if $|\mathcal{F}| = \frac{1}{d+1}\binom{n}{d}$, then $\mathcal{F}$ is a Steiner $(d, d+1, n)$-system.

Proof. Since $\mathcal{F}$ is a partial Steiner $(d, d+1, n)$-system, then any subset of V of size d is contained in at most one element of $\mathcal{F}$. On the other hand, since each $\sigma \in \mathcal{F}$ contains exactly $d+1$ subsets of size d, we obtain

$$(d+1)|\mathcal{F}| \le \binom{n}{d}. \tag{6.2}$$

Therefore,

$$|\mathcal{F}| \leq \left\lfloor \frac{1}{d+1}\binom{n}{d} \right\rfloor.$$

Now, assume that $|\mathcal{F}| = \frac{1}{d+1}\binom{n}{d}$. Then, equality must hold in (6.2). Thus, each subset of V of size d must be contained in exactly one set of $\mathcal{F}$. That is, $\mathcal{F}$ is a Steiner $(d, d+1, n)$-system. $\square$

Lemma 6.4.2. *Let $\mathcal{F} \subset 2^V$ be a $(d, d+1, n)$-Steiner system. Let $\tau \subset V$ be a set of size at most $d+1$ that is not contained in any set of $\mathcal{F}$. Then,*

$$|\{\sigma \in \mathcal{F} : |\tau \setminus \sigma| = 1\}| \geq d+1.$$

Proof. Since $\mathcal{F}$ forms a Steiner $(d, d+1, n)$-system, then any set of size at most d is contained in at least one set of $\mathcal{F}$. Therefore, we must have $|\tau| = d+1$. Now, let $\tau_1, \ldots, \tau_{d+1}$ be the subsets of τ of size d. Again, since $\mathcal{F}$ is a Steiner system, there exists $\sigma_1, \ldots, \sigma_{d+1} \in \mathcal{F}$ such that $\tau_i \subset \sigma_i$ for all $i \in [d+1]$.

Since τ is the only set of size $d+1$ containing two or more of the sets $\tau_1, \ldots, \tau_{d+1}$, but $\tau \notin \mathcal{F}$, we must have $\sigma_i \neq \sigma_j$ for all $i \neq j$. Thus,

$$|\{\sigma \in \mathcal{F} : |\tau \setminus \sigma| = 1\}| \geq |\{\sigma_1, \ldots, \sigma_{d+1}\}| = d+1.$$

$\square$

The last ingredient needed for the proof of Theorem 1.4.2 is the following result:

Proposition 6.4.3. *Let X be a simplicial complex on vertex set V of size n, satisfying $h(X) \leq d$. Let t be the minimum size of a family $\{\sigma_1, \ldots, \sigma_t\}$ of subsets of size $d+1$ of V satisfying $\sigma_i \notin X$ for all $i \in [t]$, such that for any missing face τ of X, there exists some $i \in [t]$ such that $|\tau \setminus \sigma_i| \leq 1$. Then,*

$$t \leq \left\lfloor \frac{1}{d+1}\binom{n}{d} \right\rfloor.$$

Moreover, if $h(X) = d \geq 2$, then $t = \frac{1}{d+1}\binom{n}{d}$ if and only if the set of missing faces of X forms a Steiner $(d, d+1, n)$-system.

Proof. Let $\mathcal{M}$ be the collection of all subsets of V of size $d+1$ that are not simplices of X.

Let $\mathcal{A} \subset \mathcal{M}$ be a maximal (with respect to inclusion) partial Steiner $(d, d+1, n)$-system. By Lemma 6.4.1, we have

$$|\mathcal{A}| \leq \left\lfloor \frac{1}{d+1}\binom{n}{d} \right\rfloor.$$

We will show that for any missing face τ of X, there exists some $\sigma \in \mathcal{A}$ such that $|\tau \setminus \sigma| \leq 1$. Assume for contradiction that there exists some missing face τ of X

such that $|\tau \setminus \sigma| > 1$ for all $\sigma \in \mathcal{A}$. Let σ_0 be some set in $\mathcal{M}$ containing τ. Then $|\sigma_0 \setminus \sigma| \geq |\tau \setminus \sigma| > 1$ for all $\sigma \in \mathcal{A}$. Let $\mathcal{A}' = \mathcal{A} \cup \{\sigma_0\}$. Let $\eta \subset V$ be a set of size d. If $\eta \not\subset \sigma_0$, then, since $\mathcal{A}$ is a partial Steiner $(d, d+1, n)$-system, η is contained in at most one set in $\mathcal{A}'$. If $\eta \subset \sigma_0$, then assume for contradiction that $\eta \subset \sigma$ for some $\sigma \in \mathcal{A}$. Since $|\sigma_0 \setminus \sigma| > 1$, we have $|\sigma_0 \cap \sigma| \leq d-1$. But this is a contradiction to the fact that η is a set of size d contained in $\sigma_0 \cap \sigma$. So, η is not contained in any set of $\mathcal{A}$. In both cases, η is contained in at most one set of $\mathcal{A}'$. Therefore, $\mathcal{A}' \subset \mathcal{M}$ is a partial Steiner $(d, d+1, n)$-system. But this is a contradiction to the maximality of $\mathcal{A}$.

Therefore, for any missing face τ of X there exists some $\sigma \in \mathcal{A}$ such that $|\tau \setminus \sigma| \leq 1$. Hence,

$$t \leq |\mathcal{A}| \leq \left\lfloor \frac{1}{d+1} \binom{n}{d} \right\rfloor.$$

Now, assume $t = \frac{1}{d+1}\binom{n}{d}$. Then, we must have $|\mathcal{A}| = t = \frac{1}{d+1}\binom{n}{d}$. By Lemma 6.4.1, $\mathcal{A}$ is a Steiner $(d, d+1, n)$-system.

Assume that $h(X) = d \geq 2$. We will show that $\mathcal{A}$ is exactly the set of missing faces of X:

We may assume that $n \geq d+2$. Otherwise, since $h(X) = d$, X must contain a unique missing face of size $d+1$ (that is, X is a Steiner $(d, d+1, d+1)$-system).

First, we will show that $\mathcal{A} = \mathcal{M}$. Assume for contradiction that there exists some $\tilde{\tau} \in \mathcal{M} \setminus \mathcal{A}$. By Lemma 6.4.2, there exist $\sigma_1, \sigma_2 \in \mathcal{A}$ such that $|\tilde{\tau} \setminus \sigma_1| = |\tilde{\tau} \setminus \sigma_2| = 1$. Since $|\tilde{\tau}| = d+1$, we also have $|\sigma_1 \setminus \tilde{\tau}| = |\sigma_2 \setminus \tilde{\tau}| = 1$. Let

$$\mathcal{A}' = \mathcal{A} \cup \{\tilde{\tau}\} \setminus \{\sigma_1, \sigma_2\}.$$

Let τ be a missing face of X. We will show that there exists some $\sigma \in \mathcal{A}'$ such that $|\tau \setminus \sigma| \leq 1$. We divide into the following cases:

1. If τ is not contained in any set of $\mathcal{A}$, then, by Lemma 6.4.2, we have

$$\begin{aligned} |\{\sigma \in \mathcal{A}' : |\tau \setminus \sigma| = 1\}| &\geq |\{\sigma \in \mathcal{A} : |\tau \setminus \sigma| = 1\}| - 2 \\ &\geq d + 1 - 2 = d - 1 \geq 1. \end{aligned}$$

   Therefore, there exists some $\sigma \in \mathcal{A}'$ such that $|\tau \setminus \sigma| = 1$.

2. If τ is contained in some $\sigma \in \mathcal{A} \setminus \{\sigma_1, \sigma_2\} \subset \mathcal{A}'$, then $|\tau \setminus \sigma| = 0 \leq 1$.

3. If τ is contained in σ_i for some $i \in \{1, 2\}$, then

$$|\tau \setminus \tilde{\tau}| \leq |\sigma_i \setminus \tilde{\tau}| = 1.$$

Since $|\mathcal{A}'| = t - 1$, this is a contradiction to the minimality of t. Hence, we must have $\mathcal{A} = \mathcal{M}$.

Finally, assume for contradiction that there exists some missing face τ of X of size $|\tau| \leq d$. Let η be a set of size d containing τ. Then , since we assumed $n \geq d+2$, we have

$$|\{\sigma \subset V : |\sigma| = d+1, \eta \subset \sigma\}| = n - d \geq 2.$$

Note that any $\sigma \subset V$ such that $|\sigma| = d+1$ and $\eta \subset \sigma$ is not a simplex of X (since it contains the missing face τ), and therefore belongs to $\mathcal{M} = \mathcal{A}$. Hence, η is contained in at least two sets of $\mathcal{A}$, a contradiction to $\mathcal{A}$ being a Steiner $(d, d+1, n)$-system. Thus, the set of missing faces of X is exactly $\mathcal{A}$. □

Theorem 1.4.2. *Let X be a simplicial complex with n vertices, satisfying $h(X) \leq d$. Then*

$$box_d(X) \leq \left\lfloor \frac{1}{d+1}\binom{n}{d} \right\rfloor.$$

Moreover, if $h(X) = d$, then $box_d(X) = \frac{1}{d+1}\binom{n}{d}$ if and only if the missing faces of X form a Steiner $(d, d+1, n)$-system.

Proof. Let $\{V_1, \ldots, V_t\}$ be a family of minimum size of subsets of size $d+1$ of V such that $V_i \notin X$ for all $i \in [t]$, and such that for any missing face τ of X, there exists some $i \in [t]$ satisfying $|\tau \setminus V_i| \leq 1$. By Theorem 1.4.4, we have $\text{box}_d(X) \leq t$. So, by Proposition 6.4.3, we obtain

$$\text{box}_d(X) \leq t \leq \left\lfloor \frac{1}{d+1}\binom{n}{d} \right\rfloor.$$

Now, assume that $h(X) = d$, and the set of missing faces of X does not form a Steiner $(d, d+1, n)$-system. If $d = 1$, then it is proved in [Wit80, Theorem 1] that $\text{box}_1(X) < \frac{n}{2}$. If $d \geq 2$ then, by Proposition 6.4.3, we have

$$t < \frac{1}{d+1}\binom{n}{d},$$

and therefore

$$\text{box}_d(X) \leq t < \frac{1}{d+1}\binom{n}{d}.$$

Finally, assume that the missing faces of X form a Steiner $(d, d+1, n)$-system $\mathcal{M}$. Then, by Theorem 1.4.3, we have

$$\text{box}_d(X) = |\mathcal{M}| = \frac{1}{d+1}\binom{n}{d},$$

as wanted. □

Remark. In the case $d = 1$, the proof of the upper bound in Theorem 1.4.2 reduces to the proof of Theorem 1.4.1 presented by Cozzens and Roberts in [CR83, Corollary 3.7].

6.5 Representability of complexes without large missing faces

Let X be a simplicial complex. By Lemma 6.1.2, we have for any $d \geq 1$,

$$\text{rep}(X) \leq d \cdot \text{box}_d(X).$$

In particular, for $d = 1$, we obtain as a corollary of Theorem 1.4.1:

Proposition 6.5.1. *Let G be a graph with n vertices, and let $X(G)$ be its clique complex. Then,*

$$rep(X(G)) \leq \left\lfloor \frac{n}{2} \right\rfloor.$$

Moreover, $rep(X(G)) = \frac{n}{2}$ if and only if G is the complete $\frac{n}{2}$-partite graph with all sides of size 2.

The fact that $\text{rep}(X(G)) = \frac{n}{2}$ if G is the complete $\frac{n}{2}$-partite graph with sides of size 2 does not follow directly from Theorem 1.4.1. However, it is easy to check that in this case $X(G)$ is the boundary of the $\frac{n}{2}$-dimensional cross-polytope; in particular, it has non-trivial $\left(\frac{n}{2} - 1\right)$-dimensional homology group. Thus, $X(G)$ is not $\left(\frac{n}{2} - 1\right)$-Leray, and therefore is not $\left(\frac{n}{2} - 1\right)$-representable.

We conjecture that for $d \geq 1$, the following extension of Proposition 6.5.1 holds:

Conjecture 1.4.5. *Let X be simplicial complex with n vertices, satisfying $h(X) \leq d$. Then,*

$$rep(X) \leq \left\lfloor \frac{dn}{d+1} \right\rfloor.$$

Moreover, $rep(X) = \frac{dn}{d+1}$ if and only if the missing faces of X consist of $\frac{n}{d+1}$ pairwise disjoint sets of size $d + 1$.

Analogous bounds are known to hold for Leray numbers (see [Ada14, Proposition 5.4]) and for collapsibility (see Proposition 4.2.2). Conjecture 1.4.5, if true, would imply both of these results.

The results presented here do not seem suitable for dealing with Conjecture 1.4.5. One of the simplest examples where our methods fail is the complex $X_{2,7}$, the complex whose set of missing faces forms a Steiner $(2, 3, 7)$-system (usually referred to as the Fano plane). Since any two vertices in $X_{2,7}$ are contained in a missing face, the best bound we can obtain from an application of Theorem 6.3.2 is $\text{rep}(X_{2,7}) \leq 5$, which is larger than the conjectured bound $\left\lfloor \frac{2 \cdot 7}{3} \right\rfloor = 4$. This bound can be proved, however, by the following simple method:

Lemma 6.5.2. *Let X be a d-representable simplicial complex on vertex set V. Let $\sigma_1, \sigma_2 \subset V$ such that $\sigma_1 \cap \sigma_2 \in X$. Then, the complex $X' = X \cup 2^{\sigma_1} \cup 2^{\sigma_2}$ is $(d + 1)$-representable.*

Proof. Let $e_1, \ldots, e_{d+1}$ be the standard basis for $\mathbb{R}^{d+1}$. We identify $\mathbb{R}^d$ with the hyperplane $H = \{x \in \mathbb{R}^{d+1} : x \cdot e_{d+1} = 0\}$ in $\mathbb{R}^{d+1}$.

Let $\mathcal{P} = \{P_v\}_{v\in V}$ be a representation of X in $\mathbb{R}^d$. Let $x \in \cap_{v\in\sigma_1\cap\sigma_2} P_v \subset H$ (note that $\cap_{v\in\sigma_1\cap\sigma_2} P_v \neq \emptyset$ since $\sigma_1 \cap \sigma_2 \in X$ and $\mathcal{P}$ is a representation of X). Let $x_1 = x + e_{d+1}$ and $x_2 = x - e_{d+1}$.

For $v \in V$, we define

$$P'_v = \begin{cases} \operatorname{conv}(P_v \cup \{x_1\} \cup \{x_2\}) & \text{if } v \in \sigma_1 \cap \sigma_2, \\ \operatorname{conv}(P_v \cup \{x_1\}) & \text{if } v \in \sigma_1 \setminus \sigma_2, \\ \operatorname{conv}(P_v \cup \{x_2\}) & \text{if } v \in \sigma_2 \setminus \sigma_1, \\ P_v & \text{if } v \notin \sigma_1 \cup \sigma_2. \end{cases}$$

We will show that $\mathcal{P}' = \{P'_v\}_{v\in V}$ is a representation of $X' = X \cup 2^{\sigma_1} \cup 2^{\sigma_2}$.

First, let $\sigma \in X'$. If $\sigma \in X$, then

$$\cap_{v\in\sigma} P'_v \supset \cap_{v\in\sigma} P_v \neq \emptyset,$$

since $\mathcal{P}$ is a representation of X. Otherwise, either $\sigma \subset \sigma_1$ or $\sigma \subset \sigma_2$. Assume without loss of generality that $\sigma \subset \sigma_1$. Then,

$$x_1 \in \cap_{v\in\sigma} P'_v,$$

so $\cap_{v\in\sigma} P'_v \neq \emptyset$.

For the second direction, we will need the following claim:

Claim 6.5.3. *Let $v \in V$. Then,*

$$P'_v \cap H = P_v.$$

Proof. If $v \notin \sigma_1 \cap \sigma_2$, then the claim follows immediately from the definition of P'_v. Assume that $v \in \sigma_1 \cap \sigma_2$. It is clear that $P_v \subset P'_v \cap H$. We will show that $P'_v \cap H \subset P_v$:

Let $y \in P'_v \cap H$. We can write $y = \alpha p + \beta x_1 + \gamma x_2$, where $p \in P_v$, $\alpha, \beta, \gamma \geq 0$ and $\alpha + \beta + \gamma = 1$. Since $y \in H$, we have

$$0 = y \cdot e_{d+1} = \alpha p \cdot e_{d+1} + \beta x_1 \cdot e_{d+1} + \gamma x_2 \cdot e_{d+1} = \beta - \gamma.$$

Hence,

$$y = \alpha p + \beta(x_1 + x_2) = \alpha p + 2\beta x.$$

Since $p, x \in P_v$ and P_v is convex, we obtain $y \in P_v$. So, $P'_v \cap H = P_v$, as wanted. □

Now, let $\sigma \subset V$ such that $\sigma \notin X'$. In particular, $\sigma \not\subset \sigma_1$ and $\sigma \not\subset \sigma_2$. Let $u \in \sigma \setminus \sigma_1$

and $w \in \sigma \setminus \sigma_2$. Then, we have $P'_u \subset H^-$ and $P'_w \subset H^+$, where

$$H^+ = \{x \in \mathbb{R}^{d+1} : x \cdot e_{d+1} \geq 0\}$$

and

$$H^- = \{x \in \mathbb{R}^{d+1} : x \cdot e_{d+1} \leq 0\}.$$

Therefore,

$$\cap_{v \in \sigma} P'_v \subset H^+ \cap H^- = H.$$

So, by Claim 6.5.3, we have

$$\cap_{v \in \sigma} P'_v = \cap_{v \in \sigma} P'_v \cap H = \cap_{v \in \sigma} P_v = \emptyset,$$

where the last equality follows since $\mathcal{P}$ is a representation of X and $\sigma \notin X$.

Hence, $\mathcal{P}'$ is a representation of X' in $\mathbb{R}^{d+1}$. $\square$

Proposition 6.5.4.

$$rep(X_{2,7}) \leq 4.$$

Proof. We identify the vertex set of $X_{2,7}$ with the set $[7] = \{1, 2, \ldots, 7\}$. Then, the set of missing faces of $X_{2,7}$ is the set

$$\mathcal{M} = \{\{1,2,3\}, \{1,4,5\}, \{1,6,7\}, \{2,4,7\}, \{3,4,6\}, \{2,5,6\}, \{3,5,7\}\}.$$

It is easy to check that the set of maximal faces of $X_{2,7}$ is the set whose elements are the complements of the sets in $\mathcal{M}$:

$$\{\{4,5,6,7\}, \{2,3,6,7\}, \{2,3,4,5\}, \{1,3,5,6\},$$
$$\{1,2,5,7\}, \{1,3,4,7\}, \{1,2,4,6\}\}.$$

Let X_0 be the complex on vertex set $[7]$ whose set of maximal faces is:

$$\{\{1,2,4\}, \{2,3,4,5\}, \{4,5,6,7\}\}.$$

It can be checked that the following is a representation of X_0 in $\mathbb{R}^1$:

$$P_1 = [0,1], \qquad P_2 = [1,2],$$
$$P_3 = [2,3], \qquad P_4 = [0,5],$$
$$P_5 = [2,5], \qquad P_6 = P_7 = [4,5].$$

Let $X_1 = X_0 \cup 2^{\{1,2,5,7\}} \cup 2^{\{1,2,4,6\}}$. Since $\{1,2,5,7\} \cap \{1,2,4,6\} = \{1,2\} \in X_0$ then, by Lemma 6.5.2, X_1 is 2-representable.

Let $X_2 = X_1 \cup 2^{\{1,3\}} \cup 2^{\{2,3,6,7\}}$. Since $\{1,3\} \cap \{2,3,6,7\} = \{3\} \in X_1$ then, by Lemma 6.5.2, X_2 is 3-representable.

Finally, let $X_3 = X_2 \cup 2^{\{1,3,5,6\}} \cup 2^{\{1,3,4,7\}}$. Since $\{1,3,5,6\} \cap \{1,3,4,7\} = \{1,3\} \in X_2$ then, by Lemma 6.5.2, X_3 is 4-representable. But it is easy to check that X_3 is in fact the complex $X_{2,7}$. $\square$

Lemma 6.5.2 gives non-trivial bounds only for complexes with a small number of maximal faces, so it seems unlikely that such a method will be useful in more general cases of our problem.

We conclude with the following problem, whose solution may be a (very modest) step towards Conjecture 1.4.5:

Conjecture 6.5.5. *Let $X_{2,9}$ be the simplicial complex whose missing faces form a Steiner $(2,3,9)$-system (that is, they are the lines of the affine plane of order 3). Then,*

$$rep(X_{2,9}) \leq 5.$$

Chapter 7

Complexes of line-free sets in finite affine planes

This chapter is organized as follows. In Section 7.1 we present an outline of the proof of Theorem 1.5.2. In Section 7.2 we prove Theorem 1.5.3 about stable and strongly stable blocking sets. In Section 7.3 we study the homology of certain subcomplexes of the complexes X_q and $\hat{X}_q$; this is the last step in the proof of Theorem 1.5.2. Section 7.4 deals with complexes of hyperplane-free sets in n-dimensional finite affine spaces. We present a conjecture about the top-dimensional homology groups of these complexes, extending Theorem 1.5.2. We present a possible direction for proving this conjecture, involving a conjectural characterization of strongly stable blocking sets in finite affine spaces, generalizing Theorem 1.5.3.

7.1 Proof outline

Let $L_1, \ldots, L_{q+1} \subset \mathbb{F}_q^2$ be the lines throught the origin. For $i \in [q+1]$, let $L_{i,1} = L_i, L_{i,2}, \ldots, L_{i,q}$ be the translates of L_i.

For any set V, we define the simplicial complex ∂V as

$$\partial V = \{S \subset V : S \neq V\}.$$

The complex ∂V is the boundary of the simplex V; Therefore, it is homeomorphic to a $(|V|-2)$-dimensional sphere.

For $i \in [q+1]$ and $j \in [q]$, let

$$K_{i,j} = \partial L_{i,1} * \partial L_{i,2} * \cdots * \partial L_{i,j-1} * \partial L_{i,j+1} * \cdots * \partial L_{i,q}.$$

Note that all the sets in $K_{i,j}$ are line-free. Therefore, $K_{i,j} \subset X_q$ for $i \in [q+1]$ and $j \in [q]$, and $K_{i,1} \subset \hat{X}_q$ for $i \in [q+1]$. Moreover, the complex $K_{i,j}$ is homeomorphic to a (q^2-2q)-dimensional sphere. Also, note that the sets in $K_{i,j}$ are exactly the complements of the blocking sets of size $2q-1$ containing the line $L_{i,j}$.

Let $Y_q = \cup_{i=1}^{q+1} \cup_{j=1}^{q} K_{i,j}$ and $\hat{Y}_q = \cup_{i=1}^{q+1} K_{i,1}$. Note that Y_q is a subcomplex of X_q and $\hat{Y}_q$ is a subcomplex of $\hat{X}_q$.

The first step in the proof of Theorem 1.5.2 consists on relating the top-dimensional homology of X_q and $\hat{X}_q$ to the top-dimensional homology of the subcomplexes Y_q and $\hat{Y}_q$, respectively.

Proposition 7.1.1. *Let q be a prime power. Then,*

$$\tilde{H}_{q^2-2q}(X_q) = \tilde{H}_{q^2-2q}(Y_q),$$

and

$$\tilde{H}_{q^2-2q}(\hat{X}_q) = \tilde{H}_{q^2-2q}(\hat{Y}_q).$$

The proof of Proposition 7.1.1 relies on the study of blocking sets in $\mathbb{F}_q^2$ having certain stability property.

By Proposition 7.1.1, it is enough to compute the top-dimensional homology of the subcomplexes Y_q and $\hat{Y}_q$:

Proposition 7.1.2.

$$\tilde{H}_{q^2-2q}(Y_q) = \begin{cases} \mathbb{Z}^3 & \text{if } q = 2, \\ \mathbb{Z}^{11} & \text{if } q = 3, \\ \mathbb{Z}^{q(q+1)} & \text{otherwise.} \end{cases}$$

Proposition 7.1.3. *For $q = 2$, we have*

$$\tilde{H}_i(\hat{Y}_2) = \begin{cases} \mathbb{Z}^2 & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

For $q \geq 3$,

$$\tilde{H}_i(\hat{Y}_q) = \begin{cases} \mathbb{Z} & \text{if } i = q-1, \\ \mathbb{Z}^{q+1} & \text{if } i = q^2-2q, \\ 0 & \text{otherwise.} \end{cases}$$

Note that, in the case of $\hat{Y}_q$, we understand the homology in all dimensions.

Theorem 1.5.2.

$$\tilde{H}_{q^2-2q}(X_q) = \begin{cases} \mathbb{Z}^3 & \text{if } q = 2, \\ \mathbb{Z}^{11} & \text{if } q = 3, \\ \mathbb{Z}^{q(q+1)} & \text{if } q > 3, \end{cases}$$

and

$$\tilde{H}_{q^2-2q}(\hat{X}_q) = \begin{cases} \mathbb{Z}^2 & \text{if } q = 2, \\ \mathbb{Z}^{q+1} & \text{if } q > 2. \end{cases}$$

Proof. The claim follows immediately from Propositions 7.1.1, 7.1.2 and 7.1.3. $\square$

7.2 Stable blocking sets

Our main goal in this section is to prove Proposition 7.1.1. In order to do this, we need to study blocking sets satisfying certain stability property:

Let B be a blocking set in $\mathbb{F}_q^2$ of size $2q-1$. Recall that B is called *stable* if for every point $v \notin B$ there is some $u \in B$ such that $B \cup \{v\} \setminus \{u\}$ is also a blocking set. B is called *strongly stable* if $0 \in B$ and for every point $v \notin B$ there is some $u \in B \setminus \{0\}$ such that $B \cup \{v\} \setminus \{u\}$ is also a blocking set.

Our main goal in this section is to prove the following characterizations of stable and strongly stable blocking sets:

Theorem 1.5.3. *Let B be a blocking set in $\mathbb{F}_q^2$ of size $2q-1$. Then, B is stable if and only if B contains an affine line, and it is strongly stable if and only if it contains a line through the origin.*

Before proceeding to the proof of Theorem 1.5.3, let us see how it implies Proposition 7.1.1:

Proof of Proposition 7.1.1. Since $Y_q \subset X_q$, we have $Z_{q^2-2q}(Y_q) \subset Z_{q^2-2q}(X_q)$. We are left to show that $Z_{q^2-2q}(X_q) \subset Z_{q^2-2q}(Y_q)$.

Let $0 \neq z \in Z_{q^2-2q}(X_q)$, and let $\sigma \in X_q(q^2-2q)$ be a simplex in the support of z. We will show that $\sigma \in Y_q$:

Note that for any $v \in \sigma$, there exists a vertex $u \in \mathbb{F}_q^2 \setminus \sigma$ such that $\sigma \setminus \{v\} \cup \{u\} \in X_q$. Otherwise, the simplex $\sigma \setminus \{v\}$ must belong to the support of ∂z, a contradiction to $\partial z = 0$.

This is equivalent to the set $B = \mathbb{F}_q^2 \setminus \sigma$ being a stable blocking set. Hence, by Theorem 1.5.3, B contains an affine line. That is, $\sigma \in Y_q$. So, $Z_{q^2-2q}(X_q) \subset Z_{q^2-2q}(Y_q)$.

We obtain

$$\tilde{H}_{q^2-2q}(X_q) = Z_{q^2-2q}(X_q) = Z_{q^2-2q}(Y_q) = \tilde{H}_{q^2-2q}(Y_q),$$

as wanted.

The proof of $\tilde{H}_{q^2-2q}(\hat{X}_q) = \tilde{H}_{q^2-2q}(\hat{Y}_q)$ is similar. $\square$

For the proof of Theorem 1.5.3, we will need the following equivalent definitions for stable and strongly stable blocking sets:

Let $B \subset \mathbb{F}_q^2$ be a blocking set of size $2q-1$. We say that an affine line L is *tangent to B at the point u* if $B \cap L = \{u\}$. Note that for any $u \in B$, there is at least one line tangent to B at u (otherwise $B \setminus \{u\}$ is a blocking set, a contradiction to the minimality of B).

Let

$$B_1 = \{u \in B : \text{ there is a unique line tangent to } B \text{ at } u\},$$

and

$$B_0 = B \setminus B_1 = \{u \in B : \text{ there are at least two lines tangent to } B \text{ at } u\}.$$

For $u \in B_1$, let $L_B(u)$ be the unique line tangent to B at u.

Lemma 7.2.1. *Let $B \subset \mathbb{F}_q^2$ be a blocking set of size $2q-1$. Let $u \in B$ and $v \in \mathbb{F}_q^2 \setminus B$. Then, $B' = B \setminus \{u\} \cup \{v\}$ is a blocking set if and only if $u \in B_1$ and $v \in L_B(u)$.*

Proof. The set B' is a blocking set if and only if v is contained in all the lines tangent to B at u. But, since through every two points passes a unique line, v can be contained in at most one line tangent to B at u. Therefore, B' is a blocking set if and only if $u \in B_1$ and $v \in L_B(u)$.

Lemma 7.2.2. *Let $B \subset \mathbb{F}_q^2$ be a blocking set of size $2q-1$. Then, B is stable if and only if*

$$\bigcup_{u \in B_1} L_B(u) \supset \mathbb{F}_q^2 \setminus B.$$

If $0 \in B$, then B is strongly stable if and only if

$$\bigcup_{u \in B_1 \setminus \{0\}} L_B(u) \supset \mathbb{F}_q^2 \setminus B.$$

Proof. Follows immediately from Lemma 7.2.1. □

Lemma 7.2.3. *Let $B \subset \mathbb{F}_q^2$ be a blocking set of size $2q-1$. Let $B' \subset B_1$ such that*

$$\bigcup_{u \in B'} L_B(u) \supset \mathbb{F}_q^2 \setminus B.$$

Then, there exists an affine line L such that $B \setminus B' \subset L \subset B$.

For the proof of Lemma 7.2.3 we will need the following dual version of the Jamison-Brouwer-Schrijver Theorem:

Theorem 7.2.4 (Jamison [Jam77], Brouwer-Schrijver [BS78])**.** *Let $\mathcal{L}$ be a family of affine lines in $\mathbb{F}_q^2$. If $\cup_{L \in \mathcal{L}} L = \mathbb{F}_q^2 \setminus \{u\}$ for some $u \in \mathbb{F}_q^2$, then $|\mathcal{L}| \geq 2(q-1)$.*

We will also need the following simple results about lines tangent to blocking sets:

Lemma 7.2.5. *Let $B \subset \mathbb{F}_q^2$ be a blocking set of size $2q-1$. Then, every $v \notin B$ intersects at least three different lines tangent to B.*

Proof. Let $v \notin B$. Let $\ell_1, \ldots, \ell_{q+1}$ be the lines passing through v. For $i \in [q+1]$, let $A_i = \ell_i \setminus \{v\}$. Note that the sets $A_1, \ldots, A_{q+1}$ are pairwise disjoint. Since B is a blocking set, and $v \notin B$, we have $B \cap A_i \neq \emptyset$ for all $i \in [q+1]$. Let t be the number of lines through v that are tangent to B. That is,

$$t = |\{i \in [q+1] : |A_i \cap B| = 1\}|.$$

Then, we have

$$2q - 1 = |B| \geq \left| B \cap \left(\cup_{i=1}^{q+1} A_i \right) \right| = \sum_{i=1}^{q+1} |B \cap A_i| \geq t + 2(q + 1 - t).$$

We obtain $t \geq 3$, as wanted. ☐

Corollary 7.2.6. *Let $B \subset \mathbb{F}_q^2$ be a blocking set of size $2q - 1$. Then, there are at least $3(q-1)$ lines tangent to B.*

Proof. There are $q^2 - (2q-1) = (q-1)^2$ points in $\mathbb{F}_q^2 \setminus B$. By Lemma 7.2.5, there are at least three tangents to B passing through each one of them. Each tangent to B contains exactly $q - 1$ points in $\mathbb{F}_q^2 \setminus B$. Therefore, there are at least $3(q-1)^2/(q-1) = 3(q-1)$ lines tangent to B. ☐

Lemma 7.2.7. *Let $q \geq 3$ be a prime power. Let B be a blocking set of $\mathbb{F}_q^2$ of size $2q-1$. Then, $B_0 \neq \emptyset$. Moreover, if $|B_0| = 1$, then B is the union of two lines passing through the unique point in B_0.*

Proof. Assume for contradiction that $B_0 = \emptyset$. Then, we have $B = B_1$. That is, for each point $u \in B$ there is a unique line tangent to B at u. But then, there must be exactly $2q - 1$ lines tangent to B. Since $q \geq 3$, we have $2q - 1 < 3(q-1)$, a contradiction to Corollary 7.2.6.

Now, assume that $|B_0| = 1$. Let u be the unique point in B_0. By Lemma 7.2.5, there are at least $3(q-1)$ lines tangent to B. Since for any $w \in B \setminus \{u\}$ there is exactly one line tangent to B at w, then there must be at least $q - 1$ lines tangent to B at u. Denote these lines by $L'_1, \ldots, L'_t$, where $t \geq q - 1$. Denote the rest of the lines through u by $L'_{t+1}, \ldots, L'_{q+1}$.

Since $\cup_{i=1}^{q+1} L'_i = \mathbb{F}_q^2$ and $B \cap L'_i = \{u\}$ for $i \in [t]$, we have

$$B \setminus \{u\} \subset \bigcup_{t+1 \leq i \leq q+1} L'_i \setminus \{u\}.$$

Thus, since $|B| = 2q - 1$, we have

$$2q - 2 = |B \setminus \{u\}| \leq \left| \bigcup_{t+1 \leq i \leq q+1} L'_i \setminus \{u\} \right| = (q + 1 - t)(q - 1).$$

We obtain $t \leq q-1$. Therefore, we have $t = q-1$ and $B \subset L'_q \cup L'_{q+1}$. Since $|B| = 2q-1 = |L'_q \cup L'_{q+1}|$, we have in fact $B = L'_q \cup L'_{q+1}$. That is, B is the union of two lines passing through the unique point in B_0. □

Proof of Lemma 7.2.3. For $q = 2$ the claim holds trivially. Therefore, we will assume that $q \geq 3$.

By Lemma 7.2.7, since $B_0 \subset B \setminus B'$, we have $B \setminus B' \neq \emptyset$. We divide into two cases:

Assume that $|B \setminus B'| = 1$. Since $\emptyset \neq B_0 \subset B \setminus B'$, we have in fact $|B_0| = |B \setminus B'| = 1$. Let u be the unique point in $B_0 = B \setminus B'$. Then, by Lemma 7.2.7, B is the union of two lines passing through u. In particular, B contains a line L that contains $B \setminus B' = \{u\}$.

Now, assume that $|B \setminus B'| \geq 2$. First, we will show that there is a line L such that $B \setminus B' \subset L$. If $|B \setminus B'| = 2$, this holds trivially. Otherwise, assume that $|B \setminus B'| \geq 3$, and assume for contradiction that the points in $B \setminus B'$ are not contained in a line. Then, there are three points $u_1, u_2, u_3 \in B \setminus B'$ such that the line L' that passes through u_1 and u_2 does not contain u_3. For each $u \in B \setminus (B' \cup \{u_1, u_2, u_3\})$, let L'_u be a line passing through u that does not contain u_3. Then, the family

$$\mathcal{L} = \{L_B(u)\}_{u \in B'} \cup \{L'_u : u \in B \setminus (B' \cup \{u_1, u_2, u_3\})\} \cup \{L'\}$$

is a family of lines satisfying

$$\bigcup_{\ell \in \mathcal{L}} \ell = \mathbb{F}_q^2 \setminus \{u_3\}.$$

By Theorem 7.2.4, we must have

$$|\mathcal{L}| \geq 2(q-1).$$

On the other hand, we have

$$|\mathcal{L}| = |B'| + (|B \setminus B'| - 3) + 1 = |B| - 2 = 2q - 3 < 2(q-1),$$

a contradiction. Therefore, all the points in $B \setminus B'$ are contained in some affine line L.

Next, we show that L is contained in B. Let $W = L \setminus B$. We want to show that $W = \emptyset$. Let $\mathcal{L}$ be the set of lines parallel to L (other than L). Let $\mathcal{T} \subset \mathcal{L}$ consist of the lines tangent to B and $\mathcal{N} \subset \mathcal{L}$ consist of the lines that are not tangent to B.

For $w \in W$, let $\mathcal{T}_w$ be the set of lines tangent to B at w. By Lemma 7.2.5, $|\mathcal{T}_w| \geq 3$. Note that $L \notin \cup_{w \in W} \mathcal{T}_w$, since $|L \cap B| \geq |B \setminus B'| \geq 2$. Therefore, each line in $\cup_{w \in W} \mathcal{T}_w$ intersects B at some point in $B \setminus L$.

Any two lines in $\cup_{w \in W} \mathcal{T}_w$ intersect $B \setminus L$ at a different point; otherwise, assume there exists $L', L'' \in \cup_{w \in W} \mathcal{T}_w$ that intersect at the point $u \in B \setminus L$. Then, since there are at least two lines tangent to B at u, we have $u \in B_0$. But $B_0 \subset B \setminus B' \subset L$, a contradiction to $u \in B \setminus L$.

Moreover, each line $L' \in \cup_{w \in W} \mathcal{T}_w$ must intersect $B \setminus L$ at a point lying in one of the lines of $\mathcal{N}$. Otherwise, let u be the unique point in $B \cap L'$, and assume that $u \in L''$, for some $L'' \in \mathcal{T}$. Then, both L' and L'' are lines tangent to B at u. Thus, $u \in B_0$, again a contradiction to $B_0 \subset L$.

Let t be the number of points of B that are contained in one of the lines in $\mathcal{N}$. Then, we obtain

$$t \geq |\cup_{w\in W} \mathcal{T}_w| \geq 3|W|.$$

Also, since each line in $\mathcal{N}$ contains at least 2 points of B, we have

$$t \geq 2|\mathcal{N}|,$$

On the other hand, we have

$$\begin{aligned} t \leq |B \setminus L| - |\mathcal{T}| &= |B| - |L| + |L \setminus B| - |\mathcal{T}| \\ &= (2q-1) - q + |W| - |\mathcal{T}| = |W| + (q - 1 - |\mathcal{T}|) = |W| + |\mathcal{N}|. \end{aligned}$$

We obtain

$$2|\mathcal{N}| + 3|W| \leq 2t \leq 2|\mathcal{N}| + 2|W|.$$

Therefore,

$$|W| \leq 0.$$

Thus, $W = \emptyset$, as wanted. $\square$

Proof of Theorem 1.5.3. First, assume that B contains an affine line L. Let $L'_1, \dots, L'_{q-1}$ be the lines parallel to L (other than L). Then, we must have $B = L \cup \{u_i\}_{i=1}^{q-1}$, where $u_i \in L'_i$ for all $i \in [q-1]$. Let $v \notin B$. Then, $v \in L'_i$ for some $i \in [q-1]$, so the set $B \cup \{v\} \setminus \{u_i\}$ is also a blocking set. Hence, B is stable.

Similarly, if B contains a complete line through the origin, it is strongly stable.

Now, assume that B is stable. By Lemma 7.2.2,

$$\bigcup_{u \in B_1} L_B(u) \supset \mathbb{F}_q^2 \setminus B.$$

By Lemma 7.2.3, there exists an affine line L such that $L \subset B$.

Finally, assume that B is strongly stable. Then, by Lemma 7.2.2,

$$\bigcup_{u \in B_1 \setminus \{0\}} L_B(u) \supset \mathbb{F}_q^2 \setminus B.$$

By Lemma 7.2.3, there exists an affine line L such that $B \setminus (B_1 \setminus \{0\}) \subset L \subset B$. In particular, $0 \in L$. That is, L is a line through the origin. $\square$

7.3 The homology of the subcomplexes Y_q and $\hat{Y}_q$

In this section we prove Propositions 7.1.2 and 7.1.3.

Proof of Proposition 7.1.2. If $q = 2$ then Y_2 consists of just 4 isolated vertices; hence, $\tilde{H}_0(Y_2) = \mathbb{Z}^3$. For $q = 3$ and $q = 4$, it may be verified by computer that $\tilde{H}_3(Y_3) = \mathbb{Z}^{11}$ and $\tilde{H}_8(Y_4) = \mathbb{Z}^{20}$.

Let $q \geq 5$. Let $m = q^2 - 2q$. Let Z be the subcomplex of Y_q whose maximal faces are the simplices of the form $\mathbb{F}_q^2 \setminus (L \cup L')$, where L and L' are two non-parallel lines in $\mathbb{F}_q^2$.

Claim 7.3.1. *Let $i \in [q+1]$ and $j \in [q]$. Let $\mathcal{L}$ be the set of affine lines that are not parallel to the line $L_{i,j}$. Then,*

$$Z \cap K_{i,j} = \bigcup_{L \in \mathcal{L}} 2^{\mathbb{F}_q^2 \setminus (L_{i,j} \cup L)}.$$

Proof. First, note that $\mathbb{F}_q^2 \setminus (L_{i,j} \cup L) \in Z \cap K_{i,j}$ for every $L \in \mathcal{L}$. Hence, $Z \cap K_{i,j} \supset \cup_{L \in \mathcal{L}} 2^{\mathbb{F}_q^2 \setminus (L_{i,j} \cup L)}$.

On the other direction, let $\sigma \in Z \cap K_{i,j}$. Then, since $\sigma \in Z$, we have $\sigma \subset \mathbb{F}_q^2 \setminus (L' \cup L'')$ for some non-parallel lines L', L''. Moreover, since $\sigma \in K_{i,j}$, we have $\sigma \cap L_{i,j} = \emptyset$. That is, $\sigma \subset \mathbb{F}_q^2 \setminus (L' \cup L'' \cup L_{i,j})$. At least one of the lines L' or L'' is not parallel to $L_{i,j}$. Assume without loss of generality that L' is not parallel to $L_{i,j}$. Then, $\sigma \subset \mathbb{F}_q^2 \setminus (L_{i,j} \cup L') \in \bigcup_{L \in \mathcal{L}} 2^{\mathbb{F}_q^2 \setminus (L_{i,j} \cup L)}$. Thus, $Z \cap K_{i,j} \subset \cup_{L \in \mathcal{L}} 2^{\mathbb{F}_q^2 \setminus (L_{i,j} \cup L)}$. $\square$

Claim 7.3.2. *Let k be an integer, and let $L_1', L_1'', \ldots, L_k', L_k''$ be a family of affine lines, such that $L_i' \neq L_i''$ for all $i \in [k]$. Then,*

$$\tilde{H}_j \left(\cup_{i=1}^k 2^{\mathbb{F}_q^2 \setminus (L_i' \cup L_i'')} \right) = 0$$

for $j \geq m - 1$. In particular,

$$\tilde{H}_j(Z) = 0$$

and (by Claim 7.3.1)

$$\tilde{H}_j(Z \cap K_{i,r}) = 0$$

for $j \geq m - 1$, $i \in [q+1]$ and $r \in [q]$.

Proof. We argue by induction on k. For $k = 1$, the complex $2^{\mathbb{F}_q^2 \setminus (L_1' \cup L_1'')}$ is contractible. In particular, $\tilde{H}_j(2^{\mathbb{F}_q^2 \setminus (L_1' \cup L_1'')}) = 0$ for all j.

Let $k \geq 2$. Let

$$K = \cup_{i=1}^k 2^{\mathbb{F}_q^2 \setminus (L_i' \cup L_i'')}$$

and

$$K' = \cup_{i=1}^{k-1} 2^{\mathbb{F}_q^2 \setminus (L_i' \cup L_i'')}.$$

By the induction hypothesis, we have $\tilde{H}_j(K') = 0$ for $j \geq m-1$.

We may assume that $\{L_k, L'_k\} \neq \{L_i, L'_i\}$ for all $i < k$ (otherwise $K = K'$, and by the induction hypothesis the claim holds). By the Mayer-Vietoris exact sequence (Theorem 2.2.1), we have a long exact sequence

$$\cdots \to \tilde{H}_j(K') \bigoplus \tilde{H}_j(2^{\mathbb{F}_q^2 \setminus (L'_k \cup L''_k)}) \to \tilde{H}_j(K) \to \tilde{H}_{j-1}(K' \cap 2^{\mathbb{F}_q^2 \setminus (L'_k \cup L''_k)}) \to \cdots .$$

Let $\sigma \in K' \cap 2^{\mathbb{F}_q^2 \setminus (L'_k \cup L''_k)}$. Then, $\sigma \subset \mathbb{F}_q^2 \setminus (L'_i \cup L''_i \cup L'_k \cup L''_k)$ for some $i < k$. Since $\{L'_k, L''_k\} \neq \{L'_i, L''_i\}$, then at least three of these lines are distinct. Without loss of generality, assume that L'_i, L''_i, L'_k are pairwise distinct lines. Then,

$$|\sigma| \leq |\mathbb{F}_q^2 \setminus (L'_i \cup L''_i \cup L'_k)| \leq q^2 - 3q + 3 = m - q + 3.$$

Hence, $\dim(K' \cap 2^{\mathbb{F}_q^2 \setminus (L'_k \cup L''_k)}) \leq m - q + 2 \leq m - 3$. In particular, for $j \geq m-1$, we have

$$\tilde{H}_{j-1}(K' \cap 2^{\mathbb{F}_q^2 \setminus (L'_k \cup L''_k)}) = 0.$$

By the induction hypothesis, we have, for $j \geq m-1$,

$$\tilde{H}_j(K') \bigoplus \tilde{H}_j(2^{\mathbb{F}_q^2 \setminus (L'_k \cup L''_k)}) = 0.$$

Therefore, we obtain $\tilde{H}_j(K) = 0$ for $j \geq m-1$, as wanted. □

Claim 7.3.3. *We have*

$$\tilde{H}_m(Y_q) \cong H_m(Y_q, Z)$$

and, for $i \in [q+1]$, $j \in [q]$,

$$H_m(K_{i,j}, Z \cap K_{i,j}) \cong \mathbb{Z}.$$

Proof. We have a long exact sequence of a pair (Theorem 2.2.6)

$$\cdots \to \tilde{H}_m(Z) \to \tilde{H}_m(Y_q) \to H_m(Y_q, Z) \to \tilde{H}_{m-1}(Z) \to \cdots$$

By Claim 7.3.2, $\tilde{H}_m(Z) = \tilde{H}_{m-1}(Z) = 0$. Therefore, we obtain

$$H_m(Y_q, Z) \cong \tilde{H}_m(Y_q).$$

Similarly, we have a long exact sequence

$$\cdots \to \tilde{H}_m(Z \cap K_{i,j}) \to \tilde{H}_m(K_{i,j}) \to H_m(K_{i,j}, Z \cap K_{i,j}) \to \tilde{H}_{m-1}(Z \cap K_{i,j}) \to \cdots$$

By Claim 7.3.2, $\tilde{H}_m(Z \cap K_{i,j}) = \tilde{H}_{m-1}(Z \cap K_{i,j}) = 0$. Therefore, we obtain

$$H_m(K_{i,j}, Z \cap K_{i,j}) \cong \tilde{H}_m(K_{i,j}) \cong \mathbb{Z}.$$

Claim 7.3.4. *Let $(i_1, j_1), \dots, (i_k, j_k)$ be a family of pairwise distinct pairs of indices, such that $i_r \in [q+1]$ and $j_r \in [q]$ for all $r \in [k]$. Then,*

$$H_m\left(\cup_{r=1}^{k} K_{i_r,j_r}, \cup_{r=1}^{k} K_{i_r,j_r} \cap Z\right) = \mathbb{Z}^k.$$

In particular,

$$H_m\left(Y_q, Z\right) = \mathbb{Z}^{q(q+1)}.$$

Proof. We argue by induction on k. For $k = 1$ we have, by Claim 7.3.3, $H_m\left(K_{i_1,j_1}, Z \cap K_{i_1,j_1}\right) = \mathbb{Z}$, as wanted. Let $k \geq 2$. Let

$$K = \cup_{r=1}^{k} K_{i_r,j_r}$$

and

$$K' = \cup_{r=1}^{k-1} K_{i_r,j_r}$$

By the induction hypothesis, we have $H_m\left(K', K' \cap Z\right) = \mathbb{Z}^{k-1}$.

By the relative version of Mayer-Vietoris (Theorem 2.2.7), we have the long exact sequence:

$$\begin{aligned} \cdots \to H_m\left(K' \cap K_{i_k,j_k}, K' \cap K_{i_k,j_k} \cap Z\right) \to \\ \to H_m\left(K', K' \cap Z\right) \bigoplus H_m\left(K_{i_k,j_k}, K_{i_k,j_k} \cap Z\right) \to \\ \to H_m\left(K, K \cap Z\right) \to H_{m-1}\left(K' \cap K_{i_k,j_k}, K' \cap K_{i_k,j_k} \cap Z\right) \to \cdots \end{aligned} \tag{7.1}$$

Let $\sigma \in K' \cap K_{i_k,j_k}$. Since $\sigma \in K'$, there exist some $r < k$ such that $\sigma \in K_{i_r,j_r}$. In particular, $\sigma \subset \mathbb{F}_q^2 \setminus L_{i_r,j_r}$. Similarly, since $\sigma \in K_{i_k,j_k}$, we have $\sigma \subset \mathbb{F}_q^2 \setminus L_{i_k,j_k}$. Therefore,

$$\sigma \subset \mathbb{F}_q^2 \setminus (L_{i_r,j_r} \cup L_{i_k,j_k}).$$

If $\dim(\sigma) = m$, then L_{i_r,j_r} and L_{i_k,j_k} must be non-parallel (otherwise $|\sigma| \leq m$, a contradiction to $\dim(\sigma) = m$), and therefore $\sigma \in Z$. Thus, $(K' \cap K_{i_k,j_k}) \setminus Z$ does not contain m-dimensional simplices. Hence,

$$H_m\left(K' \cap K_{i_k,j_k}, K' \cap K_{i_k,j_k} \cap Z\right) = 0.$$

Assume that $\dim(\sigma) = m - 1$. Also in this case the lines L_{i_r,j_r} and L_{i_k,j_k} must be non-parallel; otherwise, since $|\sigma| = m = q^2 - 2q$, we must have $\sigma = \mathbb{F}_q^2 \setminus (L_{i_r,j_r} \cup L_{i_k,j_k})$, a contradiction to $\sigma \in K_{i_k,j_k}$. Therefore, L_{i_r,j_r} and L_{i_k,j_k} are non-parallel and $\sigma = \mathbb{F}_q^2 \setminus (L_{i_r,j_r} \cup L_{i_k,j_k} \cup \{p\})$ for some $p \in \mathbb{F}_q^2 \setminus (L_{i_r,j_r} \cup L_{i_k,j_k})$. In particular, $\sigma \in Z$. Thus, $(K' \cap K_{i_k,j_k}) \setminus Z$ does not contain $(m-1)$-dimensional simplices either. Therefore, we have

$$H_{m-1}\left(K' \cap K_{i_k,j_k}, K' \cap K_{i_k,j_k} \cap Z\right) = 0.$$

Hence, by (7.1) and the induction hypothesis, we obtain

$$H_m(K, K\cap Z) \cong H_m\left(K', K'\cap Z\right) \bigoplus H_m\left(K_{i_k,j_k}, K_{i_k,j_k}\cap Z\right) \cong \mathbb{Z}^{k-1} \bigoplus \mathbb{Z} = \mathbb{Z}^k,$$

as wanted. $\square$

Finally, by Claim 7.3.3 and Claim 7.3.4, we obtain $\tilde{H}_m(Y_q) \cong H_m(Y_q, Z) \cong \mathbb{Z}^{q(q+1)}$. $\square$

Proof of Proposition 7.1.3. If $q = 2$, then $\hat{Y}_2$ consists of just 3 isolated vertices; hence,

$$\tilde{H}_i(\hat{Y}_2) = \begin{cases} \mathbb{Z}^2 & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Now, assume that $q \geq 3$. For $i \in [q+1]$, let

$$Z_i = \cup_{j\leq i} K_{j,1}.$$

Note that $Z_1 = K_{1,1}$, $Z_{q+1} = \hat{Y}_q$ and $Z_i = Z_{i-1} \cup K_{i,1}$ for $2 \leq i \leq q+1$.

First, we will show that for $i \leq q$,

$$\tilde{H}_j(Z_i) = \begin{cases} \mathbb{Z}^i & \text{if } j = q^2 - 2q, \\ 0 & \text{otherwise.} \end{cases}$$

We argue by induction on i:

For $i = 1$ the claim holds, since $Z_1 = K_{1,1}$ is a $(q^2 - 2q)$-dimensional sphere.

Now, assume that $i > 1$. We have

$$Z_{i-1} \cap K_{i,1} = \cup_{j=1}^{i-1}(K_{j,1} \cap K_{i,1}).$$

For all $\emptyset \neq I \subset [i-1]$, $\cap_{j\in I} K_{j,1} \cap K_{i,1}$ is just the complete complex on vertex set $\mathbb{F}_q^2 \setminus \cup_{j\in I\cup\{i\}} L_j$. In particular, it is either empty or acyclic. Therefore, by the Nerve Theorem (Theorem 2.2.4), the homology groups of $Z_{i-1} \cap K_{i,1}$ are the same as those of the nerve

$$\begin{aligned} N(\{K_{j,1} \cap K_{i,1}\}_{j=1}^{i-1}) &= \{\sigma \subset [i-1] : \cap_{j\in\sigma} K_{j,1} \cap K_{i,1} \neq \{\emptyset\}\} \\ &= \{\sigma \subset [i-1] : (\cup_{j\in\sigma} L_j) \cup L_i \neq \mathbb{F}_q^2\}. \end{aligned}$$

Let $\tau \subset [q+1]$. The lines $\{L_j\}_{j\in\tau}$ cover $\mathbb{F}_q^2$ if and only if $\tau = [q+1]$. Therefore, since $2 \leq i \leq q$, we have

$$N(\{K_{j,1} \cap K_{i,1}\}_{j=1}^{i-1}) = 2^{[i-1]}$$

In particular, $N(\{K_{j,1} \cap K_{i,1}\}_{j=1}^{i-1})$ is contractible. Therefore, we obtain

$$\tilde{H}_j(Z_{i-1} \cap K_{i,1}) = 0$$

for all j. By Mayer-Vietoris (Theorem 2.2.1), we have a long exact sequence:

$$\cdots \to \tilde{H}_j(Z_{i-1} \cap K_{i,1}) \to \tilde{H}_j(Z_{i-1}) \bigoplus \tilde{H}_j(K_{i,1}) \to \tilde{H}_j(Z_i) \to \tilde{H}_{j-1}(Z_{i-1} \cap K_{i,1}) \to \cdots .$$

So, we obtain

$$\tilde{H}_j(Z_i) \cong \tilde{H}_j(Z_{i-1}) \bigoplus \tilde{H}_j(K_{i,1})$$

for all j. Using the fact that $K_{i,1}$ is a (q^2-2q)-dimensional sphere and the induction hypothesis, we obtain

$$\tilde{H}_j(Z_i) = \begin{cases} \mathbb{Z}^i & \text{if } j = q^2 - 2q, \\ 0 & \text{otherwise,} \end{cases}$$

as wanted.

Now, let $i = q+1$. Similarly as before, we have

$$Z_q \cap K_{q+1,1} = \cup_{j=1}^{q} (K_{j,1} \cap K_{q+1,1}).$$

By the Nerve Theorem (Theorem 2.2.4), the homology groups of $Z_q \cap K_{q+1,1}$ are the same as those of the nerve

$$\begin{aligned} N(\{K_{j,1} \cap K_{q+1,1}\}_{j=1}^{q}) &= \{\sigma \subset [q] : \cap_{j\in\sigma} K_{j,1} \cap K_{q+1,1} \neq \{\emptyset\}\} \\ &= \{\sigma \subset [q] : (\cup_{j\in\sigma} L_j) \cup L_{q+1} \neq \mathbb{F}_q^2\} = \{\sigma \subset [q] : \sigma \neq [q]\}. \end{aligned}$$

That is, the nerve is the boundary of a $(q-1)$-dimensional simplex; so, it is a $(q-2)$-dimensional sphere. Hence,

$$\tilde{H}_j(Z_q \cap K_{q+1,1}) = \begin{cases} \mathbb{Z} & \text{if } j = q-2, \\ 0 & \text{otherwise,} \end{cases}$$

By the long exact sequence

$$\cdots \to \tilde{H}_j(Z_q \cap K_{q+1,1}) \to \tilde{H}_j(Z_q) \bigoplus \tilde{H}_j(K_{q+1,1}) \to \tilde{H}_j(Z_{q+1}) \to \tilde{H}_{j-1}(Z_q \cap K_{q+1,1}) \to \cdots , \tag{7.2}$$

we obtain

$$\tilde{H}_j(Z_{q+1}) \cong \tilde{H}_j(Z_q) \bigoplus \tilde{H}_j(K_{q+1,1})$$

for $j \notin \{q-2, q-1\}$. Since $K_{q+1,1}$ is a (q^2-2q)-dimensional sphere, we obtain

$$\tilde{H}_{q^2-2q}(Z_{q+1}) = \mathbb{Z}^{q+1}$$

(note that, since $q \geq 3$, we have $q^2 - 2q > q - 1$), and

$$\tilde{H}_j(Z_{q+1}) = 0$$

for $j \notin \{q-1, q-2, q^2-2q\}$.

Since $q^2 - 2q > q-1$, we have $\tilde{H}_j(Z_q) \bigoplus \tilde{H}_j(K_{q+1,1}) = 0$ for $j \leq q-1$. Hence, we obtain from the same exact sequence

$$\tilde{H}_j(Z_{q+1}) \cong \tilde{H}_{j-1}(Z_q \cap K_{q+1,1})$$

for $j \in \{q-1, q-2\}$. That is,

$$\tilde{H}_{q-1}(Z_{q+1}) = \mathbb{Z}$$

and

$$\tilde{H}_{q-2}(Z_{q+1}) = 0.$$

Thus,

$$\tilde{H}_i(\hat{Y}_q) = \tilde{H}_i(Z_{q+1}) = \begin{cases} \mathbb{Z} & \text{if } i = q-1, \\ \mathbb{Z}^{q+1} & \text{if } i = q^2 - 2q, \\ 0 & \text{otherwise,} \end{cases}$$

as wanted. □

Remark. Using the stronger version of the Nerve Theorem, Theorem 2.A.2, we can obtain a shorter proof of Proposition 7.1.3:

Assume $q \geq 3$. We have $\hat{Y}_q = \cup_{j=1}^{q+1} K_{j,1}$. Let $N = N\left(\{K_{j,1}\}_{j=1}^{q+1}\right)$. For $I \subset [q+1]$ such that $|I| \geq 2$, the complex $\cap_{i \in I} K_{i,1}$ is just the complete complex on vertex set $\mathbb{F}_q^2 \setminus (\cup_{i \in I} L_i)$. In particular, we have $\tilde{H}_k\left(\cap_{i \in I} K_{i,1}\right) = 0$. Moreover, since $\mathbb{F}_q^2 \setminus (\cup_{i \in I} L_i) = \emptyset$ if and only if $I = [q+1]$, we have $N = \{\sigma \subset [q+1] : \sigma \neq [q+1]\}$. That is, N is a $(q-1)$-dimensional sphere.

By Theorem 2.A.2, we have a long exact sequence

$$\cdots \to \tilde{H}_{k+1}(N) \to \bigoplus_{i=1}^{q+1} \tilde{H}_k(K_{i,1}) \to \tilde{H}_k\left(\hat{Y}_q\right) \to \tilde{H}_k(N) \to \cdots.$$

Note that this is essentially the same as the exact sequence (7.2). Hence, the rest of the proof follows similarly as before.

7.4 Complexes of hyperplane-free sets

Let q be a prime power and $n \geq 2$ an integer. A set $\sigma \subset \mathbb{F}_q^n$ is called *hyperplane-free* if it does not contain any affine hyperplane.

We define the simplicial complex

$$\hat{X}_{q,n} = \left\{\sigma \subset \mathbb{F}_q^n \setminus \{0\} : \sigma \text{ is hyperplane-free}\right\}.$$

Note that, for $n = 2$, $\hat{X}_{q,2} = \hat{X}_q$.

A *blocking set* in $\mathbb{F}_q^n$ is a set that intersects all the affine hyperplanes. One can build a blocking set of size $n(q-1)+1$ by taking the union of all the lines passing through some point. In fact, there are no smaller blocking sets:

Theorem 7.4.1 (Jamison [Jam77], Brouwer-Schrijver [BS78])**.** *The minimum size of a blocking set in $\mathbb{F}_q^n$ is $n(q-1)+1$.*

Note that a set $\sigma \subset \mathbb{F}_q^n \setminus \{0\}$ is hyperplane-free if and only if its complement is a blocking set containing the origin. Therefore, by Theorem 7.4.1, we have

$$\dim(\hat{X}_{q,n}) = q^n - n(q-1) - 2.$$

For $q = 2$ we understand the homology of these complexes completely:

Proposition 7.4.2.

$$\tilde{H}_i(\hat{X}_{2,n}) = \begin{cases} \mathbb{Z}^{2^{\binom{n}{2}}} & \text{if } i = 2^n - n - 2, \\ 0 & \text{otherwise.} \end{cases}$$

For the proof we will need the following result:

Theorem 7.4.3 (Solomon-Tits (see [CL82]))**.** *Let $Fl_{n,q}$ be the simplicial complex whose vertices correspond to the non-trivial linear subspaces of $\mathbb{F}_q^n$ and whose simplices are the sets $\{V_1, \ldots, V_k\}$ that form a flag $V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_k$. Then,*

$$\tilde{H}_i(Fl_{n,q}) = \begin{cases} \mathbb{Z}^{q^{\binom{n}{2}}} & \text{if } i = n - 2, \\ 0 & \text{otherwise.} \end{cases}$$

Proof of Proposition 7.4.2. We look at the Alexander dual of $\hat{X}_{2,n}$:

$$\hat{X}_{2,n}^V = \{\sigma \subset \mathbb{F}_2^n \setminus \{0\} : \mathbb{F}_2^n \setminus (\sigma \cup \{0\}) \text{ contains an affine hyperplane } \}.$$

Since the complement of a hyperplane in $\mathbb{F}_2^n$ is also a hyperplane, we obtain

$$\hat{X}_{2,n}^V = \{\sigma \subset \mathbb{F}_2^n \setminus \{0\} : \sigma \subset H \text{ for some linear hyperplane } H\}.$$

So, $\hat{X}_{2,n}^V$ is the simplicial complex on vertex set $\mathbb{F}_2^n \setminus \{0\}$ whose maximal faces are the linear hyperplanes in $\mathbb{F}_2^n$.

Now, we can write

$$\mathrm{Fl}_{n,2} = \bigcup_{u \in \mathbb{F}_2^n \setminus \{0\}} \mathrm{st}(\mathrm{Fl}_{n,2}, \mathrm{span}(u)).$$

Note that, for any $\sigma \subset \mathbb{F}_2^n \setminus \{0\}$,

$$\bigcap_{u\in\sigma} \mathrm{st}(\mathrm{Fl}_{n,2}, \mathrm{span}(u)) = \begin{cases} \mathrm{st}(\mathrm{Fl}_{n,2}, \mathrm{span}(\sigma)) & \text{if } \mathrm{span}(\sigma) \neq \mathbb{F}_2^n, \\ \{\emptyset\} & \text{if } \mathrm{span}(\sigma) = \mathbb{F}_2^n, \end{cases}$$
$$= \begin{cases} \mathrm{st}(\mathrm{Fl}_{n,2}, \mathrm{span}(\sigma)) & \text{if } \sigma \text{ is contained in a linear hyperplane,} \\ \{\emptyset\} & \text{otherwise.} \end{cases}$$

Therefore,

$$N\left(\{\mathrm{st}(\mathrm{Fl}_{n,2}, \mathrm{span}(u))\}_{u\in\mathbb{F}_2^n\setminus\{0\}}\right) = \hat{X}_{2,n}^V.$$

Moreover, since $\mathrm{st}(\mathrm{Fl}_{n,2}, \mathrm{span}(\sigma))$ is contractible for any $\sigma \subset \mathbb{F}_2^n\setminus\{0\}$ with $\mathrm{span}(\sigma) \neq \mathbb{F}_2^n$, we obtain by the Nerve Theorem (Theorem 2.2.4),

$$\tilde{H}_k(\hat{X}_{2,n}^V) \cong \tilde{H}_k(\mathrm{Fl}_{n,2})$$

for all k. By Theorem 7.4.3, we obtain

$$\tilde{H}_i(\hat{X}_{2,n}^V) = \begin{cases} \mathbb{Z}^{2^{\binom{n}{2}}} & \text{if } i = n-2, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, by Alexander duality (Theorem 2.2.10), we obtain

$$\tilde{H}^i(\hat{X}_{2,n}) = \begin{cases} \mathbb{Z}^{2^{\binom{n}{2}}} & \text{if } i = 2^n - n - 2, \\ 0 & \text{otherwise.} \end{cases}$$

Since the cohomology of $\hat{X}_{2,n}$ is torsion-free, we obtain by Lemma 2.2.9,

$$\tilde{H}_i(\hat{X}_{2,n}) = \begin{cases} \mathbb{Z}^{2^{\binom{n}{2}}} & \text{if } i = 2^n - n - 2, \\ 0 & \text{otherwise,} \end{cases}$$

as wanted. □

Remark. The fact that $\tilde{H}_*(\hat{X}_{2,n}^V) \cong \tilde{H}_*(\mathrm{Fl}_{n,2})$ is a special case of Folkman's Cross-Cut Theorem (see [Fol66]).

For $q \geq 3$, we focus again on the top-dimensional homology groups. We conjecture the following.

Conjecture 7.4.4. *Let $q \geq 3$ be a prime power. Then,*

$$\tilde{H}_{q^n-n(q-1)-2}(\hat{X}_{q,n}) = \mathbb{Z}^{\prod_{i=1}^n \frac{q^i-1}{q-1}}.$$

Note that, for $n = 2$, this is the statement of Theorem 1.5.2.

We propose an approach similar to the one used for Theorem 1.5.2: Let $\mathcal{F}_{q,n}$ be the collection of all flags $F = \{V_1, \dots, V_{n-1}\}$, where V_i is an i-dimensional linear subspace of $\mathbb{F}_q^n$ for all $i \in [n-1]$. Let $F = \{V_1, \dots, V_{n-1}\} \in \mathcal{F}_{q,n}$. For each $i \in [n-1]$, let $\{V_i^j\}_{j\in[q-1]}$ be the affine i-dimensional subspaces of $\mathbb{F}_q^n$ parallel to V_i (other than V_i) that are contained in V_{i+1} (where, for $i = n-1$, we define $V_{i+1} = V_n = \mathbb{F}_q^n$).

We define the subcomplex $K_F \subset \hat{X}_{q,n}$ as

$$K_F = *_{i=1}^{n} \left(*_{j=1}^{q-1} \partial V_i^j \right).$$

The complex K_F is homeomorphic to a $(q^n - n(q-1) - 2)$-dimensional sphere. Let

$$\hat{Y}_{q,n} = \bigcup_{F \in \mathcal{F}_{q,n}} K_F.$$

Conjecture 7.4.4 would follow from the following two conjectures:

Conjecture 7.4.5. *Let $q \geq 3$ be a prime power and $n \geq 2$ be an integer. Then,*

$$\tilde{H}_{q^n - n(q-1)-2}(\hat{X}_{q,n}) \cong \tilde{H}_{q^n - n(q-1)-2}(\hat{Y}_{q,n}).$$

Conjecture 7.4.6. *Let $q \geq 3$ be a prime power and $n \geq 2$ be an integer. Then,*

$$\tilde{H}_{q^n - n(q-1)-2}(\hat{Y}_{q,n}) = \mathbb{Z}^{\prod_{i=1}^{n} \frac{q^i - 1}{q-1}}.$$

We propose the following approach for proving Conjecture 7.4.5:

Let $B \subset \mathbb{F}_q^n$ be a blocking set of size $n(q-1)+1$ containing the origin. B is called *strongly stable* if for every point $v \in \mathbb{F}_q^n \setminus B$ there is some $u \in B \setminus \{0\}$ such that $B \cup \{v\} \setminus \{u\}$ is also a blocking set. We conjecture the following characterization of strongly stable blocking sets, generalizing the characterization in the $n = 2$ case in Theorem 1.5.3:

Conjecture 7.4.7. *Let $B \subset \mathbb{F}_q^n$ be a blocking set of size $n(q-1)+1$ containing the origin. Then, B is strongly stable if and only if there is a flag $V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_{n-1}$ of linear subspaces of $\mathbb{F}_q^n$ such that $\dim(V_k) = k$ and $|B \cap V_k| = k(q-1)+1$ for all $k \in [n-1]$.*

Finally, we will need the following simple Lemma:

Lemma 7.4.8. *Let $B \subset \mathbb{F}_q^n$ be a blocking set of size $n(q-1)+1$ containing the origin. Let $F = \{V_1, \dots, V_{n-1}\}$ be a flag such that $V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_{n-1}$ and $\dim(V_k) = k$ for all $k \in [n-1]$. Then, $\mathbb{F}_q^n \setminus B \in K_F$ if and only if $|B \cap V_k| = k(q-1)+1$ for all $k \in [n-1]$.*

Proof. Let $B \subset \mathbb{F}_q^n$ be a blocking set of size $n(q-1)+1$ containing the origin, and let $F = \{V_1, \dots, V_{n-1}\}$ be a flag such that $V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_{n-1}$ and $\dim(V_k) = k$ for all $k \in [n-1]$.

First, note that, by the definition of K_F, if $\mathbb{F}_q^n \setminus B \in K_F$ then $|B \cap V_k| = k(q-1)+1$ for all $k \in [n-1]$.

Now, assume that $|B \cap V_k| = k(q-1)+1$ for all $k \in [n-1]$. We will show that $\mathbb{F}_q^n \setminus B \in K_F$. We argue by induction on n. For $n = 2$ the claim holds: if $F = \{V_1\}$, where V_1 is a line through the origin, we have $\mathbb{F}_q^2 \setminus B \in K_F$ if and only if B contains the line V_1, that is, if and only if $|B \cap V_1| = 1 \cdot (q-1)+1 = q$.

Let $n \geq 3$. Let $F' = \{V_1, \ldots, V_{n-2}\}$. Note that, by the definition of K_F, we have $\mathbb{F}_q^n \setminus B \in K_F$ if and only if $V_{n-1} \setminus (B \cap V_{n-1}) \in K_{F'}$ and $|B \cap V_{n-1}^j| = 1$ for all $j \in [q-1]$.

Since $|B| = n(q-1)+1$ and $|B \cap V_{n-1}| = (n-1)(q-1)+1$, we obtain $|B \setminus V_{n-1}| = q-1$. Since B is a blocking set, it must intersect each of the hyperplanes V_{n-1}^j, for $j \in [q-1]$. Therefore, we must have $|B \cap V_{n-1}^j| = 1$ for $j \in [q-1]$.

$B \cap V_{n-1}$ is a blocking set in V_{n-1}: otherwise, assume for contradiction that $B \cap V_{n-1}$ is not a blocking set in V_{n-1}. Then, there is some $(n-2)$-dimensional subspace U of V_{n-1} that is disjoint from $B \cap V_{n-1}$. Let $H_1, \ldots, H_q$ be the linear hyperplanes in $\mathbb{F}_q^n$ containing U, other than V_{n-1}. Since B is a blocking set and $B \cap V_{n-1} \cap H_i = B \cap U = \emptyset$ for all $i \in [q]$, we must have $(B \setminus V_{n-1}) \cap H_i \neq \emptyset$ for all $i \in [q]$. The sets $H_i \setminus V_{n-1} = H_i \setminus U$ are pairwise disjoint, therefore we must have $|B \setminus V_{n-1}| \geq q$, a contradiction to $|B \setminus V_{n-1}| = q-1$.

So, $B \cap V_{n-1}$ is a blocking set of size $(n-1)(q-1)+1$ in $V_{n-1} \cong \mathbb{F}_q^{n-1}$, and $|(B \cap V_{n-1}) \cap V_k| = |B \cap V_k| = k(q-1)+1$ for all $k \in [n-2]$. So, by the induction hypothesis, $V_{n-1} \setminus (B \cap V_{n-1}) \in K_{F'}$. Thus, $\mathbb{F}_q^n \setminus B \in K_F$. $\square$

From Conjecture 7.4.7 and Lemma 7.4.8 it follows that, for a blocking set $B \subset \mathbb{F}_q^n$ of size $n(q-1)+1$ containing the origin, B is strongly stable if and only if $\mathbb{F}_q^n \setminus B \in K_F$ for some flag $F = \{V_1, \ldots, V_{n-1}\}$. Hence, Conjecture 7.4.5 would follow from Conjecture 7.4.7 (following the same argument as in the proof of Proposition 7.1.1).

Chapter 8

Laplacian eigenvalues of complexes of flags

This chapter is organized as follows. In Section 8.1 we present some facts about q-binomial coefficients that we will need later. In Section 8.3 we introduce the "subspace inclusion matrices" A_{ij}, we study some of their properties and explain their relation to the Laplacian matrix $L_0^+(\mathrm{Fl}_{n,q})$. In Section 8.3.1 we finish the proof of our main result, Theorem 1.6.2.

8.1 q-Binomial coefficients

Let q be a prime power, and let a, b be integers. We define the *q-binomial coefficient* $\binom{a}{b}_q$ to be the number of b-dimensional subspaces contained in $\mathbb{F}_q^a$. More explicitly, we have

$$\binom{a}{b}_q = \frac{\prod_{i=1}^{a}(q^i - 1)}{\prod_{i=1}^{b}(q^i - 1)\prod_{i=1}^{a-b}(q^i - 1)} = \frac{\prod_{i=a-b+1}^{a}(q^i - 1)}{\prod_{i=1}^{b}(q^i - 1)} \tag{8.1}$$

if $a \geq b \geq 0$, and $\binom{a}{b}_q = 0$ otherwise.

Lemma 8.1.1. *Let $0 \leq r \leq k \leq n$. Let U be an r-dimensional subspace of $\mathbb{F}_q^n$. Then, the number of k-dimensional subspaces of $\mathbb{F}_q^n$ that contain U is $\binom{n-r}{k-r}_q$.*

Proof. The number of k-dimensional subspaces of $\mathbb{F}_q^n$ containing U is equal to the number of $(k-r)$-dimensional subspaces of $\mathbb{F}_q^n/U \cong \mathbb{F}_q^{n-r}$. Hence, there are $\binom{n-r}{k-r}_q$ such subspaces. □

We will need the following simple results about the behavior of the q-binomial coefficients as q tends to infinity.

Lemma 8.1.2. *Let $a, b \geq 0$ be integers. Then,*

$$\lim_{q\to\infty} \frac{\binom{a+b}{b}_q}{q^{ab}} = 1.$$

Proof. By Equation (8.1), we have

$$\frac{\binom{a+b}{b}_q}{q^{ab}} = \frac{f(q)}{g(q)},$$

where f is a monic polynomial of degree

$$\sum_{i=a+1}^{a+b} i = \frac{1}{2}b(2a+b+1) = ab + \frac{1}{2}b(b+1)$$

and g is a monic polynomial of degree

$$ab + \sum_{i=1}^{b} i = ab + \frac{1}{2}b(b+1).$$

Therefore, we have

$$\lim_{q\to\infty} \frac{\binom{a+b}{b}_q}{q^{ab}} = \lim_{q\to\infty} \frac{f(q)}{g(q)} = 1,$$

as wanted. □

Lemma 8.1.3. *Let $k < b < a$. Then*

$$\lim_{q\to\infty} \binom{a-k}{b-k}_q \Big/ \binom{a}{b}_q = 0.$$

Proof. By Lemma 8.1.2, we have

$$\lim_{q\to\infty} \frac{\binom{a-k}{b-k}_q}{\binom{a}{b}_q} = \lim_{q\to\infty} \frac{\binom{a-k}{b-k}_q}{q^{(a-b)(b-k)}} \cdot \frac{q^{(a-b)b}}{\binom{a}{b}_q} \cdot q^{-(a-b)k} = 1 \cdot 1 \cdot 0 = 0.$$

We will also use the following inversion formula due to Carlitz:

Lemma 8.1.4 (Carlitz [Car73]). *Let $\{a_n\}_{n=0}^m$ and $\{b_n\}_{n=0}^m$ be two sequences such that*

$$a_n = \sum_{k=0}^{n} \binom{n}{k}_q b_k$$

for all $0 \le n \le m$. Then,

$$b_n = \sum_{k=0}^{n} (-1)^{n-k} q^{\binom{k+1}{2}+\binom{n}{2}-kn} \binom{n}{k}_q a_k$$

for all $0 \le n \le m$.

8.2 The weight function

Let $V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_k$ be a flag in $\mathbb{F}_q^n$, and let $\sigma = \{V_1, \ldots, V_k\} \in \mathrm{Fl}_{n,q}$. Recall that we defined the weight function $w(\sigma)$ to be the number of complete flags extending σ or, equivalently, the number of maximal faces of $\mathrm{Fl}_{n,q}$ containing σ. In this section we discuss some useful properties of the weight function w.

Let $\mathrm{GL}(n, q)$ be the group of invertible $n \times n$ matrices over $\mathbb{F}_q$. For $g \in \mathrm{GL}(n, q)$ and a subspace $V \subset \mathbb{F}_q^n$, let $gV = \{gv : v \in V\}$. Note that gV is also a subspace of $\mathbb{F}_q^n$ and that $\dim(gV) = \dim(V)$.

Let F be the flag $V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_k$. Then, we denote by gF the flag

$$gV_1 \subsetneq gV_2 \subsetneq \cdots \subsetneq gV_k.$$

First, we show that w depends only on the dimensions of the subspaces forming the flag:

Lemma 8.2.1. *Let $V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_k$ and $W_1 \subsetneq W_2 \subsetneq \cdots \subsetneq W_k$ be two flags in $\mathbb{F}_q^n$. Let $\sigma = \{V_1, \ldots, V_k\}$ and $\sigma' = \{W_1, \ldots, W_k\}$. If $\dim(V_i) = \dim(W_i)$ for all $i \in [k]$, then $w(\sigma) = w(\sigma')$.*

Proof. Let $e_1, \ldots, e_n$ be a basis of $\mathbb{F}_q^n$ such that, for any $i \in [k]$, the first $\dim(V_i)$ vectors in the basis form a basis for V_i. Similarly, let $v_1, \ldots, v_n$ be a basis of $\mathbb{F}_q^n$ such that, for any $i \in [k]$, the first $\dim(W_i) = \dim(V_i)$ vectors in the basis form a basis for W_i.

Let $g \in \mathrm{GL}(n, q)$ be the linear isomorphism that maps e_i to v_i for all $i \in [n]$. Then, for any $i \in [k]$, we have $gV_i = W_i$.

Let $\mathcal{F}$ be the set of complete flags extending σ, and let $\mathcal{F}'$ be the set of complete flags extending σ'. We have a map $\tilde{g} : \mathcal{F} \to \mathcal{F}'$ defined by $\tilde{g}(F) = gF$ for any $F \in \mathcal{F}$. Since $F = g^{-1}gF$, $\tilde{g}$ is injective. Hence, $|\mathcal{F}| \leq |\mathcal{F}'|$. By symmetry, we have $w(\sigma) = |\mathcal{F}| = |\mathcal{F}'| = w(\sigma')$. □

Lemma 8.2.2. *Let k be an integer. Let $V_1 \subsetneq \cdots \subsetneq V_{i-1} \subsetneq V_{i+1} \subsetneq \cdots \subsetneq V_k$ be a flag in $\mathbb{F}_q^n$, and let $\tau = \{V_1, \ldots, V_{i-1}, V_{i+1}, \ldots, V_k\}$. Let $\dim(V_{i-1}) < d < \dim(V_{i+1})$. Let $\mathcal{U}$ be the set of d-dimensional subspaces $U \subset \mathbb{F}_q^n$ satisfying $V_{i-1} \subset U \subset V_{i+1}$. For $U \in \mathcal{U}$, let $\sigma_U = \{V_1, \ldots, V_{i-1}, U, V_{i+1}, \ldots, V_k\}$. Then, we have*

$$w(\tau) = \sum_{U \in \mathcal{U}} w(\sigma_U).$$

Proof. Notice that, for any $U \in \mathcal{U}$, any complete flag extending σ_U extends also τ. Moreover, each complete flag extending τ extends exactly one of the flags σ_U. Therefore, we have

$$w(\tau) = \sum_{U \in \mathcal{U}} w(\sigma_U)$$

as wanted. □

Lemma 8.2.3. *Let k be an integer, and $1 \leq i \leq k$. Let $V_1 \subsetneq \cdots \subsetneq V_k$ be a flag in $\mathbb{F}_q^n$. Let $\sigma = \{V_1, \ldots, V_k\}$ and $\tau = \{V_1, \ldots, V_{i-1}, V_{i+1}, \ldots, V_k\}$. Let $V_0 = \{0\}$ and $V_{k+1} = \mathbb{F}_q^n$. Let $r = \dim V_{i+1} - \dim V_{i-1}$ and $t = \dim V_i - \dim V_{i-1}$. Then,*

$$\frac{w(\sigma)}{w(\tau)} = \binom{r}{t}_q^{-1}.$$

Proof. Let $\mathcal{U}$ be the family of subspaces $U \subset \mathbb{F}_q^n$ satisfying $V_{i-1} \subset U \subset V_{i+1}$ and $\dim(U) = \dim(V_i)$. By Lemma 8.1.1, we have $|\mathcal{U}| = \binom{r}{t}_q$.

For any $U \in \mathcal{U}$, let $\sigma_U = \{V_1, \ldots, V_{i-1}, U, V_{i+1}, \ldots, V_k\}$. Then, we have $\sigma = \sigma_{V_i}$, and, by Lemma 8.2.1, $w(\sigma_U) = w(\sigma)$ for all $U \in \mathcal{U}$.

Therefore, by Lemma 8.2.2, we have

$$w(\tau) = \sum_{U \in \mathcal{U}} w(\sigma_U) = |\mathcal{U}| \cdot w(\sigma) = \binom{r}{t}_q \cdot w(\sigma).$$

We obtain $\frac{w(\sigma)}{w(\tau)} = \binom{r}{t}_q^{-1}$, as wanted. □

Lemma 8.2.4. *Let $\sigma \in X_{n,q}(k)$. Then*

$$L_k^+(Fl_{n,q})(\sigma, \sigma) = n - k - 2.$$

Proof. Let $\sigma = \{V_1, \ldots, V_{k+1}\}$, and let $d_i = \dim(V_i)$ for all $i \in [k+1]$.

By Lemma 2.2.15, we have

$$\begin{aligned} L_k^+(\mathrm{Fl}_{n,q})(\sigma, \sigma) &= \frac{1}{w(\sigma)} \sum_{v \in \mathrm{lk}(\mathrm{Fl}_{n,q}, \sigma)} w(v\sigma) \\ &= \frac{1}{w(\sigma)} \sum_{\substack{d \in [n-1], \\ d \notin \{d_1, \ldots, d_{k+1}\}}} \sum_{\substack{v \in \mathrm{lk}(\mathrm{Fl}_{n,q}, \sigma) \\ \dim(v) = d}} w(v\sigma). \end{aligned}$$

By Lemma 8.2.2, we have for any $d \in [n-1] \setminus \{d_1, \ldots, d_{k+1}\}$:

$$\sum_{\substack{v \in \mathrm{lk}(\mathrm{Fl}_{n,q}, \sigma) \\ \dim(v) = d}} w(v\sigma) = w(\sigma).$$

So, we obtain

$$\begin{aligned} L_k^+(\mathrm{Fl}_{n,q})(\sigma, \sigma) = \frac{1}{w(\sigma)} &\sum_{\substack{d \in [n-1], \\ d \notin \{d_1, \ldots, d_{k+1}\}}} \sum_{\substack{v \in \mathrm{lk}(\mathrm{Fl}_{n,q}, \sigma) \\ \dim(v) = d}} w(v\sigma) \\ &= \frac{1}{w(\sigma)} \sum_{\substack{d \in [n-1], \\ d \notin \{d_1, \ldots, d_{k+1}\}}} w(\sigma) = n - k - 2, \end{aligned}$$

as wanted. □

8.3 Subspace inclusion matrices

For $i \in \{0, \dots, n\}$, denote by $S(i)$ the collection of subspaces of $\mathbb{F}_q^n$ of dimension i.

Let $1 \le i \le n-1$ and $U \in S(i)$. Define the cochain $1_U \in C^0(X_{n,q})$ as

$$1_U(V) = \begin{cases} 1 & \text{if } U = V, \\ 0 & \text{otherwise.} \end{cases}$$

We call the basis $\cup_{i=1}^{n-1}\{1_U : U \in S(i)\}$ the *standard basis* for $C^0(X_{n,q})$.

Let $0 \le i, j \le n$. Let A_{ij} be the $S(i) \times S(j)$ matrix

$$A_{ij}(U, V) = \begin{cases} 1 & \text{if } U \subset V \text{ or } V \subset U, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $A_{ij} = A_{ji}^t$, and that A_{ii} is just the identity matrix $I_{S(i)}$. Also, for all $0 \le j \le n$, $A_{0j} \in \mathbb{R}^{1 \times S(j)} = \mathbb{R}^{S(j)}$ is the all-1 vector.

Using the matrices A_{ij}, we can give the following explicit description for $L_0^+(\mathrm{Fl}_{n,q})$:

Lemma 8.3.1. *Let $L = L_0^+(Fl_{n,q})$. We identify L with its matrix representation with respect to the standard basis. Then, we can write L as an $(n-1) \times (n-1)$ block matrix*

$$L = \begin{pmatrix} L_{1,1} & \cdots & L_{1,n-1} \\ \vdots & & \vdots \\ L_{n-1,1} & \cdots & L_{n-1,n-1} \end{pmatrix},$$

where, for $(i, j) \in [n-1] \times [n-1]$, L_{ij} is the $S(i) \times S(j)$ matrix

$$L_{ij} = \begin{cases} (n-2)I & \text{if } i = j, \\ -\binom{n-i}{j-i}_q^{-1} A_{ij} & \text{if } i < j, \\ -\binom{i}{j}_q^{-1} A_{ij} & \text{if } i > j. \end{cases}$$

Proof. Let $i, j \in [n-1]$, and let $U \in S(i)$ and $V \in S(j)$. Then, by Corollary 2.2.16 and Lemma 8.2.4, we have

$$L(U, V) = \begin{cases} n-2 & \text{if } U = V, \\ -\frac{w(\{U,V\})}{w(U)} & \text{if } U \subsetneq V \text{ or } V \subsetneq U, \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 8.2.3, we have

$$\frac{w(\{U, V\})}{w(U)} = \binom{n-i}{j-i}_q^{-1}$$

if $U \subset V$ and

$$\frac{w(\{U,V\})}{w(U)} = \binom{i}{j}_q^{-1}$$

if $V \subset U$.

So, for $i = j$, we obtain

$$L(U,V) = \begin{cases} n-2 & \text{if } U = V, \\ 0 & \text{otherwise} \end{cases} = (n-2)I(U,V),$$

where I is the $S(i) \times S(i)$ identity matrix.

For $i < j$, we obtain

$$L(U,V) = \begin{cases} -\binom{n-i}{j-i}_q^{-1} & \text{if } U \subset V, \\ 0 & \text{otherwise} \end{cases} = -\binom{n-i}{j-i}_q^{-1} A_{ij}(U,V).$$

For $i > j$, we obtain

$$L(U,V) = \begin{cases} -\binom{i}{j}_q^{-1} & \text{if } V \subset U, \\ 0 & \text{otherwise} \end{cases} = -\binom{i}{j}_q^{-1} A_{ij}(U,V),$$

as wanted. □

We will need the following results regarding products of subspace inclusion matrices:

Lemma 8.3.2 (Kantor [Kan72]). *Let $k \leq j \leq i$. Then*

$$A_{ij}A_{jk} = \binom{i-k}{j-k}_q A_{ik}.$$

Proof. Let $U \in S(k)$. Then,

$$A_{ij}A_{jk}1_U = A_{ij}\left(\sum_{\substack{V \in S(j): \\ U \subset V}} 1_V\right) = \sum_{\substack{V \in S(j): \\ U \subset V}} A_{ij}1_V = \sum_{\substack{V \in S(j): \\ U \subset V}} \sum_{\substack{W \in S(i): \\ V \subset W}} 1_W.$$

Let $W \in S(i)$ such that $U \subset W$. By Lemma 8.1.1, the number of j-dimensional subspaces of W containing U is $\binom{i-k}{j-k}_q$. So, we obtain

$$A_{ij}A_{jk}1_U = \sum_{\substack{W \in S(i): \\ U \subset W}} \binom{i-k}{j-k}_q 1_W = \binom{i-k}{j-k}_q A_{ik}1_U.$$

Thus, $A_{ij}A_{jk} = \binom{i-k}{j-k}_q A_{ik}$. □

Lemma 8.3.3. *Let $0 \leq k \leq i$, and let $U \in S(k)$. Then, for any sequence $\{\alpha_m\}_{m=0}^k$, we can write*

$$\sum_{m=0}^{k} \alpha_m \sum_{\substack{V \in S(m): \\ V \subset U}} \sum_{\substack{W \in S(i): \\ V \subset W}} 1_W = \sum_{m=0}^{k} \sum_{\substack{W \in S(i): \\ \dim(U \cap W) = m}} \beta_m 1_W, \tag{8.2}$$

where

$$\beta_m = \sum_{r=0}^{m} \binom{m}{r}_q \alpha_r$$

for all $0 \leq m \leq k$.

Proof. Let $0 \leq m \leq k$, and let $W \in S(i)$ such that $\dim(U \cap W) = m$. The coefficient of 1_W on the right-hand side of Equation (8.2) is β_m.

Let $0 \leq r \leq k$. The number of r-dimensional subspaces of $U \cap W$ is $\binom{m}{r}_q$ if $r \leq m$ and 0 otherwise. Therefore, the coefficient of 1_W on the left-hand side of Equation (8.2) is

$$\sum_{r=0}^{m} \binom{m}{r}_q \alpha_r.$$

We obtain

$$\beta_m = \sum_{r=0}^{m} \binom{m}{r}_q \alpha_r$$

for all $0 \leq m \leq k$, as wanted. ☐

Lemma 8.3.4. *Let $0 \leq k \leq i \leq j \leq n$. Then,*

$$A_{ij} A_{jk} = \sum_{m=0}^{k} c_{ijkm} A_{im} A_{mk},$$

where, for all $0 \leq m \leq k$, the coefficients c_{ijkm} satisfy the relations

$$\sum_{r=0}^{m} \binom{m}{r}_q c_{ijkr} = \binom{n-i-k+m}{j-i-k+m}_q. \tag{8.3}$$

Proof. Let $U \in S(k)$. Then, we can write

$$A_{ij} A_{jk} 1_U = \sum_{\substack{V \in S(j): \\ U \subset V}} \sum_{\substack{W \in S(i): \\ W \subset V}} 1_W.$$

Let $0 \leq m \leq k$ and let $W \in S(i)$ such that $\dim(U \cap W) = m$. Then, by Lemma 8.1.1, since $\dim(U + V) = \dim(U) + \dim(W) - \dim(U \cap W) = k + i - m$, the number of j-dimensional subspaces of $\mathbb{F}_q^n$ containing both U and W is

$$\binom{n-(k+i-m)}{j-(k+i-m)}_q = \binom{n-i-k+m}{j-i-k+m}_q.$$

So, we can write

$$A_{ij}A_{jk}1_U = \sum_{m=0}^{k} \sum_{\substack{W\in S(i):\\ \dim(U\cap W)=m}} \binom{n-i-k+m}{j-i-k+m}_q 1_W.$$

Using Lemma 8.3.3 and the fact that $\sum_{\substack{V\in S(m):\\ V\subset U}} \sum_{\substack{W\in S(i):\\ V\subset W}} 1_W = A_{im}A_{mk}1_U$, we obtain

$$\begin{aligned} A_{ij}A_{jk}1_U &= \sum_{m=0}^{k} \sum_{\substack{W\in S(i):\\ \dim(U\cap W)=m}} \binom{n-i-k+m}{j-i-k+m}_q 1_W \\ &= \sum_{m=0}^{k} c_{ijkm} \sum_{\substack{V\in S(m):\\ V\subset U}} \sum_{\substack{W\in S(i):\\ V\subset W}} 1_W \\ &= \sum_{m=0}^{k} c_{ijkm} A_{im}A_{mk}1_U, \end{aligned}$$

where the coefficients c_{ijkm} satisfy the relations

$$\sum_{r=0}^{m} \binom{m}{r}_q c_{i,j,k,r} = \binom{n-i-k+m}{j-i-k+m}_q$$

for all $0 \le m \le k$. □

Remark. Let $0 \le m \le k \le i \le j \le n$. By Lemma 8.1.4, we have the following explicit formula for the coefficient c_{ijkm}:

$$c_{ijkm} = \sum_{r=0}^{m} (-1)^{m-r} q^{\binom{r+1}{2}+\binom{m}{2}-rm} \binom{m}{r}_q \binom{n-i-k+r}{j-i-k+r}_q. \tag{8.4}$$

We will need the following Lemma about the asymptotic behaviour of the numbers c_{ijkm}.

Lemma 8.3.5. *Let $k \le i \le j \le n-k$. Then, for all $0 \le m \le k$,*

$$\lim_{q\to\infty} \frac{c_{ijkm}}{\binom{n-i-k+m}{j-i-k+m}_q} = 1. \tag{8.5}$$

Proof. We argue by induction on m. For $m = 0$ we have, by Equation (8.4):

$$c_{ijk0} = \binom{n-i-k}{j-i-k}_q,$$

so the claim holds trivially. Now, let $m > 0$. Since

$$\binom{n-i-k+m}{j-i-k+m}_q = \binom{n-j+(j-i-k+m)}{j-i-k+m}_q,$$

by Lemma 8.1.2, Equation (8.5) is equivalent to

$$\lim_{q\to\infty} \frac{c_{ijkm}}{q^{(n-j)(j-i-k+m)}} = 1.$$

By Equation (8.3), we have

$$c_{ijkm} = \binom{n-i-k+m}{j-i-k+m}_q - \sum_{r=0}^{m-1} \binom{m}{r}_q c_{ijkr}.$$

Dividing by $q^{(n-j)(j-i-k+m)}$, we obtain

$$\frac{c_{ijkm}}{q^{(n-j)(j-i-k+m)}} = \frac{\binom{n-i-k+m}{j-i-k+m}_q}{q^{(n-j)(j-i-k+m)}} - \sum_{r=0}^{m-1} \frac{1}{q^{(m-r)(n-j-r)}} \frac{\binom{m}{r}_q}{q^{(m-r)r}} \frac{c_{ijkr}}{q^{(n-j)(j-i-k+r)}}.$$

Since $r \leq m-1 \leq k-1 \leq n-j-1$, we obtain (by the induction hypothesis and Lemma 8.1.2),

$$\lim_{q\to\infty} \frac{c_{ijkm}}{q^{(n-j)(j-i-k+m)}} = 1,$$

as wanted. □

For $i \in \{0, 1, \ldots, n\}$, let

$$E^i = \mathrm{span}(\{1_U : U \in S(i)\}).$$

We will need the following Theorem of Kantor:

Theorem 8.3.6 (Kantor [Kan72])**.** *Let $0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$ and $k \leq i \leq n-k$. Then*

$$\mathit{rank}(A_{ik}) = |S(k)| = \binom{n}{k}_q.$$

In particular, the linear map $A_{ik} : E^k \to E^i$ is injective.

Let $\tilde{E}^0 = E^0$, and, for $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$, let $\tilde{E}^k$ be the orthogonal complement in E^k of the subspace $A_{k,k-1}E^{k-1}$.

Proposition 8.3.7. *Let $0 \leq i \leq n$. Then,*

$$E^i = \bigoplus_{0\leq k\leq \left\lfloor \frac{n}{2} \right\rfloor,\, k\leq i\leq n-k} A_{ik}\tilde{E}^k.$$

Proof. Assume first $i \leq \lfloor \frac{n}{2} \rfloor$. We argue by induction on i. For $i = 0$ the claim holds trivially. Let $i > 0$. Note that, by Lemma 8.3.2, we have for $k \leq i - 1$,

$$A_{i,i-1}A_{i-1,k}\tilde{E}^k = A_{ik}\tilde{E}^k.$$

By the induction hypothesis, we obtain

$$\begin{aligned}
E^i &= A_{i,i-i}E^{i-1} \bigoplus \tilde{E}^i \\
&= A_{i,i-1}\left(\bigoplus_{0 \leq k \leq \lfloor \frac{n}{2} \rfloor, k \leq i-1} A_{i-1,k}\tilde{E}^k \right) \bigoplus \tilde{E}^i \\
&= \left(\bigoplus_{0 \leq k \leq \lfloor \frac{n}{2} \rfloor, k \leq i-1} A_{ik}\tilde{E}^k \right) \bigoplus \tilde{E}^i \\
&= \bigoplus_{0 \leq k \leq \lfloor \frac{n}{2} \rfloor, k \leq i} A_{ik}\tilde{E}^k = \bigoplus_{0 \leq k \leq \lfloor \frac{n}{2} \rfloor, k \leq i \leq n-k} A_{ik}\tilde{E}^k.
\end{aligned}$$

Now, let $i > \lfloor \frac{n}{2} \rfloor$. By Lemma 8.3.2, we have for $k \leq i \leq n - k$,

$$A_{i,n-i}A_{n-i,k}\tilde{E}^k = A_{ik}\tilde{E}^k.$$

By Theorem 8.3.6, we have $E^i = A_{i,n-i}E^{n-i}$. Therefore, since $n - i \leq \lfloor \frac{n}{2} \rfloor$, we obtain

$$\begin{aligned}
E^i = A_{i,n-i}E^{n-i} &= A_{i,n-i}\left(\bigoplus_{0 \leq k \leq \lfloor \frac{n}{2} \rfloor, k \leq i \leq n-k} A_{n-i,k}\tilde{E}^k \right) \\
&= \bigoplus_{0 \leq k \leq \lfloor \frac{n}{2} \rfloor, k \leq i \leq n-k} A_{ik}\tilde{E}^k.
\end{aligned}$$

Lemma 8.3.8. *Let $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$ and let $v \in \tilde{E}^k$. Then, $A_{jk}v = 0$ for all $0 \leq j < k$.*

Proof. By definition, we have

$$v(A_{k,k-1}u) = 0$$

for all $u \in E^{k-1}$. Hence, $vA_{k,k-1} = 0$. That is,

$$A_{k-1,k}v = (vA_{k,k-1})^t = 0.$$

Now, let $0 \leq j \leq k - 1$. By Lemma 8.3.2 we have

$$A_{k,k-1}A_{k-1,j} = \binom{k-j}{k-j-1}_q A_{kj}.$$

Transposing the equation, we obtain

$$A_{jk}v = \binom{k-j}{k-j-1}_q^{-1} A_{j,k-1}(A_{k-1,k}v) = 0,$$

as wanted. $\square$

Lemma 8.3.9. *Let* $0 \le k \le \lfloor \frac{n}{2} \rfloor$. *Let* $0 \le i < k \le j \le n$. *Let* $v \in \tilde{E}^k$. *Then,*

$$A_{ij}A_{jk}v = 0.$$

Proof. By Lemma 8.3.4, we have

$$A_{kj}A_{ji} = \sum_{m=0}^{i} c_{kjim}A_{km}A_{mi}.$$

Transposing the equation, we obtain

$$A_{ij}A_{jk}v = \sum_{m=0}^{i} c_{kjim}A_{im}A_{mk}v.$$

By Lemma 8.3.8, we have $A_{mk}v = 0$ for all $m \le i < k$. Therefore, we obtain

$$A_{ij}A_{jk}v = 0,$$

as wanted. $\square$

8.3.1 Proof of Theorem 1.6.2

For $0 \le k \le \lfloor \frac{n}{2} \rfloor$, let B_k be a basis for $\tilde{E}^k$. Then, by Proposition 8.3.7,

$$B = \bigcup_{k=0}^{\lfloor \frac{n}{2} \rfloor} \bigcup_{v \in B_k} \{A_{ik}v : \max\{1,k\} \le i \le \min\{n-1, n-k\}\}$$

is a basis for $C^0(X_{n,q})$.

Theorem 8.3.10. *Let* $L = L_0^+(Fl_{n,q})$. *Then, the matrix representation of* L *with respect to the basis* B *is a block diagonal matrix*

$$\begin{pmatrix} L_0 & & \\ & \ddots & \\ & & L_{\lfloor \frac{n}{2} \rfloor} \end{pmatrix}$$

with blocks

$$L_k = I_{\dim(\tilde{E}^k)} \otimes \tilde{L}_k,$$

where $\tilde{L}_0$ is the $(n-1)\times(n-1)$ matrix with entries

$$(\tilde{L}_0)_{ij} = \begin{cases} n-2 & \text{if } i=j, \\ -1 & \text{if } i\neq j, \end{cases} \tag{8.6}$$

for $1\leq i,j\leq n-1$, and, for $1\leq k\leq \left\lfloor \frac{n}{2}\right\rfloor$, $\tilde{L}_k$ is the $(n-2k+1)\times(n-2k+1)$ matrix with entries

$$(\tilde{L}_k)_{ij} = \begin{cases} n-2 & \text{if } i=j, \\ -c_{ijkk}\binom{n-i}{j-i}_q^{-1} & \text{if } i<j, \\ -\binom{i-k}{j-k}_q\binom{i}{j}_q^{-1} & \text{if } i>j \end{cases} \tag{8.7}$$

for $k\leq i,j\leq n-k$.

Proof. Let $0\leq k\leq \left\lfloor \frac{n}{2}\right\rfloor$. Let $v\in B_k$ and $\max\{1,k\}\leq j\leq \min\{n-1,n-k\}$. By Lemma 8.3.1, we have

$$LA_{jk}v = -\sum_{i=1}^{j-1}\binom{n-i}{j-i}_q^{-1} A_{ij}A_{jk}v + (n-2)A_{jk}v - \sum_{i=j+1}^{n-1}\binom{i}{j}_q^{-1} A_{ij}A_{jk}v.$$

Let $1\leq i\leq n-1$. If $i<k$, we have by Lemma 8.3.9,

$$A_{ij}A_{jk}v = 0.$$

Next, let $i>n-k$. Then, we have $n-i<k\leq j\leq n-k<i$. So, we have, by Lemma 8.3.4 (after transposing the equation),

$$A_{n-i,i}A_{ij} = \sum_{m=0}^{n-i} c_{j,i,n-i,m}A_{n-i,m}A_{mj}.$$

Therefore, by Lemma 8.3.9, we obtain

$$A_{n-i,i}A_{ij}A_{jk}v = \sum_{m=0}^{n-i} c_{j,i,n-i,m}A_{n-i,m}A_{mj}A_{jk}v = 0.$$

Since $A_{n-i,i}$ is invertible (by Theorem 8.3.6), we obtain

$$A_{ij}A_{jk}v = 0.$$

Now, assume $k\leq i\leq n-k$. If $i<j$, we have by Lemma 8.3.4 and Lemma 8.3.8,

$$A_{ij}A_{jk}v = \sum_{m=0}^{k} c_{ijkm}A_{im}A_{mk}v = c_{ijkk}A_{ik}v.$$

If $i > j$, we have by Lemma 8.3.2,

$$A_{ij}A_{jk}v = \binom{i-k}{j-k}_q A_{ik}v.$$

Therefore, we obtain

$$LA_{jk}v = -\sum_{i=\max\{1,k\}}^{j-1} c_{ijkk}\binom{n-i}{j-i}_q^{-1} A_{ik}v + (n-2)A_{jk}v \\ - \sum_{i=j+1}^{\min\{n-1,n-k\}} \binom{i}{j}_q^{-1}\binom{i-k}{j-k}_q A_{ik}v.$$

Thus, the subspace spanned by the vectors

$$\{A_{ik}v : \max\{1,k\} \le i \le \min\{n-1, n-k\}\}$$

is invariant under L, and the matrix representation of the restriction of L on this subspace is exactly $\tilde{L}_k$ (for the case $k = 0$, note that, by Equation (8.4), we have $c_{ijkk} = \binom{n-i}{j-i}_q$). Therefore, the representation of L with respect to the basis B is the diagonal block matrix

$$\begin{pmatrix} L_0 & & \\ & \ddots & \\ & & L_{\lfloor \frac{n}{2} \rfloor} \end{pmatrix}$$

where

$$L_k = I_{\dim(\tilde{E}^k)} \otimes \tilde{L}_k = \begin{pmatrix} \tilde{L}_k & & \\ & \ddots & \\ & & \tilde{L}_k \end{pmatrix}$$

for all $0 \le k \le \lfloor \frac{n}{2} \rfloor$.

Now we can prove our main result, Theorem 1.6.2.

Theorem 1.6.2. *Let $n \ge 3$ and let q be a prime power. Then, for any $\epsilon > 0$ there is an integer q_0 such that, for $q \ge q_0$, any eigenvalue $\lambda \ne 0, n-1$ of $L_0^+(Fl_{n,q})$ satisfies*

$$|\lambda - (n-2)| < \epsilon.$$

That is, as q tends to infinity, all nonzero eigenvalues of $L_0^+(Fl_{n,q})$ either are equal to $n-1$ or tend to $n-2$.

Proof. By Theorem 8.3.10, the set of eigenvalues of $L = L_0^+(\mathrm{Fl}_{n,q})$ is the union of the sets of eigenvalues of the matrices $\tilde{L}_k$, for $0 \le k \le \lfloor \frac{n}{2} \rfloor$.

First, let $k = 0$. By Equation (8.6), we have for $1 \leq i, j \leq n-1$,

$$(\tilde{L}_0)_{ij} = \begin{cases} n-2 & \text{if } i = j, \\ -1 & \text{if } i \neq j. \end{cases}$$

So, for all q, the eigenvalues of $\tilde{L}_0$ are 0 with multiplicity 1, and $n-1$ with multiplicity $n-2$.

Now, let $k \geq 1$, and let $k \leq i, j \leq n-k$. Then, by Equation (8.7) and Lemma 8.1.3, we have for $i > j$

$$\lim_{q\to\infty} (\tilde{L}_k)_{ij} = \lim_{q\to\infty} -\frac{\binom{i-k}{j-k}_q}{\binom{i}{j}_q} = 0.$$

For $i < j$, we have by Equation (8.7) and Lemma 8.3.5,

$$\lim_{q\to\infty} (\tilde{L}_k)_{ij} = \lim_{q\to\infty} -\frac{c_{ijkk}}{\binom{n-i}{j-i}_q} = -1.$$

Thus, the matrix $\tilde{L}_k$ tends element-wise to the upper triangular matrix:

$$\lim_{q\to\infty} (\tilde{L}_k)_{ij} = \begin{cases} n-2 & \text{if } i = j, \\ -1 & \text{if } i < j, \\ 0 & \text{if } i > j. \end{cases}$$

Therefore, as $q \to \infty$, all the eigenvalues of $\tilde{L}_k$ tend to $n-2$. That is, for any $\epsilon > 0$ there is an integer q_0 such that for $q \geq q_0$, any eigenvalue λ of $\tilde{L}_k$ satisfies $|\lambda - (n-2)| < \epsilon$.□

Another consequence of Theorem 8.3.10 is the following bound on the number of distinct eigenvalues of $L_0^+(\mathrm{Fl}_{n,q})$.

Corollary 8.3.11. *For any prime power $q \geq 2$, the number of distinct eigenvalues of $L_0^+(Fl_{n,q})$ is at most $\left\lfloor \frac{n^2}{4} \right\rfloor + 2$.*

Proof. By Theorem 8.3.10, the set of distinct eigenvalues of $L_0^+(\mathrm{Fl}_{n,q})$ is the union of the sets of eigenvalues of the matrices $\tilde{L}_k$, for $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$. The matrix $\tilde{L}_0$, defined by

$$(\tilde{L}_0)_{ij} = \begin{cases} n-2 & \text{if } i = j, \\ -1 & \text{if } i \neq j, \end{cases}$$

has only two distinct eigenvalues. For $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, the matrix $\tilde{L}_k$ is an $(n-2k+1) \times (n-2k+1)$ matrix, and therefore it has at most $n-2k+1$ distinct eigenvalues. Hence, the matrix $L_0^+(\mathrm{Fl}_{n,q})$ has at most

$$
\begin{aligned}
2+\sum_{k=1}^{\left\lfloor \frac{n}{2}\right\rfloor}(n-2k+1) &= 2+n\left\lfloor\frac{n}{2}\right\rfloor-\left(1+\left\lfloor\frac{n}{2}\right\rfloor\right)\left\lfloor\frac{n}{2}\right\rfloor+\left\lfloor\frac{n}{2}\right\rfloor \\
&= 2+\left\lfloor\frac{n}{2}\right\rfloor\left(n-\left\lfloor\frac{n}{2}\right\rfloor\right)=\left\lfloor\frac{n}{2}\right\rfloor\left\lceil\frac{n}{2}\right\rceil+2=\left\lfloor\frac{n^2}{4}\right\rfloor+2
\end{aligned}
$$

distinct eigenvalues. □

We can see Corollary 8.3.11 as a first step towards the $k=0$ case of the first part of Conjecture 1.6.1. Based on this result, we make the following refined conjecture:

Conjecture 8.3.12. *For any prime power $q \geq 2$, the number of distinct eigenvalues of $L_0^+(Fl_{n,q})$ is exactly $\left\lfloor \frac{n^2}{4}\right\rfloor + 2$.*

Chapter 9

Conclusion

In this chapter we summarize the open problems and possible directions for further investigation arising from our work.

9.1 Collapsibility of complexes from families of matrices and graphs

Let $\mathbb{F}$ be a field, and let $\mathcal{A}$ be a finite family of $m \times n$ matrices over $\mathbb{F}$. In Chapter 3 we studied the complex $\mathrm{M}_{\mathcal{A},r}$ whose simplices correspond to families $\mathcal{B} \subset \mathcal{A}$ such that any matrix in $\mathrm{span}(\mathcal{B})$ is of rank at most r. We showed that, if $\mathbb{F}$ is an infinite field, the collapsibility of $\mathrm{M}_{\mathcal{A},r}$ is at most $r(r+1)$ (Theorem 1.1.5). We don't expect the condition on $\mathbb{F}$ to be necessary or the bound to be tight. In fact, we conjecture the following:

Conjecture 3.5.15. *Let $\mathcal{A}$ be a finite family of matrices in $\mathbb{F}^{m\times n}$, and let $r \geq 1$. Then, $M_{\mathcal{A},r}$ is $2r$-collapsible.*

It may be interesting to study the collapsibility of $\mathrm{M}_{\mathcal{A},r}$ for special families of matrices. For example, we conjecture that if $\mathcal{A}$ consists of skew-symmetric matrices of rank two, the bound on the collapsibility may be reduced to $\frac{3r}{2}$ (Conjecture 3.5.13), and if $\mathcal{A}$ consists of symmetric matrices of rank two, the bound on the collapsibility may be reduced to r (Conjecture 3.5.14).

In Chapter 4 we studied the collapsibility of $I_n(G)$, the simplicial complex whose vertices are the vertices of the graph $G = (V, E)$ and whose simplices are subsets $U \subset V$ that do not contain an independent set of size n in G. Our main concern was on the question whether, for a graph G with maximum degree Δ, the bound $C(I_n(G)) \leq \left\lceil \frac{\Delta+1}{2} \right\rceil (n-1)$ holds (Question 1.2.4). We answered this question in the affirmative in the special cases $n \leq 3$ or $\Delta \leq 2$, but found examples showing that in general the answer is negative. It would be interesting to decide for which values of Δ and n the bound in Question 1.2.4 holds. The combinatorial conjecture stating that $f_G(n) \leq \left\lfloor \left(\frac{\Delta}{2}+1\right)(n-1) \right\rfloor + 1$ for graphs with maximum degree Δ (Conjecture 1.2.3) remains open.

A weaker property which may hold is the following:

Conjecture 9.1.1. *Let $G=(V,E)$ be a graph with maximum degree at most Δ, and let $n \geq 1$ be an integer. Let A be an independent set of size $n-1$ in G. Then,*

$$C(\mathrm{lk}(I_n(G),A)) \leq \left\lfloor \frac{(n-1)\Delta}{2} \right\rfloor.$$

For the subclass of claw-free graphs, this is proved in Proposition 4.4.5. Conjecture 9.1.1 would imply the bound $f_G(n) \leq \left\lfloor \left(\frac{\Delta}{2}+1\right)(n-1)\right\rfloor + 1$ (by the same argument as the one used to prove Theorem 1.2.9), settling Conjecture 1.2.3 in the case of even Δ.

Another possible direction is to focus on the family of claw-free bounded degree graphs. We showed in Theorem 1.2.9 that Conjecture 1.2.3 holds for graphs in this family when Δ is even. In the case of odd Δ, although we obtain good upper bounds for $f_G(n)$, the question remains unsettled. It would also be interesting to prove the corresponding tight upper bound on the collapsibility number of $I_n(G)$, at least for the case of even Δ.

We know, by Proposition 4.5.2, that the bound in Question 1.2.4 does not hold for graphs with maximum degree at most 3. The following problem arises:

Problem 9.1.2. *Find the smallest positive integer $g(n)$ such that the following holds: for every graph G with maximum degree at most 3,*

$$C(I_n(G)) \leq g(n).$$

By Theorem 1.2.6 and Proposition 4.5.1 we have $2(n-1) \leq g(n) \leq 3(n-1)$ for all $n \geq 1$, and, by Corollary 4.5.5, $g(8k) \geq 17k-1$ for all $k \geq 1$. Improving either the upper or lower bounds for $g(n)$ may be of interest.

9.2 Leray numbers of tolerance complexes

Recall that given a complex K on vertex set V and an integer $t \geq 0$, the t-tolerance complex $\mathcal{T}_t(K)$ is the simplicial complex on vertex set V whose simplices are the sets $U \subset V$ that contain a simplex $\sigma \in K$ of size $|\sigma| \geq |U| - t$.

In Chapter 5 we showed that for any d and t there exists an integer $h(t,d)$ such that if K is d-collapsible, then $\mathcal{T}_t(K)$ is d-Leray (Theorem 1.3.5). It would be interesting either to weaken the condition on K from being d-collapsible to being d-Leray (see Conjecture 1.3.3), or strengthening the conclusion on the tolerance complex to give a bound on its collapsibility (see Conjecture 1.3.4).

Furthermore, we don't expect the bound $h(t,d)$ to be tight (except in the case $d=1$). In particular, in the case $t=1$ we conjecture the following:

Conjecture 9.2.1. *Let K be d-collapsible. Then, $\mathcal{T}_1(K)$ is $\left(\left\lfloor \left(\frac{d+3}{2}\right)^2 \right\rfloor - 1\right)$-Leray.*

The bound in the conjecture is of the same order of magnitude, but smaller, than the bound proved in Theorem 1.3.5, $h(1,d) = d^2 + 2d$. We were able to verify this conjecture only for $d \leq 2$ (see Theorem 1.3.6).

9.3 Representability of complexes without large missing faces

In Chapter 6 we presented the following conjecture:

Conjecture 1.4.5. *Let X be simplicial complex with n vertices, satisfying $h(X) \leq d$. Then,*

$$rep(X) \leq \left\lfloor \frac{dn}{d+1} \right\rfloor .$$

Moreover, $rep(X) = \frac{dn}{d+1}$ if and only if the missing faces of X consist of $\frac{n}{d+1}$ pairwise disjoint sets of size $d+1$.

For $d = 1$ this follows almost immediately from Roberts' theorem on the boxicity of a graph (see Proposition 6.5.1). For $d = n-1$ this is a result of Wegner (Theorem 6.3.3).

As an interesting special case, we propose to focus on the family of complexes whose missing faces form a Steiner triple system. In fact, even solving the following particular case may be of interest:

Conjecture 6.5.5. *Let $X_{2,9}$ be the simplicial complex whose missing faces form a Steiner $(2,3,9)$-system (that is, they are the lines of the affine plane of order 3). Then,*

$$rep(X_{2,9}) \leq 5.$$

9.4 Complexes of hyperplane-free sets and stability of blocking sets in finite affine spaces

Let q be a prime power and $n \geq 2$ an integer. In Chapter 7 we defined $\hat{X}_{q,n}$ to be the simplicial complex on vertex set $\mathbb{F}_q^n \setminus \{0\}$ whose simplices are the subsets that do not contain any affine hyperplane. We conjecture

Conjecture 7.4.4. *Let $q \geq 3$ be a prime power. Then,*

$$\tilde{H}_{q^n - n(q-1)-2}(\hat{X}_{q,n}) = \mathbb{Z}^{\prod_{i=1}^{n} \frac{q^i-1}{q-1}}.$$

The $n = 2$ of the conjecture follows from Theorem 1.5.2.

Recall that a set $B \subset \mathbb{F}_q^n$ is a blocking set if it intersects all the affine hyperplanes, and it is called strongly stable if for every point $v \in \mathbb{F}_q^n \setminus B$ there is some $u \in B \setminus \{0\}$ such that $B \cup \{v\} \setminus \{u\}$ is also a blocking set. We conjecture the following characterization of strongly stable blocking sets of size $n(q-1)+1$, generalizing the characterization in the $n=2$ case in Theorem 1.5.3:

Conjecture 7.4.7. *Let $B \subset \mathbb{F}_q^n$ be a blocking set of size $n(q-1)+1$ containing the origin. Then, B is strongly stable if and only if there is a flag $V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_{n-1}$ of linear subspaces of $\mathbb{F}_q^n$ such that $\dim(V_k) = k$ and $|B \cap V_k| = k(q-1)+1$ for all $k \in [n-1]$.*

In addition to its interest for its own sake, we expect Conjecture 7.4.7 to be an important step towards a solution of Conjecture 7.4.4.

9.5 Papikian's conjecture on the eigenvalues of complexes of flags

Recall that $\mathrm{Fl}_{n,q}$ is the simplicial complex whose vertices correspond to non-trivial linear subspaces of $\mathbb{F}_q^n$ and whose simplices correspond to flags. Let $L_k^+(\mathrm{Fl}_{n,q})$ be the k-dimensional weighted upper Laplacian on $\mathrm{Fl}_{n,q}$. In [Pap16], Papikian conjectured the following:

Conjecture 1.6.1 (Papikian [Pap16]). *Let $n \geq 3$ and let q be a prime power. Let $0 \leq k \leq n-3$. Then, as q tends to infinity, the positive (i.e nonzero) eigenvalues of $L_k^+(Fl_{n,q})$ tend to the integers*

$$n-k-2, n-k-1, n-k, \ldots, n-1.$$

Or, more formally: for any $\epsilon > 0$ there exists an integer q_0 such that, for $q \geq q_0$, for any eigenvalue λ of $L_k^+(Fl_{n,q})$ there is some $m \in \{n-k-2, n-k-1, \ldots, n-1\}$ such that

$$|\lambda - m| < \epsilon.$$

In Chapter 8 we prove this conjecture in the special case $k=0$ (Theorem 1.6.2). A possible direction for future research is to try to extend the methods in Chapter 8 to prove the general case of the conjecture, or at least some other special cases, such as the case $k=1$.

Bibliography

[AB09] Ron Aharoni and Eli Berger. Rainbow matchings in r-partite r-graphs. *Electron. J. Combin.*, 16(1):Research Paper 119, 9, 2009.

[ABC$^+$19] Ron Aharoni, Eli Berger, Maria Chudnovsky, David Howard, and Paul Seymour. Large rainbow matchings in general graphs. *European J. Combin.*, 79:222–227, 2019.

[ABKK19] Ron Aharoni, Joseph Briggs, Jinha Kim, and Minki Kim. Rainbow independent sets in certain classes of graphs. preprint, `https://arxiv.org/abs/1909.13143`, 2019.

[Ada14] Michał Adamaszek. Extremal problems related to Betti numbers of flag complexes. *Discrete Appl. Math.*, 173:8–15, 2014.

[Aha19] Ron Aharoni. Personal communication, 2019.

[AHJ19] Ron Aharoni, Ron Holzman, and Zilin Jiang. Rainbow fractional matchings. *Combinatorica*, 39(6):1191–1202, 2019.

[AK85] Noga Alon and Gil Kalai. A simple proof of the upper bound theorem. *European J. Combin.*, 6(3):211–214, 1985.

[Alo85] Noga Alon. An extremal problem for sets with applications to graph theory. *J. Comb. Theory (A)*, 40(1):82–89, 1985.

[AY21] Karim Adiprasito and Geva Yashfe. The partition complex: an invitation to combinatorial commutative algebra. *Surveys in Combinatorics 2021*, page 1, 2021.

[Bár82] Imre Bárány. A generalization of Carathéodory's theorem. *Discrete Mathematics*, 40(2-3):141–152, 1982.

[BBM97] Anders Björner, Lynne M. Butler, and Andrey O. Matveev. Note on a combinatorial application of Alexander duality. *Journal of Combinatorial Theory, Series A*, 80(1):163–165, 1997.

[BGS17] János Barát, András Gyárfás, and Gábor N. Sárközy. Rainbow matchings in bipartite multigraphs. *Period. Math. Hungar.*, 74(1):108–111, 2017.

[BK19] Joseph Briggs and Minki Kim. Choice functions in the intersection of matroids. *Electron. J. Combin.*, 26(4):Paper No. 4.26, 11, 2019.

[Bro41] R. L. Brooks. On colouring the nodes of a network. *Proc. Cambridge Philos. Soc.*, 37:194–197, 1941.

[BS78] Andries E. Brouwer and Alexander Schrijver. The blocking number of an affine space. *Journal of Combinatorial Theory, Series A*, 24(2):251–253, 1978.

[BT82] Raoul Bott and Loring W. Tu. *Differential forms in algebraic topology*, volume 82. Springer, 1982.

[Car73] Leonard Carlitz. Some inverse relations. *Duke Mathematical Journal*, 40(4):893–901, 1973.

[CL82] Charles W. Curtis and Gustav I. Lehrer. A new proof of a theorem of Solomon-Tits. *Proceedings of the American Mathematical Society*, 85(2):154–156, 1982.

[CR83] Margaret B. Cozzens and Fred S. Roberts. Computing the boxicity of a graph by covering its complement by cointerval graphs. *Discrete Applied Mathematics*, 6(3):217–228, 1983.

[Die48] Jean Dieudonné. Sur une généralisation du groupe orthogonal à quatre variables. *Archiv der Mathematik*, 1(4):282–287, Jul 1948.

[Eck93] Jürgen Eckhoff. Helly, Radon, and Carathéodory type theorems. In *Handbook of convex geometry*, pages 389–448. Elsevier, 1993.

[Edm67] Jack Edmonds. Systems of distinct representatives and linear algebra. *J. Res. Nat. Bur. Standards Sect. B*, 71(4):241–245, 1967.

[EG61] Paul Erdős and Tibor Gallai. On the minimal number of vertices representing the edges of a graph. *Publ. Math. Inst. Hungar. Acad. Sci*, 6(18):1–203, 1961.

[Fla62] Harley Flanders. On spaces of linear transformations with bounded rank. *Journal of the London Mathematical Society*, 1(1):10–16, 1962.

[Fol66] Jon Folkman. The homology groups of a lattice. *Journal of Mathematics and Mechanics*, pages 631–636, 1966.

[Fra82] Peter Frankl. An extremal problem for two families of sets. *Eur. J. Comb.*, 3(2):125–127, 1982.

[Für84] Zoltán Füredi. Geometrical solution of an intersection problem for two hypergraphs. *Eur. J. Comb.*, 5(2):133–136, 1984.

[Hat02] Allen Hatcher. Algebraic topology. 2002. *Cambridge UP, Cambridge*, 606(9), 2002.

[HL20] Andreas Holmsen and Seunghun Lee. Leray numbers of complexes of graphs with bounded matching number. preprint, `https://arxiv.org/abs/2003.11270`, 2020.

[HS10] Liza Huijse and Kareljan Schoutens. Supersymmetry, lattice fermions, independence complexes and cohomology theory. *Advances in Theoretical and Mathematical Physics*, 14(2):643–694, 2010.

[HW14] Huy Tài Hà and Russ Woodroofe. Results on the regularity of square-free monomial ideals. *Advances in Applied Mathematics*, 58:21–36, 2014.

[Jam77] Robert E. Jamison. Covering finite fields with cosets of subspaces. *Journal of Combinatorial Theory, Series A*, 22(3):253–266, 1977.

[Jon08] Jakob Jonsson. *Simplicial complexes of graphs*. Lecture Notes in Math. 1928, Springer, Heidelberg, 2008.

[Kal84] Gil Kalai. Intersection patterns of convex sets. *Israel J. Math.*, 48(2-3):161–174, 1984.

[Kan72] William M Kantor. On incidence matrices of finite projective and affine spaces. *Mathematische Zeitschrift*, 124(4):315–318, 1972.

[Kee14] Peter Keevash. The existence of designs. preprint, `https://arxiv.org/abs/1401.3665`, 2014.

[Khm18] Igor Khmelnitsky. d-collapsibility and its applications. Master's thesis, Technion, Haifa, 2018.

[KL19] Minki Kim and Alan Lew. Complexes of graphs with bounded independence number. preprint, `https://arxiv.org/abs/1912.12605`, 2019.

[KM05] Gil Kalai and Roy Meshulam. A topological colorful Helly theorem. *Adv. Math.*, 191(2):305–311, 2005.

[KM06] Gil Kalai and Roy Meshulam. Intersections of leray complexes and regularity of monomial ideals. *Journal of Combinatorial Theory, Series A*, 113(7):1586–1592, 2006.

[LB63] C. G. Lekkerkerker and J. Ch. Boland. Representation of a finite graph by a set of intervals on the real line. *Fund. Math.*, 51:45–64, 1962/63.

[LNP80] László Lovász, Jaroslav Nešetřil, and Ales Pultr. On a product dimension of graphs. *J. Comb. Theory (B)*, 29(1):47–67, 1980.

[Lov89] László Lovász. Singular spaces of matrices and their application in combinatorics. *Boletim da Sociedade Brasileira de Matemática-Bulletin/Brazilian Mathematical Society*, 20(1):87–99, 1989.

[LSW08] Svante Linusson, John Shareshian, and Volkmar Welker. Complexes of graphs with bounded matching size. *Journal of Algebraic Combinatorics*, 27(3):331–349, 2008.

[Mat09] Jiří Matoušek. Removing degeneracy in lp-type problems revisited. *Discrete & Computational Geometry*, 42(4):517, 2009.

[Mes85] Roy Meshulam. On the maximal rank in a subspace of matrices. *The Quarterly Journal of Mathematics*, 36(2):225–229, 1985.

[Mes01] Roy Meshulam. The clique complex and hypergraph matching. *Combinatorica*, 21(1):89–94, 2001.

[MO11] Luis Montejano and Deborah Oliveros. Tolerance in helly-type theorems. *Discrete & Computational Geometry*, 45(2):348–357, 2011.

[Mon17] Luis Montejano. A variation on the homological nerve theorem. *Topology and its Applications*, 225:139–144, 2017.

[MT09] Jiří Matoušek and Martin Tancer. Dimension gaps between representability and collapsibility. *Discrete Comput. Geom.*, 42(4):631–639, 2009.

[Pap16] Mihran Papikian. On Garland's vanishing theorem for SL_n. *European Journal of Mathematics*, 2(3):579–613, 2016.

[Rob69] Fred S. Roberts. On the boxicity and cubicity of a graph. *Recent Progresses in Combinatorics*, pages 301–310, 1969.

[Röd85] Vojtěch Rödl. On a packing and covering problem. *European Journal of Combinatorics*, 6(1):69–78, 1985.

[Tan10a] Martin Tancer. d-collapsibility is NP-complete for $d \geq 4$. *Chic. J. Theoret. Comput. Sci.*, pages Article 3, 32, 2010.

[Tan10b] Martin Tancer. Non-representability of finite projective planes by convex sets. *Proceedings of the American Mathematical Society*, 138(9):3285–3291, 2010.

[Tan11] Martin Tancer. Strong d-collapsibility. *Contrib. Discrete Math.*, 6(2):32–35, 2011.

[Tan13] Martin Tancer. Intersection patterns of convex sets via simplicial complexes: a survey. In *Thirty essays on geometric graph theory*, pages 521–540. Springer, 2013.

[Tut47] William T Tutte. The factorization of linear graphs. *Journal of the London Mathematical Society*, 1(2):107–111, 1947.

[Tuz85] Zsolt Tuza. Critical hypergraphs and intersecting set-pair systems. *Journal of Combinatorial Theory, Series B*, 39(2):134–145, 1985.

[VLW01] Jacobus H. Van Lint and Richard M. Wilson. *A course in combinatorics*. Cambridge university press, 2001.

[Wat69] Mark E. Watkins. A theorem on Tait colorings with an application to the generalized Petersen graphs. *J. Combinatorial Theory*, 6:152–164, 1969.

[Weg67] Gerd Wegner. Eigenschaften der nerven homologish-einfacher familien im $\mathbb{R}^n$. *Doctorate, Göttingen*, 1967.

[Weg75] Gerd Wegner. d-collapsing and nerves of families of convex sets. *Arch. Math. (Basel)*, 26:317–321, 1975.

[Wit80] Hans S. Witsenhausen. On intersections of interval graphs. *Discrete Mathematics*, 31(2):211–216, 1980.

משפט.

$$\tilde{H}_{q^2-2q}(X_q) = \begin{cases} \mathbb{Z}^3 & \text{IF } q=2, \\ \mathbb{Z}^{11} & \text{IF } q=3, \\ \mathbb{Z}^{q(q+1)} & \text{IF } q>3, \end{cases}$$

ו־

$$\tilde{H}_{q^2-2q}(\hat{X}_q) = \begin{cases} \mathbb{Z}^2 & \text{IF } q=2, \\ \mathbb{Z}^{q+1} & \text{IF } q>2. \end{cases}$$

תהי B קבוצה חוסמת בגודל $2q-1$ ב־$\mathbb{F}_q^2$. הקבוצה B נקראת "יציבה" אם לכל נקודה $v \notin B$ קיימת נקודה $u \in B$ כך ש־$B \cup \{v\} \setminus \{u\}$ היא גם קבוצה חוסמת. B נקראת "יציבה בצורה חזקה" אם $0 \in B$, ולכל $v \notin B$ קיימת $u \in B \setminus \{0\}$ כך ש־$B \cup \{v\} \setminus \{u\}$ היא גם קבוצה חוסמת. אחד המרכיבים העיקריים בקביעת ההומולוגיה של הקומפלקסים של קבוצות חסרות־ישרים הוא המשפט הבא, המאפיין את הקבוצות החוסמות היציבות והיציבות בצורה חזקה:

משפט. תהי B קבוצה חוסמת בגודל $2q-1$ ב־$\mathbb{F}_q^2$. אזי, B היא יציבה אם ורק אם היא מכילה ישר אפיני שלם, והיא יציבה בצורה חזקה אם ורק אם היא מכילה ישר שלם העובר דרך הראשית.

אוסף תתי מרחבים וקטוריים $V_1, \ldots, V_k$ ב־$\mathbb{F}_q^n$ נקרא דגל אם $V_i \subset V_{i+1}$ לכל $i = 1, \ldots, k-1$. נגדיר את קומפלקס הדגלים $\mathrm{FL}_{n,q}$ להיות הקומפלקס שקודקודיו הם תתי המרחבים הלא־טריוויאליים של $\mathbb{F}_q^n$ והסימפלקסים שלו הם אוספי תתי מרחבים המהווים דגל.

בפרק 8 אנו חוקרים את הערכים העצמיים של אופרטורי הלפלסיאן הממושקלים על קומפלקס הדגלים. אנו מוכיחים את הטענה הבאה:

משפט. לכל $\epsilon > 0$ קיים q_0 כך שלכל חזקה ראשונית $q \geq q_0$, כל ערך עצמי $\lambda \neq 0, n-1$ של אופרטור הלפלסיאן הממושקל ה־0־מימדי על $\mathrm{FL}_{n,q}$ מקיים:

$$|\lambda - (n-2)| < \epsilon.$$

כלומר, כאשר q שואף לאינסוף, כל ערך עצמי של הלפלסיאן שאינו שווה ל־0 או ל־$n-1$ שואף לערך $n-2$.

תוצאה זאת פותרת מקרה פרטי (את המקרה ה־0־מימדי) של השערה של פפיקיאן על ההתנהגות האסימפטוטית של הערכים העצמיים של אופרטורי הלפלסיאן על קומפלקס הדגלים (ראה [Pap16]).

משפט. יהי K קומפלקס 2־מטיט. אזי, $\mathcal{T}_1(K)$ הוא 5־לירי.

בפרק השישי אנו עוסקים בשאלה הגיאומטרית הבאה: בהינתן מספר טבעי d וקומפלקס X בעל n קודקודים, האם ניתן לכתוב את X כחיתוך של מספר סופי של קומפלקסים d־יציגים? אם התשובה חיובית, מהו ה־t המינימלי עבורו ניתן לכתוב את X כחיתוך של t קומפלקסים d־יציגים? אנו קוראים ל־t המינימלי הזה ה"d־קופסאתיות" של X, ומסמנים זאת ב־$\text{BOX}_d(X)$.

פרמטר זה מכליל את מושג הקופסאתיות של גרף שנחקר לראשונה על ידי רוברטס בשנות ה־60 (ראה [Rob69]), ועל כן שמו. התוצאה העיקרית שלנו היא המשפט הבא, המכליל משפט דומה עבור גרפים מאת רוברטס:

פאה חסרה של קומפלקס X היא אוסף קודקודים τ כך ש־$\tau \notin X$ אבל $\sigma \in X$ לכל $\sigma \subsetneq \tau$. יהיו $t \le k \le n$. מערכת שטיינר מסוג (t,k,n) היא אוסף תתי קבוצות בגודל k של קבוצה V בגודל n, כך שכל תת קבוצה בגודל t של V מוכלת בדיוק באחת מהקבוצות באוסף.

משפט. יהי X קומפלקס סימפלציאלי בעל n קודקודים שהמימד המקסימלי של פאה חסרה שלו הוא d. אזי,

$$\text{BOX}_d(X) \le \left\lfloor \frac{1}{d+1}\binom{n}{d} \right\rfloor .$$

בנוסף, מתקיים שוויון $\text{BOX}_d(X) = \frac{1}{d+1}\binom{n}{d}$ אם ורק אם אוסף הפאות החסרות של X מהווה מערכת שטיינר מסוג $(d,d+1,n)$.

משפט זה, כאמור, מכליל את משפט רוברטס במקרה $d=1$, ומשפר תוצאות קודמות מאת וויטזנהאוזן ([Wit80]) עבור $d>1$.

בפרקים 7 ו־8 אנו חוקרים משפחות של קומפלקסים הקשורות להיבטים שונים של מרחבים וקטוריים סופיים. פרק 7 עוסק בקומפלקסים של קבוצות "חסרות־ישרים" במישור אפיני מעל שדה סופי. יהי $\mathbb{F}_q$ שדה בגודל q. קבוצה $\sigma \subset \mathbb{F}_q^2$ נקראת חסרת־ישרים אם היא לא מכילה ישר אפיני שלם. אנו מגדירים קומפלקסים

$$X_q = \left\{\sigma \subset \mathbb{F}_q^2 : \sigma \text{ IS LINE-FREE}\right\}$$

ו־

$$\hat{X}_q = \left\{\sigma \subset \mathbb{F}_q^2 \setminus \{0\} : \sigma \text{ IS LINE-FREE}\right\} = X_q \setminus 0.$$

קבוצה חוסמת ב־$\mathbb{F}_q^2$ היא אוסף נקודות שנוגעות בכל הישרים האפיניים. משפט של ג'מיסון ([Jam77]) ושל בראור ושריבר ([BS78]) קובע כי הגודל המינימלי של קבוצה חוסמת ב־$\mathbb{F}_q^2$ הוא $2q-1$. מכיוון שקבוצה היא חסרת ישרים אם ורק אם המשלים שלה הוא קבוצה חוסמת, אנו מקבלים כי מימד שני הקומפלקסים X_q ו־$\hat{X}_q$ הוא q^2-2q. התוצאה העיקרית שלנו בפרק זה היא קביעת ההומולוגיה במימד העליון של הקומפלקסים של קבוצות חסרות־ישרים:

החסם הנ"ל הדוק רק במקרה ש־$\Delta \leq 2$. עבור $n \leq 3$ ו־Δ כללי, אנו מוכיחים את החסמים ההדוקים הבאים:

משפט. יהי G גרף בעל דרגה מקסימלית לכל היותר Δ. אזי,

$$C(I_2(G)) \leq \left\lceil \frac{\Delta+1}{2} \right\rceil.$$

משפט. יהי G גרף בעל דרגה מקסימלית לכל היותר Δ. אזי,

$$C(I_3(G)) \leq \begin{cases} \Delta+2 & \text{IF } \Delta \text{ IS EVEN,} \\ \Delta+1 & \text{IF } \Delta \text{ IS ODD.} \end{cases}$$

כמסקנה מהתוצאות שלנו, אנו מקבלים הוכחות חדשות למספר תוצאות של אהרוני, בריגס, קים וקים [ABKK19] על קבוצות בלתי תלויות ססגוניות בגרפים. בנוסף, אנו מוכיחים את התוצאה החדשה הבאה:

משפט. יהי G גרף נטול טפרים בעל דרגה מקסימלית Δ, ויהי $n \geq 1$ מספר טבעי. יהי $t = \left\lfloor \left(\frac{\Delta}{2}+1\right)(n-1) \right\rfloor + 1$. יהיו $A_1, \ldots, A_t$ קבוצות בלתי תלויות בגודל n ב־G. אזי, קיימת קבוצה בלתי תלויה "ססגונית" $\{a_{i_1}, \ldots, a_{i_n}\}$, כאשר $1 \leq i_1 < i_2 < \cdots < i_n \leq t$ ו־$a_{i_j} \in A_{i_j}$ לכל j.

יהי K קומפלקס סימפלציאלי על קבוצת קודקודים V. בהינתן מספר טבעי t, אנו מגדירים קומפלקס

$$\begin{aligned} \mathcal{T}_t(K) &= \{\eta \cup \tau : \eta \in K, \tau \subset V, |\tau| \leq t\} \\ &= \{\sigma \subset V : \exists \eta \subset \sigma, |\sigma \setminus \eta| \leq t, \eta \in K\}. \end{aligned}$$

בפרק 5 אנו חוקרים את הקומפלקסים $\mathcal{T}_t(K)$. אנו מתארים את הקשר בין הקומפלקסים האלה לבין משפטים מסוג הלי שהוכיחו מונטחנו ואוליוורוס ב־[MO11]. העיסוק העיקרי שלנו הוא בשאלה מה ניתן להגיד על מספרי לריי או על המטיטות של $\mathcal{T}_t(K)$ אם ידוע שהקומפלקס המקורי K הוא d־מטיט או d־לריי. התוצאה העיקרית שלנו בפרק זה היא:

משפט. לכל t ו־d קיים מספר $h(t,d)$ המקיים: אם K הוא d־מטיט, אז $\mathcal{T}_t(K)$ הוא $h(t,d)$־לריי.

עבור $d = 1$, אנו מקבלים חסם הדוק $h(t,1) = 2t+1$. עבור $t = 1$, מתקיים $h(1,d) = d^2 + 2d$. במקרה זה החסם לא אופטימלי. במקרה הפרטי $d = 2, t = 1$, אנו מוכיחים את החסם ההדוק הבא:

העצב של משפחת קבוצות סופיות:

משפט. יהי X קומפלקס סימפלציאלי על קבוצת קודקודים V. יהי $S(X)$ אוסף הקבוצות $\{v_1,\ldots,v_k\}\subset V$ המקיימות את התכונה הבאה:

קיימות פאות מקסימליות $\sigma_1,\ldots,\sigma_{k+1}$ של X כך ש־

- $v_i\notin\sigma_i$ עבור $i\in[k]$,
- $v_i\in\sigma_j$ עבור $1\leq i<j\leq k+1$.

יהי $d'(X)$ הגודל המקסימלי של קבוצה ב־$S(X)$. אזי, X הוא $d'(X)$־מטיט.

לאחר מכן אנו חוקרים את המטיטות של משפחות של קומפלקסים הקשורות לתכונות שונות של גרפים והיפרגרפים. התוצאות המרכזיות שלנו בפרק 3 הן:

יהי $\mathcal{H}$ היפרגרף. נסמן ב־$\tau(\mathcal{H})$ את גודל הכיסוי המינימלי של $\mathcal{H}$. כלומר, זהו הגודל המינימלי של קבוצת קודקודים אשר חותכת את כל צלעות ההיפרגף. בהינתן מספר טבעי p, נגדיר את הקומפלקס

$$\mathrm{Cov}_{\mathcal{H},p}=\{\mathcal{F}\subset\mathcal{H}:\,\tau(\mathcal{F})\leq p\}.$$

בנוסף, נגדיר קומפלקס

$$\mathrm{Int}_{\mathcal{H}}=\{\mathcal{F}\subset\mathcal{H}:\,A\cap B\neq\emptyset\quad\forall A,B\in\mathcal{F}\}.$$

אנו מוכיחים את החסמים ההדוקים הבאים על המטיטות של הקומפלקסים הנ"ל:

משפט. יהי $\mathcal{H}$ היפרגרף בעל צלעות בגודל לכל היותר r. אזי, $\mathrm{Cov}_{\mathcal{H},p}$ הוא $\left(\binom{r+p}{r}-1\right)$־מטיט.

משפט. יהי $\mathcal{H}$ היפרגרף בעל צלעות בגודל לכל היותר r. אזי, $\mathrm{Int}_{\mathcal{H}}$ הוא $\frac{1}{2}\binom{2r}{r}$־מטיט.

יהי $G=(V,E)$ גרף. קבוצה בלתי תלויה ב־G היא אוסף קודקודים אשר אף שניים מתוכם אינם מחוברים על ידי צלע. נגדיר את $I_n(G)$ להיות הקומפלקס הסימפלציאלי שקודקודיו הם קודקודי הגרף G והסימפלקסים שלו הם קבוצות הקודקודים אשר אינן מכילות קבוצה בלתי תלויה בגודל n.

בפרק 4 אנו חוקרים את המטיטות של הקומפלקסים $I_n(G)$ עבור משפחות שונות של גרפים. אנו מתמקדים במשפחת הגרפים בעלי דרגה מקסימלית חסומה. אנו מוכיחים:

משפט. יהי G גרף בעל דרגה מקסימלית לכל היותר Δ. אזי,

$$C(I_n(G))\leq\Delta(n-1).$$

תקציר

קומפלקס סימפלציאלי הוא מרחב טופולוגי המורכב מאוסף אבני בניין פשוטות בשם סימפלקסים. לעיתים קרובות ניתן לשייך לבעיה קומבינטורית או גיאומטרית מסוימת קומפלקס סימפלציאלי מתאים. חקירת המבנה הטופולוגי של הקומפלקס חושפת לעיתים מידע על הבעיה המקורית.

בעבודה זאת אנו חוקרים היבטים טופולוגיים, קומבינטוריים וגיאומטריים שונים של קומפלקסים סימפלציאלים ואת הקשר ביניהם. שלוש תכונות מרכזיות שאנו עוסקים בהן הן תכונת d־לריי, d־מטיטות ו־d־יציגות של קומפלקס, המוגדרות באופן הבא:

יהי $\mathbb{F}$ שדה. קומפלקס X נקרא d־לריי אם $\tilde{H}_k(Y;\mathbb{F})=0$ לכל תת קומפלקס מושרה Y של X ולכל $k\geq d$. מספר לריי של X, המסומן ב־$L(X)$, הוא ה־d המינימלי עבורו X הוא d־לריי. אפשר לחשוב על מספר לריי כעל ה"מימד ההומולוגי התורשתי" של הקומפלקס.

יהי $\sigma\in X$ סימפלקס בגודל לכל היותר d המוכל בפאה מקסימלית יחידה τ ב־X. אנו אומרים כי הקומפלקס

$$X'=X\setminus\{\eta\in X:\,\sigma\subset\eta\subset\tau\}$$

מתקבל מ־X על ידי צעד d־מיטוט אלמנטרי. הקומפלקס X נקרא d־מטיט אם ניתן להסיר את כל הפאות שלו על ידי ביצוע סדרה של צעדי d־מיטוט אלמנטריים. המטיטות של X, המסומנת על ידי $C(X)$, היא המספר d המינימלי עבורו X הוא d־מטיט.

תהי $F_1,\ldots,F_n$ משפחה של קבוצות. העצב של המשפחה הוא הקומפלקס הסימפלציאלי

$$N(\{F_1,\ldots,F_n\})=\{I\subset[n]:\,\cap_{i\in I}F_i\neq\emptyset\}.$$

קומפלקס X נקרא d־יציג אם הוא איזומורפי לעצב של משפחת קבוצות קמורות ב־$\mathbb{R}^d$.

הקשר בין שלושת התכונות הנ"ל נחקר לראשונה על יד וגנר ב־[Weg75]. הוא הראה שכל קומפלקס d־יציג הוא d־מטיט, וכל קומפלקס d־מטיט הוא d־לריי. תכונות d־לריי ו־d־מטיטות משחקות תפקיד מרכזי בבעיות מטיפוס הלי (ראה למשל [Kal84, AK85, KM05]).

בשני הפרקים הראשונים (פרקים 3 ו־4) בתיזה זאת אנו עוסקים בתכונת המטיטות. אנו מפתחים מספר כלים לחסימת המטיטות של קומפלקס X. אחד הכלים הנ"ל הוא המשפט הבא, אשר הוכח על ידי מטושק וטנצר ב־ [MT09] במקרה הפרטי שבו X הוא

המחקר בוצע בהנחייתו של פרופסור רועי משולם בפקולטה למתמטיקה.

חלק מן התוצאות בחיבור זה פורסמו כמאמרים מאת המחבר ושותפיו למחקר בכנסים ובכתבי־עת במהלך תקופת מחקר הדוקטורט של המחבר, אשר גרסאותיהם העדכניות ביותר הינן:

Minki Kim and Alan Lew. Complexes of graphs with bounded independence number. *Israel J. Math.* to appear. arXiv:1912.12605.

Minki Kim and Alan Lew. Complexes of graphs with bounded independence number (extended abstract). *Sém. Lothar. Combin.*, 84B:Art. 39, 12, 2020.

Alan Lew. Collapsibility of simplicial complexes of hypergraphs. *The Electronic Journal of Combinatorics*, 26(4):P4.10, 2019.

Alan Lew. Representability and boxicity of simplicial complexes. *Discrete Comput. Geom.*, 2021.

תודות

אני מודה לפרופסור משולם על הנחייתו ועידודו.

אני מודה לטכניון על התמיכה הכספית הנדיבה בהשתלמותי

מחקרים בקומבינטוריקה טופולוגית

חיבור על מחקר

לשם מילוי חלקי של הדרישות לקבלת התואר
דוקטור לפילוסופיה

אלן לאו

הוגש לסנט הטכניון --- מכון טכנולוגי לישראל
אלול התשפ"א חיפה אוגוסט 2021

מחקרים בקומבינטוריקה טופולוגית

אלן לאו