Spectral gaps of generalized flag complexes

Alan Lew

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Alan Lew

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Abstract

Let X be a simplicial complex with n vertices. The missing faces of X are the sets $\sigma \notin X$ that are minimal with respect to inclusion. Assume that all the missing faces of X are of dimension at most d. Let L_j denote the j-Laplacian acting on real j-cochains of X and let $\mu_j(X)$ denote its minimal eigenvalue.

A classical result of Garland relates the spectral gaps of a complex with the spectral gaps of the links of its faces. Our main result is a global counterpart of Garland's result, connecting between the spectral gaps $\mu_k(X)$ for $k \geq d$ and the spectral gap $\mu_{d-1}(X)$. In particular, we establish the following vanishing result: If $\mu_{d-1}(X) > (1 - \binom{k+1}{d}^{-1})n$, then $\tilde{H}^j(X;\mathbb{R}) = 0$ for all $d-1 \leq j \leq k$. These results extend theorems of Aharoni, Berger and Meshulam for flag complexes (the case $d = 1$).

As an application we prove a fractional extension of a Hall-type theorem of Holmsen, Martínez-Sandoval and Montejano for general position sets in matroids.

We also prove a different lower bound on the spectral gaps $\mu_k(X)$, in terms of the number of vertices n and the minimal degree of a k-dimensional face. This bound follows by an application of Geršgorin's circle theorem to the k-Laplacian.

The last part of the thesis is dedicated to the study of some families of simplicial complexes arising from finite geometries, which have interesting spectral and homological properties.

Abbreviations and Notations

$[n]$	:	the set $\{1, 2, \ldots, n\}$
$\binom{V}{k}$	:	the collection of all subsets of size k of the set V
2^V	:	the collection of all subsets of the set V
$X(k)$	:	the collection of all k-dimensional simplices of the complex X
$X^{(k)}$	:	the k-dimensional skeleton of the complex X
$X[U]$	:	the subcomplex of X induced by U
$\mathrm{st}(X, \sigma)$	:	the star of the simplex σ in the complex X
$\mathrm{lk}(X, \sigma)$	:	the link of the simplex σ in the complex X
$\deg_X(\sigma)$	:	the degree of the simplex σ in the complex X
$\dim(\sigma)$	:	the dimension of the simplex σ
$\dim(X)$	:	the dimension of the complex X
$f_k(X)$	:	the k-th face number of the complex X
Δ_n	:	the complete simplicial complex on vertex set $[n+1]$
$X * Y$	:	the join of the complexes X and Y
$C^k(X)$	:	the space of real valued k-cochains of the complex X
d_k	:	the coboundary operator
$\tilde{H}^k(X; \mathbb{R})$	:	the k-th reduced cohomology group of X with real coefficients
$\tilde{H}_k(X; \mathbb{R})$	:	the k-th reduced homology group of X with real coefficients
$L_k(X)$	:	the k-dimensional Laplacian of X
$\mu_k(X)$	:	the k-th spectral gap of X
$\mathrm{Spec}_k(X)$	:	the spectrum of the k-dimensional Laplacian of X
$\mathrm{conn}_{\mathbb{R}}(X)$	:	the homological connectivity of the complex X over $\mathbb{R}$
$\eta(X)$	:	$\mathrm{conn}_{\mathbb{R}}(X) + 2$
$I(G)$	:	the independence complex of the graph G
$X(G)$	:	the clique complex of the graph G
$\tilde{\gamma}(G)$	:	the total domination number of the graph G
$\tilde{\gamma}(X)$	:	the total domination number of the complex X
$\Gamma(G)$	:	the vector domination number of the graph G
$\Gamma(X)$	:	the vector domination number of the complex X

$\rho(S)$	:	the rank of the set S in a matroid
$\mathrm{cl}(S)$	:	the closure of the set S in a matroid
$\varphi_M(S)$	:	the size of the largest subset of S in general position with respect to the matroid M
$\varphi^*_M(S)$	:	the maximum of $\sum_{v\in S} f(v)$ over all functions f in fractional general position with respect to the matroid M
$\mathbb{F}_q$	:	the finite field with q elements
$PG(n,q)$	:	the projective space of dimension n over $\mathbb{F}_q$
$AG(n,q)$	:	the affine space of dimension n over $\mathbb{F}_q$
$C_p(n,q)$	:	the simplicial complex of caps in $PG(n,q)$
$C_a(n,q)$	:	the simplicial complex of caps in $AG(n,q)$
I	:	the identity matrix
$\mathrm{Tr}(A)$	:	the trace of the matrix A

Chapter 1

Introduction

Let X be a simplicial complex on vertex set V. A simplex $\sigma \subset V$ is called a *missing face* of X if $\sigma \notin X$ but $\tau \in X$ for any $\tau \subsetneq \sigma$. The set of missing faces $\mathcal{M}_X$ of the complex X completely determines X:

$$X = \{ \tau \subset V : \quad \sigma \not\subset \tau \text{ for all } \sigma \in \mathcal{M}_X \} .$$

Let $h(X) = \max \{ \dim(\sigma) : \sigma \in \mathcal{M}_X \}$.

For $k \geq -1$ let $C^k(X)$ be the space of real valued k-cochains of the complex X and let $d_k : C^k(X) \to C^{k+1}(X)$ be the coboundary operator. For $k \geq 0$ the reduced k-dimensional Laplacian of X is defined by

$$L_k(X) = d_{k-1}d_{k-1}^* + d_k^* d_k.$$

L_k is a positive semidefinite operator from $C^k(X)$ to itself. The k-th *spectral gap* of X, denoted by $\mu_k(X)$, is the smallest eigenvalue of L_k.

Let $G = (V, E)$ be a graph on n vertices. Its clique complex (or flag complex) $X(G)$ is the simplicial complex on vertex set V whose simplices are the cliques of G. Note that clique complexes are exactly the complexes with $h(X) = 1$. Indeed, the missing faces of $X(G)$ are the edges of the complement of G. Aharoni, Berger and Meshulam [3] prove the following result:

Theorem 1.1 (Aharoni, Berger, Meshulam [3])**.** *Let $G = (V, E)$ be a graph, where $|V| = n$, and let $X = X(G)$ be its clique complex. Then for $k \geq 1$*

$$k\mu_k(X) \geq (k+1)\mu_{k-1}(X) - n.$$

Our main result is a generalization of Theorem 1.1 to complexes without large missing faces.

Theorem 1.2. *Let X be a simplicial complex with $h(X) = d$, on vertex set V, where*

$|V| = n$. Then for $k \geq d$

$$(k - d + 1)\mu_k(X) \geq (k + 1)\mu_{k-1}(X) - dn.$$

Our proof combines the approach of [3] with additional new ideas. Both results can be thought of as global variants of Garland's method, which in its original form relates the spectral gaps of a complex with the spectral gaps of the links of its faces; See [10, 20]. As a consequence of Theorem 1.2 we obtain

Theorem 1.3. *Let X be a simplicial complex with $h(X) = d$, on vertex set V, where $|V| = n$. If*

$$\mu_{d-1}(X) > \left(1 - \binom{k+1}{d}^{-1}\right) n,$$

then $\tilde{H}^j(X; \mathbb{R}) = 0$ for all $d - 1 \leq j \leq k$.

Remarks. In the case $d = 1$ it is shown in [3] that the condition in Theorem 1.3 is the best possible: Let G be the complete r-partite graph on $n = \ell r$ vertices, with all sides of size ℓ. Then $\mu_0(X(G)) = \frac{r-1}{r} n$, but $\tilde{H}^{r-1}(X(G); \mathbb{R}) \neq 0$.

For $d = 2$ we have found such extremal examples only for a few cases (see Chapter 6 for further discussion of such examples):

1. Let X be the simplicial complex whose vertices V are the points of the affine plane over $\mathbb{F}_3$, and whose missing faces are the lines of the affine plane. Let $k = 2$.
   On the one hand, one can check that $\mu_1(X) = 6 = \left(1 - \binom{k+1}{2}^{-1}\right)|V|$. On the other hand, $\tilde{H}^2(X; \mathbb{R}) = \mathbb{R} \neq 0$ (computer checked).

2. Let X be the simplicial complex whose vertices V are the points of the projective space of dimension 3 over $\mathbb{F}_3$, and whose simplices are the sets of points containing at most two points from each line (so the missing faces are the subsets of size 3 of the lines in the projective space). Let $k = 4$. One can show that $\mu_1(X) = 36 = \left(1 - \binom{k+1}{2}^{-1}\right)|V|$. On the other hand, $\tilde{H}^4(X; \mathbb{R}) \neq 0$ (computer checked).

We next give some applications of Theorem 1.2 to connectivity bounds and Hall type theorems for general simplicial complexes.

Let $\eta(X) = \text{conn}_{\mathbb{R}}(X) + 2$, where

$$\text{conn}_{\mathbb{R}}(X) = \min\{\, i : \tilde{H}^i(X; \mathbb{R}) \neq 0 \,\} - 1$$

is the *homological connectivity* of X over $\mathbb{R}$.

A subset of vertices $S \subset V$ in a graph $G = (V, E)$ is called a *totally dominating set* if for all $v \in V$ there is some $u \in S$ such that $vu \in E$. The *total domination number* of G, denoted by $\tilde{\gamma}(G)$, is the minimal size of a totally dominating set. Let $I(G)$ be the independence complex of the graph, i.e. the simplicial complex whose faces are all the independent sets $\sigma \subset V$. The total domination number gives a lower bound on the connectivity of $I(G)$ (see [18, Theorem 1.2]):

$$\eta(I(G)) \geq \tilde{\gamma}(G)/2. \tag{1.1}$$

(For additional lower bounds on $\eta(I(G))$ in terms of other domination parameters, see e.g. [4, 18]).

The inequality (1.1) had been generalized to general simplicial complexes: Let X be a complex on vertex set V. We say that a subset $S \subset V$ is *totally dominating* if for every $v \in V$ there is some $\sigma \subset S$ such that $\sigma \in X$ but $v\sigma \notin X$. The *total domination number* of X, denoted $\tilde{\gamma}(X)$, is the minimal size of a totally dominating set in X. For a graph G we have $\tilde{\gamma}(G) = \tilde{\gamma}(I(G))$ (the totally dominating sets of $I(G)$ are the same as the totally dominating sets of G). In [2] it is shown that for any simplicial complex X, $\eta(X) \geq \tilde{\gamma}(X)/2$.

Another graphical domination parameter, $\Gamma(G)$, has been introduced in [3] as follows. A *vector representation* of the graph G is an assignment $P : V \to \mathbb{R}^\ell$ such that $P(v) \cdot P(w) \geq 1$ if v and w are adjacent in G, and $P(v) \cdot P(w) \geq 0$ otherwise. A non-negative vector $\alpha \in \mathbb{R}^V$ is called *dominating* for P if $\sum_{v \in V} \alpha(v) P(v) \cdot P(w) \geq 1$ for every $w \in V$. The *value* of P is

$$|P| = \min \left\{ \sum_{v \in V} \alpha(v) \, : \, \alpha \text{ is dominating for } P \right\}.$$

Let $\Gamma(G)$ be the supremum of $|P|$ over all vector representations of G. It is easy to check that $\Gamma(G) \leq \tilde{\gamma}(G)$ (see Proposition 1.5). In [3] the following was proved:

Theorem 1.4 (Aharoni, Berger, Meshulam [3])**.**

$$\eta(I(G)) \geq \Gamma(G).$$

With a view towards generalizing Theorem 1.4 to an arbitrary simplicial complex X, we define a new domination parameter $\Gamma(X)$.

For $k \in \mathbb{N}$ let $\mathcal{M}_X(k)$ be the set of missing faces of X of dimension k. Let $J_X = \{ i \in \mathbb{N} : \mathcal{M}_X(i) \neq \emptyset \}$ be the set of dimensions of simplices in $\mathcal{M}_X$. Define $S(X) = \bigcup_{i \in J_X} \binom{V}{i-1}$.

Let $\sigma \in S(X)$ and fix $\ell = \ell(\sigma) \in \mathbb{N}$. A *vector representation of X with respect to σ* is an assignment $P_\sigma : V \to \mathbb{R}^\ell$ such that the inner product $P_\sigma(v) \cdot P_\sigma(w) \geq 1$ if $vw\sigma \in \mathcal{M}_X(|\sigma| + 1)$, and $P_\sigma(v) \cdot P_\sigma(w) \geq 0$ otherwise. We identify the representation P_σ with the matrix $P_\sigma \in \mathbb{R}^{|V| \times \ell}$ whose rows are the vectors $P_\sigma(v)$, for $v \in V$. We call the collection $P = \{ P_\sigma : \sigma \in S(X) \}$ a *vector representation* of X.

For each $\sigma \in S(X)$, let $\alpha_\sigma \in \mathbb{R}^V$ be a non-negative vector. The set $\{\alpha_\sigma : \sigma \in S(X)\}$

is called *dominating for P* if

$$\sum_{\sigma \in S(X)} \alpha_\sigma P_\sigma P_\sigma^T \geq \mathbf{1}$$

(where $\mathbf{1} \in \mathbb{R}^V$ is the all 1 vector). The *value* of P is

$$|P| = \min \left\{ \sum_{\sigma \in S(X)} \alpha_\sigma \cdot \mathbf{1} : \{\alpha_\sigma\}_{\sigma \in S(X)} \text{ is dominating for } P \right\}.$$

Let $\Gamma(X)$ be the supremum of $|P|$ over all vector representations P of X.

Remarks.

1. If $X = I(G)$ for a graph G, then $\Gamma(X)$ coincides with the parameter $\Gamma(G)$ defined in [3].
2. In the case when all the missing faces are of the same size, we can bound $\Gamma(X)$ by the total domination number $\tilde{\gamma}(X)$:

Proposition 1.5. *Let X be a simplicial complex with all its missing faces of dimension equal to d. Then*

$$\Gamma(X) \leq \binom{\tilde{\gamma}(X)}{d}.$$

Our main application of Theorem 1.2 is the following extension of Theorem 1.4.

Theorem 1.6.

$$\sum_{r \in J_X} r \binom{\eta(X)}{r} \geq \Gamma(X).$$

Let $V_1, \ldots, V_m$ be a partition of the vertex set V. We say that a subset $\sigma \subset V$ is *colorful* if $|\sigma \cap V_i| = 1$ for all $i \in [m]$. Theorem 1.6 gives rise to the following Hall-type condition for the existence of colorful simplices:

Theorem 1.7. *If for every $\emptyset \neq I \subset [m]$*

$$\Gamma(X[\cup_{i \in I} V_i]) > \sum_{r \in J_{X[\cup_{i \in I} V_i]}} r \binom{|I| - 1}{r},$$

then X has a colorful simplex.

Next we show an application of Theorem 1.7. Let M be a matroid on vertex set V with rank function ρ. Assume $\rho(V) = d + 1$. We identify M with the simplicial complex of its independent sets. For $S \subset V$, define its *closure* by $\text{cl}(S) = \{ v \in V : \rho(S) = \rho(S \cup \{v\}) \}$. A subset $F \subset V$ is a *flat* of M if $F = \text{cl}(F)$, i.e. $\rho(F \cup \{v\}) > \rho(F)$ for all $v \notin F$.

We say that a subset $S \subset V$ is in *general position* with respect to M if for any $1 \leq k \leq d$ every flat of M of rank k contains at most k points of S. This is equivalent to requiring that any $S' \subset S$ with $|S'| \leq d + 1$ is an independent set in M.

For $S \subset V$ denote by $\varphi_M(S)$ the maximal size of a subset of S in general position.

Let $V_1, \ldots, V_m$ be a partition of V. The following Hall-type theorem is proved in [13].

Theorem 1.8 (Holmsen, Martínez–Sandoval, Montejano [13])**.** *If for every $\emptyset \neq I \subset [m]$*

$$\varphi_M(\cup_{i\in I} V_i) > \begin{cases} |I|-1 & \text{if } |I| \leq d+1, \\ d\binom{2|I|-2}{d} & \text{if } |I| \geq d+2, \end{cases}$$

then V has a colorful subset in general position.

Let $S \subset V$. A weight function $f : S \to \mathbb{R}_{\geq 0}$ is in *fractional general position* with respect to M if for any $1 \leq k \leq d$ and for any flat F of M of rank k and $\sigma \subset F \cap S$ of size $k-1$,

$$\sum_{\substack{v\in S,\\ \mathrm{cl}(v\sigma)=F}} f(v) \leq d.$$

Denote by $\varphi_M^*(S)$ the maximum of $\sum_{v\in S} f(v)$ over all functions $f : S \to \mathbb{R}_{\geq 0}$ in fractional general position. We will show that

Lemma 1.9. *$\varphi_M^*(S) \geq \varphi_M(S)$.*

Here we prove the following:

Theorem 1.10. *If for every $\emptyset \neq I \subset [m]$*

$$\varphi_M^*(\cup_{i\in I} V_i) > d\sum_{r=1}^{d} r\binom{|I|-1}{r},$$

then V has a colorful subset in general position.

In particular, we obtain a strengthening of Theorem 1.8:

Theorem 1.11. *If for every $\emptyset \neq I \subset [m]$*

$$\varphi_M(\cup_{i\in I} V_i) > \begin{cases} |I|-1 & \text{if } |I| \leq d+1, \\ d\sum_{r=1}^{d} r\binom{|I|-1}{r} & \text{if } |I| \geq d+2, \end{cases}$$

then V has a colorful subset in general position.

A known result about complexes without large missing faces is the following (see e.g. [1, Prop. 5.4]):

Proposition 1.12. *Let X be a simplicial complex on n vertices with $h(X) = d$. Then $\tilde{H}^k(X;\mathbb{R}) = 0$ for all $k > \frac{d}{d+1}n - 1$.*

This will follow as a consequence of the following lower bound on the spectral gaps:

Theorem 1.13. *Let X be a simplicial complex with $h(X) = d$ on vertex set V, where $|V| = n$. Let $k \geq 0$ and let $\delta_k(X)$ denote the minimal degree of a simplex in $X(k)$. Then*

$$\mu_k(X) \geq (d+1)(\delta_k(X) + k + 1) - dn.$$

The thesis is organized as follows: Chapter 2 contains the background material needed in the following chapters. This includes an introduction to simplicial cohomology and to the Laplacian operators on simplicial complexes, and also material concerning matrices (especially eigenvalues of symmetric matrices) and the theory of matroids.

In Chapter 3 we prove our main results about the spectral gaps of the Laplacian operators. First we introduce some notation and results on complexes without large missing faces, and then, in Section 3.2, we prove our main result Theorem 1.2, and its corollary Theorem 1.3.

In Chapter 4 we show some applications of Theorem 1.2. Section 4.1 deals with the vector domination parameter $\Gamma(X)$ of the complex X. In it we prove Proposition 1.5, Theorem 1.6 and Theorem 1.7. In Section 4.2 we apply the results of the previous section in order to prove Theorems 1.10 and 1.11, that provide sufficient conditions for the existence of colorful sets in general position in a matroid.

In Chapter 5 we prove Theorem 1.13. As a consequence we obtain a new proof of Proposition 1.12. We also present examples showing that the inequalities in Theorem 1.13 are tight. In the case of $d = 1$ we characterize all such extremal cases.

In Chapter 6 we introduce some families of simplicial complexes arising from different finite geometries. Some of these complexes are extremal examples for Theorems 1.2 and 1.3. This chapter is mostly "experimental", meaning it includes many computer calculations, some conjectures based on them, and partial results supporting the conjectures.

Chapter 7 contains concluding remarks, which include the main conjectures and open questions arising from our work.

Chapter 2

Background

2.1 Matrix eigenvalues

In this section we state some known results about eigenvalues of matrices that we will use later. We refer the reader to [14] for the proofs and additional results.

Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be its eigenvalues. We will need the following characterization of the eigenvalues of A:

Theorem 2.1 (Courant-Fischer)**.** *For* $1 \leq k \leq n$

$$\lambda_k = \min_{U} \max_{0\neq x\in U} \frac{\langle Ax, x\rangle}{\langle x, x\rangle},$$

where the minimum is taken over all subspaces $U \subset \mathbb{R}^n$ *of dimension* k*.*

Denote by $\lambda_{\min}(A)$ the minimal eigenvalue of A and by $\lambda_{\max}(A)$ its maximal eigenvalue. By Theorem 2.1 (applied to the minimal and maximal eigenvalues λ_1 and λ_n) we obtain

$$\lambda_{\max}(A) = \max_{0\neq x\in\mathbb{R}^n} \frac{\langle Ax, x\rangle}{\langle x, x\rangle} \tag{2.1}$$

and

$$\lambda_{\min}(A) = \min_{0\neq x\in\mathbb{R}^n} \frac{\langle Ax, x\rangle}{\langle x, x\rangle}. \tag{2.2}$$

In the following chapters we will implicitly apply the next results:

Lemma 2.2. *Let* $A, B \in \mathbb{R}^{n\times n}$ *be symmetric matrices, then*

$$\lambda_{\max}(A + B) \leq \lambda_{\max}(A) + \lambda_{\max}(B)$$

and

$$\lambda_{\min}(A + B) \geq \lambda_{\min}(A) + \lambda_{\min}(B).$$

Lemma 2.3. *Let* $A \in \mathbb{R}^{n\times n}$ *be a symmetric matrix, and let* $B \in \mathbb{R}^{k\times k}$ *be a principal submatrix of* A *(i.e. a matrix obtained by removing some of the rows of* A *and their corresponding columns). Then* $\lambda_{\max}(B) \leq \lambda_{\max}(A)$ *and* $\lambda_{\min}(B) \geq \lambda_{\min}(A)$*.*

In particular, if A is positive semidefinite and B is a principal submatrix of A, we obtain by Lemma 2.3 that B is also positive semidefinite.

Another result about eigenvalues of matrices that we will need is the following theorem:

Theorem 2.4 (Geršgorin circle theorem)**.** *Let $A \in \mathbb{C}^{n\times n}$ and $\lambda \in \mathbb{C}$ be an eigenvalue of A. Then there is some $i \in [n]$ such that*

$$|\lambda - a_{ii}| \leq \sum_{j\neq i} |a_{ij}| .$$

Proof. Let $v = (x_1, \ldots, x_n) \in \mathbb{C}^n$ be an eigenvector of A with eigenvalue λ. Choose $k \in [n]$ such that $|x_k| = \max \{ |x_i| : i \in [n] \}$. We have $Av = \lambda v$, therefore for any $i \in [n]$, $\lambda x_i = \sum_{j=1}^{n} a_{ij}x_j$. In particular for $i = k$ we obtain

$$\lambda x_k = \sum_{j=1}^{n} a_{kj}x_j = a_{kk}x_k + \sum_{j\neq k} a_{kj}x_j.$$

Therefore

$$(\lambda - a_{kk})x_k = \sum_{j\neq k} a_{kj}x_j.$$

Since v is an eigenvector of A, we have $v \neq 0$, thus $|x_k| > 0$. By taking absolute value and dividing by $|x_k|$ we get

$$|\lambda - a_{kk}| = \frac{\left|\sum_{j\neq k} a_{kj}x_j\right|}{|x_k|} \leq \frac{\sum_{j\neq k} |a_{kj}||x_j|}{|x_k|} \leq \sum_{j\neq k} |a_{kj}|,$$

where the last inequality follows since $|x_j| \leq |x_k|$ for all $j \in [n]$, by the choice of k. □

2.2 Simplicial cohomology

Next we recall some definitions and basic facts about simplicial complexes and cohomology. For a more complete introduction see e.g. [11].

A *simplicial complex* is a family X of finite subsets of some set S, such that if $\sigma \in X$ and $\tau \subset \sigma$, then also $\tau \in X$. We call the sets $\sigma \in X$ the *simplices* of X, or the *faces* of X. For each simplex $\sigma \in X$ we define the *dimension* of σ to be $\dim(\sigma) = |\sigma| - 1$. The *dimension* of the complex X, denoted by $\dim(X)$, is the maximal dimension of a simplex in X. The *vertex set* of X is $V(X) = \bigcup_{\sigma\in X} \sigma$. We will always assume that X is finite, i.e. $V(X)$ is finite.

We denote the set of k-dimensional simplices in X by $X(k)$, and we define the *face numbers* $f_k(X) = |X(k)|$.

Y is a *subcomplex* of X if it is a simplicial complex, and each simplex of Y is also a simplex of X.

The *k-dimensional skeleton* of X is the subcomplex of X consisting of all the faces of X of dimension k or less. It is denoted by $X^{(k)}$.

For $U \subset V$, let $X[U] = \{ \sigma \in X : \sigma \subset U \}$ be the subcomplex of X *induced* by U.

For $\sigma \in X(k)$, let the *star* of σ in X be the subcomplex

$$\mathrm{st}(X, \sigma) = \{ \tau \in X : \tau \cup \sigma \in X \},$$

and the *link* of σ in X be the subcomplex

$$\mathrm{lk}(X, \sigma) = \{ \tau \in X : \tau \cup \sigma \in X, \tau \cap \sigma = \emptyset \}.$$

Let

$$\deg_X(\sigma) = |\{ \eta \in X(k+1) : \sigma \subset \eta \}| = |\{ v \in V \setminus \sigma : \sigma \cup \{v\} \in X \}|$$

be the *degree* of σ in X.

An *ordered simplex* is a simplex with a linear ordering of its vertices. For two ordered simplices $\sigma \in X$ and $\tau \in \mathrm{lk}(X, \sigma)$ denote by $[\sigma, \tau]$, or simply by $\sigma\tau$, their ordered union. Similarly, for $v \in V$ denote by $v\sigma$ the ordered union of $\{v\}$ and σ.

For $\tau \subset \sigma$, both given an order on their vertices, we define $(\sigma : \tau)$ to be the sign of the permutation on the vertices of σ that maps the ordered simplex σ to the ordered simplex $[\sigma \setminus \tau, \tau]$ (where the order on the vertices of $\sigma \setminus \tau$ is the one induced by the order on σ).

A simplicial k-cochain is a real valued skew-symmetric function on all ordered k-simplices. That is, ϕ is a k-cochain if for any two k-simplices $\sigma, \tilde{\sigma}$ in X that are equal as sets, it satisfies $\phi(\tilde{\sigma}) = (\tilde{\sigma} : \sigma)\phi(\sigma)$.

For $k \geq 0$ let $C^k(X)$ denote the space of k-cochains on X. For $k = -1$ we define $C^{-1}(X) = \mathbb{R}$.

We will use the following lemma implicitly in future calculations.

Lemma 2.5. *Let $\tau, \eta \in X(k)$ and $\phi \in C^k(X)$. Let $\sigma, \theta \in X$ be ordered simplices such that $\tau, \eta \subset \sigma$ and $\theta \subset \tau \cap \eta$, and let $\tilde{\sigma}$, $\tilde{\tau}$, $\tilde{\eta}$, $\tilde{\theta}$ be equal as sets to σ, τ, η and θ respectively. Then*

1. $(\sigma : \tau) = (\sigma : \tilde{\tau}) \cdot (\tilde{\tau} : \tau)$, *and if* $|\sigma \setminus \tau| = 1$ *then* $(\sigma : \tau) = (\sigma : \tilde{\sigma}) \cdot (\tilde{\sigma} : \tau)$.

2. $\phi(\tau)^2 = \phi(\tilde{\tau})^2$.

3. $(\sigma : \tau)\phi(\tau) = (\sigma : \tilde{\tau})\phi(\tilde{\tau})$, *and if* $|\tau \setminus \theta| = 1$ *then*

$$(\tau : \theta)\phi(\tau) = (\tilde{\tau} : \theta)\phi(\tilde{\tau}).$$

4. *If* $|\sigma \setminus \tau| = 1$ *and* $|\sigma \setminus \eta| = 1$ *then*

$$(\sigma : \tau)(\sigma : \eta)\phi(\tau)\phi(\eta) = (\tilde{\sigma} : \tilde{\tau})(\tilde{\sigma} : \tilde{\eta})\phi(\tilde{\tau})\phi(\tilde{\eta}).$$

5. *If* $|\tau \setminus \theta| = 1$ *and* $|\eta \setminus \theta| = 1$ *then*

$$(\tau : \theta)(\eta : \theta)\phi(\tau)\phi(\eta) = (\tilde{\tau} : \tilde{\theta})(\tilde{\eta} : \tilde{\theta})\phi(\tilde{\tau})\phi(\tilde{\eta}).$$

Proof.

1. Let π_1 be the permutation on the vertices of σ that maps σ to $[\sigma \setminus \tilde{\tau}, \tilde{\tau}] = [\sigma \setminus \tau, \tilde{\tau}]$, and let π_2 be the permutation on the vertices of τ that maps $\tilde{\tau}$ to τ. Extend π_2 to a permutation $\tilde{\pi}_2$ on the vertices of σ, that maps $[\sigma \setminus \tau, \tilde{\tau}]$ to $[\sigma \setminus \tau, \tau]$. It satisfies $\text{sign}(\pi_2) = \text{sign}(\tilde{\pi}_2)$. Define $\pi = \tilde{\pi}_2 \circ \pi_1$. π maps σ to $[\sigma \setminus \tau, \tau]$, therefore

$$(\sigma : \tau) = \text{sign}(\pi) = \text{sign}(\tilde{\pi}_2) \cdot \text{sign}(\pi_1) = \text{sign}(\pi_2) \cdot \text{sign}(\pi_1) = (\tilde{\tau} : \tau) \cdot (\sigma : \tilde{\tau}).$$

   Assume now that $|\sigma \setminus \tau| = 1$ and let $\{v\} = \sigma \setminus \tau$. Let π_3 be the permutation on the vertices of σ that maps σ to $\tilde{\sigma}$, and π_4 be the permutation that maps $\tilde{\sigma}$ to $[\tilde{\sigma} \setminus \tau, \tau] = v\tau = [\sigma \setminus \tau, \tau]$. Then the permutation $\pi' = \pi_4 \circ \pi_3$ maps σ to $[\sigma \setminus \tau, \tau]$, therefore

$$(\sigma : \tau) = \text{sign}(\pi') = \text{sign}(\pi_4) \cdot \text{sign}(\pi_3) = (\sigma : \tilde{\sigma}) \cdot (\tilde{\sigma} : \tau).$$

2. Since ϕ is a cochain, we have $\phi(\tau)^2 = (\tau : \tilde{\tau})^2 \phi(\tilde{\tau})^2 = \phi(\tilde{\tau})^2$.

3. By the first part of this lemma

$$(\sigma : \tau)\phi(\tau) = (\sigma : \tilde{\tau})(\tilde{\tau} : \tau)\phi(\tau),$$

   and since ϕ is a cochain

$$(\sigma : \tilde{\tau})(\tilde{\tau} : \tau)\phi(\tau) = (\sigma : \tilde{\tau})\phi(\tilde{\tau}).$$

   The second equality is similar: By the first part of the lemma

$$(\tau : \theta)\phi(\tau) = (\tau : \tilde{\tau})(\tilde{\tau} : \theta)\phi(\tau),$$

   and since ϕ is a cochain

$$(\tau : \tilde{\tau})(\tilde{\tau} : \theta)\phi(\tau) = (\tau : \tilde{\tau})(\tilde{\tau} : \theta)(\tau : \tilde{\tau})\phi(\tilde{\tau}) = (\tilde{\tau} : \theta)\phi(\tilde{\tau}).$$

4. By part 3 of this lemma we have

$$(\sigma : \tau)(\sigma : \eta)\phi(\tau)\phi(\eta) = (\sigma : \tilde{\tau})(\sigma : \tilde{\eta})\phi(\tilde{\tau})\phi(\tilde{\eta}).$$

Then by part 1

$$(\sigma:\tilde{\tau})(\sigma:\tilde{\eta})\phi(\tilde{\tau})\phi(\tilde{\eta}) = (\sigma:\tilde{\sigma})(\tilde{\sigma}:\tilde{\tau})(\sigma:\tilde{\sigma})(\tilde{\sigma}:\tilde{\eta})\phi(\tilde{\tau})\phi(\tilde{\eta})$$
$$= (\tilde{\sigma}:\tilde{\tau})(\tilde{\sigma}:\tilde{\eta})\phi(\tilde{\tau})\phi(\tilde{\eta}).$$

5. The proof is similar to the proof of part 4. □

For $k \geq 0$ let the coboundary operator $d_k : C^k(X) \to C^{k+1}(X)$ be the linear operator defined by

$$d_k\phi(\sigma) = \sum_{i=0}^{k+1}(-1)^i\phi(\sigma_i),$$

where for an ordered $(k+1)$-simplex $\sigma = [v_0, \dots, v_{k+1}]$, σ_i is the ordered simplex obtained by removing the vertex v_i, that is $\sigma_i = [v_0, \dots, \hat{v}_i, \dots, v_{k+1}]$. Equivalently, we can write

$$d_k\phi(\sigma) = \sum_{\tau\in\sigma(k)}(\sigma:\tau)\phi(\tau),$$

where $\sigma(k) \subset X(k)$ is the set of all k-dimensional faces of σ, each given some fixed order on its vertices.

For $k = -1$ we define $d_{-1} : C^{-1}(X) = \mathbb{R} \to C^0(X)$ by $d_{-1}a(v) = a$, for every $a \in \mathbb{R}$, $v \in V$.

For each $k \geq -1$ we define an inner product on $C^k(X)$ by

$$\langle\phi, \psi\rangle = \sum_{\sigma\in X(k)}\phi(\sigma)\psi(\sigma).$$

This induces a norm on $C^k(X)$:

$$\|\phi\| = \left(\sum_{\sigma\in X(k)}\phi(\sigma)^2\right)^{1/2}.$$

Let $d_k^* : C^{k+1}(X) \to C^k(X)$ be the adjoint of d_k with respect to this inner product. We can write $d_k^*\phi$ explicitly: For $\phi \in C^k(X)$ and $\psi \in C^{k+1}(X)$,

$$\begin{aligned}
\langle d_k\phi, \psi\rangle &= \sum_{\sigma\in X(k+1)} d_k\phi(\sigma)\psi(\sigma) = \sum_{\sigma\in X(k+1)}\sum_{\tau\in\sigma(k)}(\sigma:\tau)\phi(\tau)\psi(\sigma) \\
&= \sum_{\tau\in X(k)}\sum_{\sigma\in X(k+1),\tau\subset\sigma}(\sigma:\tau)\phi(\tau)\psi(\sigma) = \sum_{\tau\in X(k)}\sum_{v\in \mathrm{lk}(X,\tau)}(v\tau:\tau)\phi(\tau)\psi(v\tau) \\
&= \sum_{\tau\in X(k)}\left(\phi(\tau)\sum_{v\in\mathrm{lk}(X,\tau)}\psi(v\tau)\right) = \left\langle\phi, \sum_{v\in\mathrm{lk}(X,\tau)}\psi(v\tau)\right\rangle.
\end{aligned}$$

Thus we obtain

$$d_k^*\psi(\tau) = \sum_{v\in \mathrm{lk}(X,\tau)} \psi(v\tau).$$

Let $k \geq 0$. Let $\tilde{H}^k(X;\mathbb{R}) = \mathrm{Ker}(d_k)/\mathrm{Im}(d_{k-1})$ be the k-th reduced *cohomology group* of X with real coefficients.

Let $\partial_k = d_{k-1}^*$, and let $\tilde{H}_k(X;\mathbb{R}) = \mathrm{Ker}(\partial_k)/\mathrm{Im}(\partial_{k+1})$ be the k-th reduced *homology group* of X with real coefficients.

For $k = -1$ we define $\tilde{H}^{-1}(X;\mathbb{R}) = \mathrm{Ker}(d_{-1})$ and $\tilde{H}_{-1}(X;\mathbb{R}) = C^{-1}(X)/\mathrm{Im}(\partial_0)$. We have $\tilde{H}^{-1}(X;\mathbb{R}) = \tilde{H}_{-1}(X;\mathbb{R}) = 0$ if $X \neq \{\emptyset\}$, and $\tilde{H}^{-1}(X;\mathbb{R}) = \tilde{H}_{-1}(X;\mathbb{R}) = \mathbb{R}$ if $X = \{\emptyset\}$.

For all (finite) simplicial complexes X we have $\tilde{H}_k(X;\mathbb{R}) \cong \tilde{H}^k(X;\mathbb{R})$ for all $k \geq -1$.

Also, if X is homotopy equivalent to another simplicial complex Y, then $\tilde{H}^k(X;\mathbb{R}) \cong \tilde{H}^k(Y;\mathbb{R})$ for all $k \geq -1$.

Let Δ_{n-1} be the complete simplicial complex on vertex set $[n]$, i.e. the complex whose simplices are all the subsets $\sigma \subset [n]$. This is an $(n-1)$-dimensional complex. Then its k-skeleton $\Delta_{n-1}^{(k)}$ is the simplicial complex on vertex set $[n]$ whose simplices are all the subsets of $[n]$ of size at most $k+1$. The following result is well known:

Claim 2.6.

$$\tilde{H}^i\left(\Delta_{n-1}^{(k)};\mathbb{R}\right) = \begin{cases} 0 & \text{if } -1 \leq i \leq k-1, \\ \mathbb{R}^{\binom{n-1}{k+1}} & \text{if } i = k. \end{cases}$$

Let X and Y be simplicial complexes on disjoint vertex sets. We define a new simplicial complex:

$$X * Y = \{\, \sigma \uplus \tau : \sigma \in X,\ \tau \in Y \,\}.$$

$X * Y$ is called the *join* of X and Y.

If $X = \{v\} * X'$ (where we view $\{v\}$ as the simplicial complex with only one vertex v), then we say that X is a *cone over* v. In this case we have, since X is homotopy equivalent to a point, $\tilde{H}^k(X;\mathbb{R}) = 0$ for all $k \geq -1$.

We will denote by $X * X$ the join of X with a disjoint copy of itself. Also, we will denote the complex $X * X * \cdots * X$ (k times) by X^{*k}.

We will need the following theorem:

Theorem 2.7 (Mayer-Vietoris exact sequence)**.** *Let A, B, X be simplicial complexes such that $X = A \cup B$. Then the following sequence is exact:*

$$\cdots \to \tilde{H}^{k-1}(A\cap B;\mathbb{R}) \to \tilde{H}^k(X;\mathbb{R}) \to \tilde{H}^k(A;\mathbb{R}) \oplus \tilde{H}^k(B;\mathbb{R}) \to \cdots$$

One special case of the Mayer-Vietoris sequence is the following. Let V be the vertex set of X, and let $v \in V$. Define $A = X[V \setminus \{v\}]$ and $B = \mathrm{st}(X,v)$. We have $X = A \cup B$ and $A \cap B = \mathrm{lk}(X,v)$. Also, since $B = \{v\} * \mathrm{lk}(X,v)$ (i.e. B is a cone) , we have $\tilde{H}^k(B;\mathbb{R}) = 0$ for all $k \geq -1$. Therefore by Theorem 2.7 we obtain

Proposition 2.8. *The following sequence is exact:*

$$\cdots \to \tilde{H}^{k-1}\left(\mathrm{lk}(X,v);\mathbb{R}\right) \to \tilde{H}^{k}\left(X;\mathbb{R}\right) \to \tilde{H}^{k}\left(X[V\setminus\{v\}];\mathbb{R}\right) \to \cdots$$

Another consequence of Theorem 2.7 is the following result of Meshulam (see [18, Theorem 1.1]).

Theorem 2.9. *Let X be a clique complex that satisfies $\tilde{H}_k\left(X;\mathbb{R}\right) \neq 0$. Then for all $j \geq 0$*

$$f_j(X) \geq 2^{j+1}\binom{k+1}{j+1}.$$

2.3 Higher Laplacians

For $k \geq 0$ define the lower k-Laplacian of X by $L_k^-\left(X\right) = d_{k-1}d_{k-1}^*$ and the upper k-Laplacian of X by $L_k^+\left(X\right) = d_k^* d_k$. The reduced k-Laplacian of X is the positive semidefinite operator on $C^k(X)$ given by

$$L_k\left(X\right) = L_k^-\left(X\right) + L_k^+\left(X\right).$$

For $k=-1$ we define $L_{-1}\left(X\right)(a) = d_{-1}^* d_{-1}(a) = n \cdot a$, where $n = |V(X)|$.

Let $k \geq 0$ and $\sigma \in X(k)$. We define the k-cochain 1_σ by

$$1_\sigma(\tau) = \begin{cases} (\sigma:\tau) & \text{if } \sigma=\tau \text{ (as sets)},\\ 0 & \text{otherwise.}\end{cases}$$

The set $\{1_\sigma\}_{\sigma\in X(k)}$ forms a basis of the space $C^k(X)$, that we will call the standard basis.

For a linear operator $T: C^k(X) \to C^k(X)$, let $[T]$ be the matrix representation of T with respect to the standard basis. We denote by $[T]_{\sigma,\tau}$ the matrix element of $[T]$ at index $(1_\sigma, 1_\tau)$.

One can write explicitly the matrix representation of the Laplacian operators in the standard basis:

Claim 2.10. *For $k \geq 0$*

$$\left[L_k^-\right]_{\sigma,\tau} = \begin{cases} k+1 & \text{if } \sigma=\tau,\\ (\sigma:\sigma\cap\tau)\cdot(\tau:\sigma\cap\tau) & \text{if } |\sigma\cap\tau|=k,\\ 0 & \text{otherwise,}\end{cases}$$

$$\left[L_k^+\right]_{\sigma,\tau} = \begin{cases} \deg_X(\sigma) & \text{if } \sigma=\tau,\\ -(\sigma:\sigma\cap\tau)\cdot(\tau:\sigma\cap\tau) & \text{if } |\sigma\cap\tau|=k, \sigma\cup\tau\in X(k+1),\\ 0 & \text{otherwise,}\end{cases}$$

and

$$[L_k]_{\sigma,\tau} = \begin{cases} k+1+\deg_X(\sigma) & \text{if } \sigma=\tau, \\ (\sigma:\sigma\cap\tau)\cdot(\tau:\sigma\cap\tau) & \text{if } |\sigma\cap\tau|=k, \sigma\cup\tau\notin X(k+1), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. First we write out L_k^+ and L_k^- explicitly. For $\phi \in C^k(X)$ and $\tau \in X(k)$ we have

$$L_k^+\phi(\tau) = d_k^* d_k \phi(\tau) = \sum_{v\in \mathrm{lk}(X,\tau)} d_k\phi(v\tau) = \sum_{v\in \mathrm{lk}(X,\tau)} \sum_{\substack{\theta\subset v\tau,\\ \theta\in X(k)}} (v\tau:\theta)\phi(\theta)$$
$$= \sum_{v\in \mathrm{lk}(X,\tau)} \left((v\tau:\tau)\phi(\tau) + \sum_{\eta\in\tau(k-1)} (v\tau:v\eta)\phi(v\eta) \right).$$

Let $v \in \mathrm{lk}(X,\tau)$ and $\eta \in \tau(k-1)$. Let $\{u\} = \tau\setminus\eta$, and let π be the permutation on the vertices of τ that maps τ to $u\eta$. We can extend π to a permutation $\tilde{\pi}$ on the vertices of $v\tau$ by mapping v to itself (and we have $\operatorname{sign}\pi = \operatorname{sign}\tilde{\pi}$). $\tilde{\pi}$ maps $v\tau$ to $vu\eta$. By composing $\tilde{\pi}$ with a single transposition, we get a permutation that maps $v\tau$ to $uv\eta$, therefore

$$(v\tau : v\eta) = -\operatorname{sign}(\pi) = -(\tau:\eta).$$

So we have

$$L_k^+\phi(\tau) = \sum_{v\in \mathrm{lk}(X,\tau)} \phi(\tau) - \sum_{v\in \mathrm{lk}(X,\tau)} \sum_{\eta\in\tau(k-1)} (\tau:\eta)\phi(v\eta).$$

We have a one-to-one correspondence

$$\left\{(v,\eta) : {\scriptstyle v\in \mathrm{lk}(X,\tau), \atop \eta\in\tau(k-1)}\right\} \leftrightarrow \left\{\theta : {\scriptstyle \theta\in X(k),\, |\theta\cap\tau|=k, \atop \theta\cup\tau\in X(k+1)}\right\}$$

defined by $(v,\eta)\mapsto \{v\}\cup\eta$ (its inverse being: $\theta \mapsto (\theta\setminus\tau, \theta\cap\tau)$).

Let $\theta \in X(k)$ such that $|\theta\cap\tau| = k$ and $\theta\cup\tau \in X(k+1)$. Let $\{v\} = \theta\setminus\tau$ and $\eta = \theta\cap\tau$. We have $(v\eta:\theta) = (\theta:\eta)$ (since they are, correspondingly, the sign of the permutation mapping $v\eta$ to θ, and the sign of its inverse). Thus we have

$$(\tau:\eta)\phi(v\eta) = (\tau:\eta)(v\eta:\theta)\phi(\theta) = (\tau:\eta)(\theta:\eta)\phi(\theta)$$
$$= (\tau:\tau\cap\theta)(\tau\cap\theta:\eta)(\theta:\tau\cap\theta)(\tau\cap\theta:\eta)\phi(\theta) = (\tau:\tau\cap\theta)(\theta:\tau\cap\theta)\phi(\theta).$$

Therefore we obtain

$$L_k^+\phi(\tau) = \deg_X(\tau)\phi(\tau) - \sum_{\substack{\theta\in X(k),\\ |\tau\cap\theta|=k,\\ \tau\cup\theta\in X(k+1)}} (\tau:\tau\cap\theta)(\theta:\tau\cap\theta)\phi(\theta).$$

Similarly,

$$\begin{aligned} L_k^- \phi(\tau) = d_{k-1} d_{k-1}^* \phi(\tau) &= \sum_{\eta \in \tau(k-1)} (\tau : \eta) d_{k-1}^* \phi(\eta) = \sum_{\eta \in \tau(k-1)} (\tau : \eta) \sum_{v \in \mathrm{lk}(X,\eta)} \phi(v\eta) \\ &= \sum_{\eta \in \tau(k-1)} (\tau : \eta)(\tau : \eta)\phi(\tau) + \sum_{\eta \in \tau(k-1)} \sum_{\substack{v \in \mathrm{lk}(X,\eta), \\ v \notin \tau}} (\tau : \eta)\phi(v\eta) \\ &= (k+1)\phi(\tau) + \sum_{\substack{\theta \in X(k), \\ |\tau \cap \theta| = k}} (\tau : \tau \cap \theta)(\theta : \tau \cap \theta)\phi(\theta). \end{aligned}$$

So

$$L_k \phi(\tau) = (k+1+\deg_X(\tau))\phi(\tau) + \sum_{\substack{\theta \in X(k), \\ |\tau \cap \theta| = k, \\ \tau \cup \theta \notin X(k+1)}} (\tau : \tau \cap \theta)(\theta : \tau \cap \theta)\phi(\theta).$$

For any operator T on $C^k(X)$, $[T]_{\tau,\sigma} = T(1_\sigma)(\tau)$. Therefore plugging in $\phi = 1_\sigma$ we obtain

$$\begin{aligned} \left[L_k^+\right]_{\tau,\sigma} = L_k^+ 1_\sigma(\tau) &= \deg_X(\tau) 1_\sigma(\tau) - \sum_{\substack{\theta \in X(k), \\ |\tau \cap \theta| = k, \\ \tau \cup \theta \in X(k+1)}} (\tau : \tau \cap \theta)(\theta : \tau \cap \theta) 1_\sigma(\theta) \\ &= \begin{cases} \deg_X(\sigma) & \text{if } \sigma = \tau, \\ -(\sigma : \sigma \cap \tau)(\tau : \sigma \cap \tau) & \text{if } |\sigma \cap \tau| = k,\ \sigma \cup \tau \in X(k+1), \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

$$\begin{aligned} \left[L_k^-\right]_{\tau,\sigma} = L_k^- 1_\sigma(\tau) &= (k+1) 1_\sigma(\tau) + \sum_{\substack{\theta \in X(k), \\ |\tau \cap \theta| = k}} (\tau : \tau \cap \theta)(\theta : \tau \cap \theta) 1_\sigma(\theta) \\ &= \begin{cases} k+1 & \text{if } \sigma = \tau, \\ (\sigma : \sigma \cap \tau)(\tau : \sigma \cap \tau) & \text{if } |\sigma \cap \tau| = k, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

and therefore

$$[L_k]_{\tau,\sigma} = \begin{cases} k+1+\deg_X \sigma & \text{if } \sigma = \tau, \\ (\sigma : \sigma \cap \tau)(\tau : \sigma \cap \tau) & \text{if } |\sigma \cap \tau| = k,\ \sigma \cup \tau \notin X(k+1), \\ 0 & \text{otherwise.} \end{cases}$$

□

Let X be a simplicial complex on vertex set V, with $|V| = n$. The following upper bound on the eigenvalues of the Laplacian is implicit in [8]:

Lemma 2.11. *Let $k \geq 0$ and let λ be an eigenvalue of $L_k(X)$. Then*

$$\lambda \leq n.$$

Here we give another proof. We will use the following lemma, which will be also needed later, for the proof of Claim 4.2:

Let $k \geq 0$ and let $\bar{X}$ be the $(k+1)$-dimensional simplicial complex on vertex set V with full k-dimensional skeleton, whose $(k+1)$-dimensional faces are the sets $\sigma \in \binom{V}{k+2}$ such that $\sigma \notin X$. Denote by L the matrix obtained from $L_k^+(\bar{X})$ by keeping only the rows and columns corresponding to faces of X.

Lemma 2.12. $L = nI - [L_k(X)]$.

Proof. By Claim 2.10 we have for $\sigma, \tau \in X(k)$

$$\left[L_k^+(\bar{X})\right]_{\sigma,\tau} = \begin{cases} \deg_{\bar{X}}(\sigma) & \text{if } \sigma = \tau, \\ -(\sigma : \sigma \cap \tau) \cdot (\tau : \sigma \cap \tau) & \text{if } {}^{|\sigma \cap \tau| = k,}_{\sigma \cup \tau \in \bar{X}(k+1),} \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} n - k - 1 - \deg_X(\sigma) & \text{if } \sigma = \tau, \\ -(\sigma : \sigma \cap \tau) \cdot (\tau : \sigma \cap \tau) & \text{if } {}^{|\sigma \cap \tau| = k,}_{\sigma \cup \tau \notin X(k+1),} \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} n - [L_k(X)]_{\sigma,\tau} & \text{if } \sigma = \tau, \\ -[L_k(X)]_{\sigma,\tau} & \text{otherwise.} \end{cases}$$

Therefore $L = nI - [L_k(X)]$. □

Proof of Lemma 2.11. Let $\phi \in C^k(X)$ be an eigenvector of $L_k(X)$ with eigenvalue λ. Then, by Lemma 2.12, its coordinate vector $[\phi]$ is also an eigenvector of L with eigenvalue $n - \lambda$. But $\left[L_k^+(\bar{X})\right]$ is a positive semidefinite matrix, hence L is positive semidefinite too (as a principal submatrix of a positive semidefinite matrix), therefore $n - \lambda \geq 0$, so $\lambda \leq n$. □

Let $\text{Spec}_k(X)$ be the spectrum of $L_k(X)$, i.e. a multiset whose elements are the eigenvalues of the Laplacian. The following theorem allows us to compute the spectrum of the join of simplicial complexes (see [8, Theorem 4.10]).

Theorem 2.13. *Let $X = X_1 * \cdots * X_m$. Then*

$$\text{Spec}_k(X) = \bigcup_{\substack{i_1+\ldots+i_m=k-m+1, \\ -1\leq i_j \leq \dim(X_j)\ \forall j\in[m]}} \text{Spec}_{i_1}(X_1) + \cdots + \text{Spec}_{i_m}(X_m),$$

2.3.1 The simplicial Hodge theorem

The following discrete version of Hodge's theorem had been observed by Eckmann in [9].

Theorem 2.14 (Simplicial Hodge theorem)**.**

$$\tilde{H}^k(X;\mathbb{R}) \cong \operatorname{Ker} L_k(X).$$

To prove Theorem 2.14 we will need the following results:

Claim 2.15. $\operatorname{Ker} L_k = \operatorname{Ker} d_k \cap \operatorname{Ker} d_{k-1}^*$.

Proof. Let $\phi \in \operatorname{Ker} d_k \cap \operatorname{Ker} d_{k-1}^*$. Then

$$L_k\phi = d_{k-1}d_{k-1}^*\phi + d_k^* d_k \phi = 0 + 0 = 0,$$

therefore $\phi \in \operatorname{Ker} L_k$. On the other hand, if $\phi \in \operatorname{Ker} L_k$, then

$$\begin{aligned} 0 = \langle L_k\phi, \phi\rangle &= \left\langle d_{k-1}d_{k-1}^*\phi + d_k^* d_k\phi, \phi\right\rangle = \left\langle d_{k-1}d_{k-1}^*\phi, \phi\right\rangle + \langle d_k^* d_k\phi, \phi\rangle \\ &= \left\langle d_{k-1}^*\phi, d_{k-1}^*\phi\right\rangle + \langle d_k\phi, d_k\phi\rangle = \left\|d_{k-1}^*\phi\right\|^2 + \|d_k\phi\|^2. \end{aligned}$$

We obtain that $\left\|d_{k-1}^*\phi\right\| = \|d_k\phi\| = 0$, therefore $d_{k-1}^*\phi = 0$ and $d_k\phi = 0$, so $\phi \in \operatorname{Ker} d_k \cap \operatorname{Ker} d_{k-1}^*$. □

Theorem 2.16 (Simplicial Hodge decomposition)**.**

$$C^k(X) = \operatorname{Ker} d_k \oplus \operatorname{Im} d_k^* = \operatorname{Ker} L_k \oplus \operatorname{Im} d_{k-1} \oplus \operatorname{Im} d_k^*.$$

Proof. We have the following decompositions of $C^k(X)$:

$$C^k(X) = \operatorname{Im} d_k^* \oplus (\operatorname{Im} d_k^*)^\perp = \operatorname{Im} d_k^* \oplus \operatorname{Ker} d_k,$$

and

$$C^k(X) = \operatorname{Im} d_{k-1} \oplus (\operatorname{Im} d_{k-1})^\perp = \operatorname{Im} d_{k-1} \oplus \operatorname{Ker} d_{k-1}^*.$$

We can apply the second decomposition on the subspace $\operatorname{Ker} d_k$:

$$\operatorname{Ker} d_k = (\operatorname{Im} d_{k-1} \cap \operatorname{Ker} d_k) \oplus \left(\operatorname{Ker} d_{k-1}^* \cap \operatorname{Ker} d_k\right).$$

By Claim 2.15, $\operatorname{Ker} d_{k-1}^* \cap \operatorname{Ker} d_k = \operatorname{Ker} L_k$. Also, since $d_k d_{k-1} = 0$, $\operatorname{Im} d_{k-1} \subseteq \operatorname{Ker} d_k$, hence $\operatorname{Im} d_{k-1} \cap \operatorname{Ker} d_k = \operatorname{Im} d_{k-1}$. Therefore we have

$$\operatorname{Ker} d_k = \operatorname{Im} d_{k-1} \oplus \operatorname{Ker} L_k,$$

hence $C^k(X) = \operatorname{Im} d_k^* \oplus \operatorname{Ker} d_k = \operatorname{Im} d_k^* \oplus \operatorname{Im} d_{k-1} \oplus \operatorname{Ker} L_k$, as wanted. □

We obtain Theorem 2.14 as a corollary:

Proof of Theorem 2.14.

$$\tilde{H}^k(X;\mathbb{R}) = \frac{\operatorname{Ker} d_k}{\operatorname{Im} d_{k-1}} = \frac{\operatorname{Im} d_{k-1} \oplus \operatorname{Ker} L_k}{\operatorname{Im} d_{k-1}} \cong \operatorname{Ker} L_k.$$

□

Since $L_k(X)$ is positive semidefinite, we obtain as a consequence of Theorem 2.14:

Corollary 2.17. *$\tilde{H}^k(X;\mathbb{R}) = 0$ if and only if $\mu_k(X) > 0$.*

For example, we can use the simplicial Hodge theorem to easily compute the Laplacian spectra of the complete k-dimensional skeleton $\Delta_{n-1}^{(k)}$:

Claim 2.18.

$$\operatorname{Spec}_i(\Delta_{n-1}^{(k)}) = \begin{cases} \{\underbrace{n, n, \ldots, n}_{\binom{n}{i+1} \text{ times}}\} & \text{if } -1 \le i \le k-1, \\ \{\underbrace{0, 0, \ldots, 0}_{\binom{n-1}{k+1} \text{ times}}, \underbrace{n, n, \ldots, n}_{\binom{n-1}{k} \text{ times}}\} & \text{if } i = k. \end{cases}$$

Proof. For $i \le k-1$ we can check by Claim 2.10 that $L_i\left(\Delta_{n-1}^{(k)}\right) = nI$, therefore all the eigenvalues are equal to n.

For $i = k$, we have by Claim 2.6 $\tilde{H}^k\left(\Delta_{n-1}^{(k)};\mathbb{R}\right) = \mathbb{R}^{\binom{n-1}{k+1}}$, therefore, by Theorem 2.14, 0 is an eigenvalue of the k-th Laplacian of multiplicity $\binom{n-1}{k+1}$. It is easy to check by Claim 2.10 that

$$\operatorname{Tr}\left(L_k\left(\Delta_{n-1}^{(k)}\right)\right) = (k+1)\binom{n}{k+1} = n\binom{n-1}{k}.$$

But there are $\binom{n}{k+1} - \binom{n-1}{k+1} = \binom{n-1}{k}$ non-zero eigenvalues, all of them at most n (by Lemma 2.11), therefore they must all be equal to n. □

2.4 Matroids

In this section we recall some definitions and basic results about matroids. For additional results about matroids see [19], and see [6] for more results on the topology of the independence complex of a matroid.

Definition 2.19. A matroid is a pair $M = (V, \mathcal{I})$, where V is a finite set, and $\mathcal{I}$ is a collection of subsets of V such that:

1. $\mathcal{I} \neq \emptyset$.

2. If $I \in \mathcal{I}$, then $I' \in \mathcal{I}$ for all $I' \subset I$.

3. For every $A, B \in \mathcal{I}$, if $|A| > |B|$, then there exists $v \in A \setminus B$ such that $B \cup \{v\} \in \mathcal{I}$.

We call the sets in $\mathcal{I}$ the *independent sets* of M. Note that the first two axioms in the definition mean that $\mathcal{I}$ is a (non void) simplicial complex. The last axiom is called the *augmentation property*.

The maximal sets in $\mathcal{I}$ (with respect to inclusion) are called the *bases* of M.

Claim 2.20. *Let $A, B \in \mathcal{I}$ be bases of M. Then $|A| = |B|$.*

Proof. Assume for contradiction that $|A| > |B|$. Then by the augmentation property there is some $v \in A \setminus B$ such that $B \cup \{v\} \in \mathcal{I}$. But this is a contradiction to the maximality of B. □

Let $\rho : 2^V \to \mathbb{N}$ be the function defined by $\rho(S) = \max\{\,|I| : I \subset S, I \in \mathcal{I}\,\}$ for each $S \subset V$. We call ρ the *rank function* of the matroid M. We also define $\rho(M) = \rho(V)$, the *rank* of the matroid M.

The following properties of the rank function are easy to check from the definition:

Claim 2.21. *Let $S \subset V$ and $v \in V$. Then*

1. *$\rho(S) \leq |S|$, and $\rho(S) = |S|$ if and only if $S \in \mathcal{I}$.*

2. *$\rho(S) \leq \rho(S \cup \{v\}) \leq \rho(S) + 1$.*

For $S \subset V$ let $\mathrm{cl}(S) = \{\, v \in V : \rho(S \cup \{v\}) = \rho(S)\,\}$ be the *closure* of S.

Claim 2.22. *Let $S \subset V$. Then*

1. *$S \subset \mathrm{cl}(S)$.*

2. *$\rho(S) = \rho(\mathrm{cl}(S))$*

Proof.

1. For each $v \in S$, $S = S \cup \{v\}$, therefore $\rho(S) = \rho(S \cup \{v\})$. Hence $v \in \mathrm{cl}(S)$.

2. Assume on the contrary that $\rho(\mathrm{cl}(S)) > \rho(S)$. Let $I \subset S$ be an independent set such that $|I| = \rho(S)$, and $J \subset \mathrm{cl}(S)$ an independent set such that $|J| = \rho(\mathrm{cl}(S))$. So $|J| > |I|$, hence by the augmentation property, there exists some $v \in J \setminus I$ such that $I \cup \{v\} \in \mathcal{I}$. Therefore we obtain $\rho(S \cup \{v\}) \geq \rho(I \cup \{v\}) = |I \cup \{v\}| > |I| = \rho(S)$, a contradiction to $v \in J \subset \mathrm{cl}(S)$. □

A *flat* of M is a set $F \subset V$ such that $\mathrm{cl}(F) = F$.

Claim 2.23. *Let $F_1 \subset F_2$ be flats of M. If $\rho(F_1) = \rho(F_2)$, then $F_1 = F_2$.*

Proof. Assume that there is some $v \in F_2 \setminus F_1$. Since $v \notin F_1 = \mathrm{cl}(F_1)$, we have $\rho(F_1 \cup \{v\}) > \rho(F_1)$, therefore $\rho(F_2) \geq \rho(F_1 \cup \{v\}) > \rho(F_1)$, a contradiction. □

A *loop* in M is a vertex $v \in V$ such that for any $I \in \mathcal{I}$, $v \notin I$. We say that M is *loopless* if it does not contain any loop. From now on we will always assume that M is loopless. Also, we will identify M with $\mathcal{I}$ and treat it as a simplicial complex on vertex set V.

Claim 2.24.

1. *Let $S \subset V$. Then $M[S]$ is a matroid.*

2. *Let $\sigma \in M$. Then $\mathrm{lk}(M, \sigma)$ is a matroid.*

Proof. Both $M[S]$ and $\mathrm{lk}(M, \sigma)$ are subcomplexes of M, therefore we only have to check that the augmentation property is satisfied in both cases.

Let $A, B \in M[S]$ such that $|A| > |B|$. By the augmentation property in M, there exists $v \in A \setminus B$ such that $B \cup \{v\} \in M$. But $v \in A \subset S$, therefore $B \cup \{v\} \in M[S]$. So $M[S]$ is a matroid.

Let $A, B \in \mathrm{lk}(M, \sigma)$ such that $|A| > |B|$. Then $A \cup \sigma, B \cup \sigma \in M$, and $|A \cup \sigma| > |B \cup \sigma|$, therefore by the augmentation property for M there exists $v \in (A \cup \sigma) \setminus (B \cup \sigma) = A \setminus B$ such that $B \cup \{v\} \cup \sigma \in M$. But this means that $B \cup \{v\} \in \mathrm{lk}(M, \sigma)$. So $\mathrm{lk}(M, \sigma)$ is also a matroid. $\square$

The following result on the topology of matroids is well known:

Theorem 2.25. *Let M be a matroid. Then $\tilde{H}^k(M; \mathbb{R}) = 0$ for all $k \leq \rho(M) - 2$.*

Proof. Denote $r = \rho(M)$. We argue by induction on the size of V, the vertex set of M.

If $|V| = 1$ then M is just a point and therefore $\tilde{H}^k(M; \mathbb{R}) = 0$ for all $k \geq -1$.

Let $|V| = n$ for $n > 1$. If $\rho(M) = n$, then $V \in M$, therefore M is the complete complex, and $\tilde{H}^k(M; \mathbb{R}) = 0$ for all $k \geq -1$. Otherwise, let $J \in M$ be a basis of M, and choose some $v \in V \setminus J$. Then $J \in M[V \setminus \{v\}]$, hence $\rho(M[V \setminus \{v\}]) = r$.

Let I be a basis of M containing v. So $|I| = r$ and $I \setminus \{v\} \in \mathrm{lk}(M, v)$, therefore $\rho(\mathrm{lk}(M, v)) \geq r - 1$.

By Proposition 2.8 we have the following exact sequence

$$\cdots \to \tilde{H}^{k-1}(\mathrm{lk}(M, v); \mathbb{R}) \to \tilde{H}^k(M; \mathbb{R}) \to \tilde{H}^k(M[V \setminus \{v\}]; \mathbb{R}) \to \cdots$$

By the induction hypothesis we have $\tilde{H}^k(M[V \setminus \{v\}]; \mathbb{R}) = 0$ for all $k \leq r - 2$, and $\tilde{H}^k(\mathrm{lk}(M, v); \mathbb{R}) = 0$ for all $k \leq \rho(\mathrm{lk}(M, v)) - 2$, therefore $\tilde{H}^{k-1}(\mathrm{lk}(M, v); \mathbb{R}) = 0$ for all $k \leq r - 2$. Hence $\tilde{H}^k(M; \mathbb{R}) = 0$ for all $k \leq r - 2$. $\square$

Remark. It is in fact known that M is homotopy equivalent to a bouquet of $(\rho(M) - 1)$-dimensional spheres (see [6]).

Chapter 3

Spectral gaps of complexes without large missing faces

In this chapter we prove our main results on the spectral gaps of generalized flag complexes. Section 3.1 contains some definitions and results that we will need, and Section 3.2 contains the proofs of Theorems 1.2 and 1.3.

3.1 Missing faces and sums of degrees

Let X be a complex on vertex set V with $h(X) = d$. Let $k \geq d$ and $\theta \in \binom{V}{k+1}$. Define

$$T(\theta) = \left\{ \tau \in \binom{\theta}{d+1} : \tau \notin X(d) \right\}.$$

So $T(\theta)$ is the set of all d-dimensional simplices in θ that do not belong to X, and $\theta \in X$ if and only if $T(\theta) = \emptyset$. Let

$$\mathrm{Mis}\,(\theta) = \bigcap_{\tau \in T(\theta)} \tau$$

and

$$m\,(\theta) = \left| \bigcap_{\tau \in T(\theta)} \tau \right|.$$

Since every $\tau \in T(\theta)$ has $d+1$ vertices it follows that $m\,(\theta) \leq d+1$. Another simple observation is the following:

Lemma 3.1. *Let $\sigma, \tau \in X(k)$ such that $|\tau \cap \sigma| = k$. Then if $\sigma \cup \tau \in X(k+1)$, $m\,(\sigma \cup \tau) = 0$, otherwise $2 \leq m\,(\sigma \cup \tau) \leq d+1$.*

Proof. Denote $\sigma \setminus \tau = \{v\}$ and $\tau \setminus \sigma = \{w\}$. If $\sigma \cup \tau \in X(k+1)$ then $T(\sigma \cup \tau) = \emptyset$, therefore $m\,(\sigma \cup \tau) = 0$. If $\sigma \cup \tau \notin X(k+1)$, then every $\eta \in T(\sigma \cup \tau)$ must contain both v and w (otherwise η will be contained in σ or in τ, a contradiction to $\eta \notin X(d)$). Therefore $m\,(\sigma \cup \tau) \geq 2$. $\square$

The following is a known result on clique complexes (see [3, Claim 3.4], [5]):

Lemma 3.2. *Let X be a clique complex with n vertices and let $\sigma \in X(k)$. Then*

$$\sum_{\tau\in\sigma(k-1)} \deg_X(\tau) - k\deg_X(\sigma) \le n.$$

We will need a version of this lemma for complexes without large missing faces:

Lemma 3.3. *Let X be a simplicial complex on vertex set V with $h(X) = d$. Let $k \ge d$ and $\sigma \in X(k)$. Then*

$$\sum_{\tau\in\sigma(k-1)} \deg_X(\tau) = k+1+(k+1)\deg_X(\sigma) + \sum_{r=2}^{d+1}(r-1)\cdot |\{\, v\in V : m\,(v\sigma) = r\,\}|\,.$$

Proof.

$$\begin{aligned}\sum_{\tau\in\sigma(k-1)} \deg_X(\tau) &= \sum_{\tau\in\sigma(k-1)}\ \sum_{v\in \mathrm{lk}(X,\tau)} 1 = \sum_{v\in V}\ \sum_{\substack{\tau\in\sigma(k-1),\\ \tau\in\mathrm{lk}(X,v)}} 1\\ &= \sum_{v\in\sigma}\ \sum_{\substack{\tau\in\sigma(k-1),\\ \tau\in\mathrm{lk}(X,v)}} 1 + \sum_{v\in\mathrm{lk}(X,\sigma)}\ \sum_{\substack{\tau\in\sigma(k-1),\\ \tau\in\mathrm{lk}(X,v)}} 1 + \sum_{\substack{v\in V\setminus\sigma,\\ v\notin\mathrm{lk}(X,\sigma)}}\ \sum_{\substack{\tau\in\sigma(k-1),\\ \tau\in\mathrm{lk}(X,v)}} 1. \quad (3.1)\end{aligned}$$

We consider separately the three summands on the right hand side of (3.1):

1. For $v \in \sigma$, there is only one $\tau \in \sigma(k-1)$ such that $\tau \in \mathrm{lk}(X, v)$, namely $\tau = \sigma \setminus \{v\}$. Thus the first summand is $k + 1$.

2. For $v \in \mathrm{lk}(X, \sigma)$, any $\tau \in \sigma(k-1)$ is in $\mathrm{lk}(X, v)$, therefore the second summand is $(k+1)\deg_X(\sigma)$.

3. Let $v \in V \setminus \sigma$ such that $v \notin \mathrm{lk}(X, \sigma)$. Let $\tau \in \sigma(k-1)$ and let u be the unique vertex in $\sigma \setminus \tau$. If $\tau \in \mathrm{lk}(X, v)$ then every missing face of X contained in $v\sigma$ must contain u, so $u \in \mathrm{Mis}\,(v\sigma)$. If $\tau \notin \mathrm{lk}(X, v)$, then there is a missing face of X contained in $v\tau$, and therefore it doesn't contain the vertex u. Hence, $u \notin \mathrm{Mis}\,(v\sigma)$. Since $v \in \mathrm{Mis}\,(v\sigma)$, the number of $\tau \in \sigma(k-1)$ such that $\tau \in \mathrm{lk}(X, v)$ is exactly $m\,(v\sigma) - 1$. Hence the third summand is

$$\sum_{\substack{v\in V\setminus\sigma,\\ v\notin\mathrm{lk}(X,\sigma)}} (m\,(v\sigma)-1) = \sum_{r=2}^{d+1}(r-1)|\,\{\, v\in V :\ m\,(v\sigma) = r\,\}\,|.$$

We obtain

$$\sum_{\tau\in\sigma(k-1)} \deg_X(\tau) = k+1+(k+1)\deg_X(\sigma) + \sum_{r=2}^{d+1}(r-1)\,|\{\, v\in V : m\,(v\sigma) = r\,\}|\,.$$

A slightly different version of the previous lemma is the following:

Lemma 3.4. *Let X be a simplicial complex with $h(X) = d$ on vertex set V, where $|V| = n$. Let $k \geq 0$ and $\sigma \in X(k)$. Then*

$$\sum_{\tau \in \sigma(k-1)} \deg_X(\tau) - (k-d+1)\deg_X(\sigma) \leq dn - (d-1)(k+1).$$

Proof. As in the proof of Lemma 3.3, we have

$$\sum_{\tau \in \sigma(k-1)} \deg_X(\tau) = (k+1)(\deg_X(\sigma)+1) + \sum_{\substack{v \in V \setminus \sigma, \\ v \notin \mathrm{lk}(X,\sigma)}} \sum_{\substack{\tau \in \sigma(k-1), \\ \tau \in \mathrm{lk}(X,v)}} 1.$$

Let $v \in V \setminus \sigma$ such that $v \notin \mathrm{lk}(X,\sigma)$. For each $\tau \in \sigma(k-1)$ such that $\tau \in \mathrm{lk}(X,v)$, let u be the unique vertex in $\sigma \setminus \tau$. Since $v\tau \in X$ but $v\sigma \notin X$, u must belong to every missing face of X contained in $v\sigma$. Also v must belong to every such missing face, since $\sigma \in X$. Therefore, since $h(X) = d$, we can have at most d such different vertices u, therefore

$$|\{\, \tau \in \sigma(k-1) :\, \tau \in \mathrm{lk}(X,v) \,\}| \leq d.$$

Thus we obtain

$$\begin{aligned}
\sum_{\tau \in \sigma(k-1)} \deg_X(\tau) &\leq (k+1)(\deg_X(\sigma)+1) + \sum_{\substack{v \in V \setminus \sigma, \\ v \notin \mathrm{lk}(X,\sigma)}} d \\
&\leq (k+1)(\deg_X(\sigma)+1) + (n-k-1-\deg_X(\sigma))d \\
&= dn - (d-1)(k+1) + (k-d+1)\deg_X(\sigma).
\end{aligned}$$

3.2 Spectral gaps

In this section we prove Theorems 1.2 and 1.3.

Let X be a simplicial complex with $h(X) = d$ on vertex set V, where $|V| = n$, and let $k \geq d$. For $\phi \in C^k(X)$ and $u \in V$ we define $\phi_u \in C^{k-1}(X)$ by

$$\phi_u(\tau) = \begin{cases} \phi(u\tau) & \text{if } u \in \mathrm{lk}(X,\tau), \\ 0 & \text{otherwise.} \end{cases}$$

Let $B_k : C^k(X) \to C^k(X)$ be the linear transformation whose matrix representation in the standard basis is

$$[B_k]_{\tau,\sigma} = \begin{cases} k\deg_X(\sigma) - \sum_{\eta \in \sigma(k-1)} \deg_X(\eta) & \text{if } \sigma = \tau, \\ (m(\sigma \cup \tau) - 2) \cdot (\sigma : \sigma \cap \tau) \cdot (\tau : \sigma \cap \tau) & \text{if } \substack{|\sigma \cap \tau| = k, \\ \sigma \cup \tau \notin X(k+1),} \\ 0 & \text{otherwise.} \end{cases}$$

Let $R_k = (d-1)L_k - B_k$, and let λ_k be the largest eigenvalue of R_k.

The proof of Theorem 1.2 depends on the following two ingredients:

Proposition 3.5. *Let $\phi \in C^k(X)$. Then*

$$(k-d+1)\langle L_k\phi, \phi\rangle = \sum_{u\in V} \langle L_{k-1}\phi_u, \phi_u\rangle - \langle R_k\phi, \phi\rangle .$$

Proposition 3.6. $\lambda_k \le dn$.

We postpone the proof of these propositions to the end of this section, and first show how they imply Theorem 1.2.

Theorem 1.2. *For $k \ge d$*

$$(k-d+1)\mu_k(X) \ge (k+1)\mu_{k-1}(X) - dn.$$

Proof. Let $0 \ne \phi \in C^k(X)$ be an eigenvector of L_k with eigenvalue $\mu_k(X)$. By Proposition 3.5 we obtain

$$\begin{aligned}(k-d+1)\mu_k(X)\|\phi\|^2 &= (k-d+1)\langle L_k\phi, \phi\rangle \\ &= \sum_{u\in V} \langle L_{k-1}\phi_u, \phi_u\rangle - \langle R_k\phi, \phi\rangle \ge \mu_{k-1}(X)\sum_{u\in V}\|\phi_u\|^2 - \lambda_k\|\phi\|^2 .\end{aligned}$$

But

$$\begin{aligned}\sum_{u\in V}\|\phi_u\|^2 = \sum_{u\in V}\sum_{\tau\in X(k-1)} \phi_u(\tau)^2 &= \sum_{\tau\in X(k-1)}\sum_{u\in \mathrm{lk}(X,\tau)} \phi(u\tau)^2 \\ &= (k+1)\sum_{\sigma\in X(k)}\phi(\sigma)^2 = (k+1)\|\phi\|^2 .\end{aligned}$$

Therefore

$$(k-d+1)\mu_k(X) \ge (k+1)\mu_{k-1}(X) - \lambda_k,$$

and by Proposition 3.6

$$(k-d+1)\mu_k(X) \ge (k+1)\mu_{k-1}(X) - dn.$$

□

For the proof of Theorem 1.3 we will need the following result, which will also be used in Section 4.1.

Claim 3.7. *For $k \ge d-1$,*

$$\mu_k(X) \ge \binom{k+1}{d}\mu_{d-1}(X) - \left(\binom{k+1}{d} - 1\right)n. \tag{3.2}$$

If in addition X has complete $(d-1)$-dimensional skeleton, then there is equality in (3.2) *for* $0 \le k \le d-1$.

Proof. We argue by induction on k. The case $k = d-1$ is clear. Let $k \ge d$. By Theorem 1.2 and the induction hypothesis we obtain

$$\begin{aligned}\mu_k(X) &\ge \frac{k+1}{k-d+1}\mu_{k-1}(X) - \frac{d}{k-d+1}n \\ &\ge \frac{k+1}{k-d+1}\left[\binom{k}{d}\mu_{d-1}(X) - \left(\binom{k}{d}-1\right)n\right] - \frac{d}{k-d+1}n \\ &= \binom{k+1}{d}\mu_{d-1}(X) - \left(\binom{k+1}{d}-1\right)n.\end{aligned}$$

Now assume that X has complete $(d-1)$-dimensional skeleton, and let $k < d-1$. Then we have $\binom{k+1}{d} = 0$, therefore the inequality in the claim is just $\mu_k(X) \ge n$. But one can see by Claim 2.10 that in this case L_k is the scalar matrix with diagonal elements n, thus $\mu_k(X) = n$. $\square$

Theorem 1.3. *If*

$$\mu_{d-1}(X) > \left(1 - \binom{k+1}{d}^{-1}\right)n,$$

then $\tilde{H}^j(X;\mathbb{R}) = 0$ *for all* $d-1 \le j \le k$.

Proof. Let $d-1 \le j \le k$. We have by Claim 3.7

$$\begin{aligned}\mu_j(X) &\ge \binom{j+1}{d}\mu_{d-1}(X) - \left(\binom{j+1}{d}-1\right)n \\ &> \binom{j+1}{d}\cdot\left(1-\binom{k+1}{d}^{-1}\right)n - \left(\binom{j+1}{d}-1\right)n \\ &\ge \binom{j+1}{d}\cdot\left(1-\binom{j+1}{d}^{-1}\right)n - \left(\binom{j+1}{d}-1\right)n = 0.\end{aligned}$$

Thus, by Corollary 2.17, $\tilde{H}^j(X;\mathbb{R}) = 0$. $\square$

In order to prove Proposition 3.5 we will need the following claims.

Claim 3.8 (see [3, Claim 3.1])**.** *For* $\phi \in C^k(X)$

$$\|d_k\phi\|^2 = \sum_{\sigma\in X(k)} \deg_X(\sigma)\phi(\sigma)^2 - 2\sum_{\eta\in X(k-1)}\sum_{vw\in \mathrm{lk}(X,\eta)} \phi(v\eta)\phi(w\eta).$$

Proof.

$$\begin{aligned}\|d_k\phi\|^2 &= \sum_{\tau\in X(k+1)} d_k\phi(\tau)^2 = \sum_{\tau\in X(k+1)} \left(\sum_{\theta_1\in\tau(k)} (\tau:\theta_1)\phi(\theta_1)\right)\left(\sum_{\theta_2\in\tau(k)} (\tau:\theta_2)\phi(\theta_2)\right)\\ &= \sum_{\tau\in X(k+1)}\sum_{\sigma\in\tau(k)}\phi(\sigma)^2 + \sum_{\tau\in X(k+1)}\sum_{\theta_1\in\tau(k)}\sum_{\substack{\theta_2\in\tau(k),\\ \theta_2\neq\theta_1}} (\tau:\theta_1)(\tau:\theta_2)\phi(\theta_1)\phi(\theta_2)\\ &= \sum_{\sigma\in X(k)}\deg(\sigma)\phi(\sigma)^2 + \sum_{\tau\in X(k+1)}\sum_{\theta_1\in\tau(k)}\sum_{\substack{\theta_2\in\tau(k),\\ \theta_2\neq\theta_1}} (\tau:\theta_1)(\tau:\theta_2)\phi(\theta_1)\phi(\theta_2).\end{aligned}$$

Now look at the map

$$\left\{(\eta,v,w) : \substack{\eta\in X(k-1),\\ v,w\in V,\ v\neq w,\\ vw\in\mathrm{lk}(X,\eta)}\right\} \to \left\{(\tau,\theta_1,\theta_2) : \substack{\tau\in X(k+1),\\ \theta_1,\theta_2\in\tau(k),\,\theta_1\neq\theta_2}\right\}$$

defined by $(\eta,v,w)\mapsto(vw\eta,v\eta,w\eta)$. For each (τ,θ_1,θ_2) in the codomain, let $\eta=\theta_1\cap\theta_2$, $\{v\}=\theta_1\setminus\theta_2$ and $\{w\}=\theta_2\setminus\theta_1$. (η,v,w) is the unique element sent to (τ,θ_1,θ_2). So the map is a bijection, therefore we obtain

$$\begin{aligned}\|d_k\phi\|^2 &= \sum_{\sigma\in X(k)}\deg_X(\sigma)\phi(\sigma)^2\\ &\quad+\sum_{\eta\in X(k-1)}\sum_{v\in V}\sum_{\substack{w\in V\setminus\{v\},\\ vw\in\mathrm{lk}(X,\eta)}}(vw\eta:v\eta)(vw\eta:w\eta)\phi(v\eta)\phi(w\eta)\\ &=\sum_{\sigma\in X(k)}\deg_X(\sigma)\phi(\sigma)^2-2\sum_{\eta\in X(k-1)}\sum_{vw\in\mathrm{lk}(X,\eta)}\phi(v\eta)\phi(w\eta).\end{aligned}$$

□

Claim 3.9. *For $\phi\in C^k(X)$*

$$\begin{aligned}\sum_{u\in V}\|d_{k-1}\phi_u\|^2 &= \sum_{\sigma\in X(k)}\sum_{\tau\in\sigma(k-1)}\deg_X(\tau)\phi(\sigma)^2-2k\sum_{\tau\in X(k-1)}\sum_{vw\in\mathrm{lk}(X,\tau)}\phi(v\tau)\phi(w\tau)\\ &\quad-2\sum_{\eta\in X(k-2)}\sum_{vw\in\mathrm{lk}(X,\eta)}\sum_{\substack{u\in\mathrm{lk}(X,v\eta)\cap\mathrm{lk}(X,w\eta)\\ u\notin\mathrm{lk}(X,vw\eta)}}\phi(vu\eta)\phi(wu\eta).\end{aligned}$$

Proof. First we apply Claim 3.8 to $\phi_u\in C^{k-1}(X)$:

$$\|d_{k-1}\phi_u\|^2=\sum_{\tau\in X(k-1)}\deg_X(\tau)\phi_u(\tau)^2-2\sum_{\eta\in X(k-2)}\sum_{vw\in\mathrm{lk}(X,\eta)}\phi_u(v\eta)\phi_u(w\eta).$$

Summing over all vertices we obtain

$$\begin{aligned}\sum_{u\in V}\|d_{k-1}\phi_u\|^2 &= \sum_{u\in V}\sum_{\tau\in X(k-1)}\deg_X(\tau)\phi_u(\tau)^2\\ &- 2\sum_{u\in V}\sum_{\eta\in X(k-2)}\sum_{vw\in \mathrm{lk}(X,\eta)}\phi_u(v\eta)\phi_u(w\eta)\\ &= \sum_{u\in V}\sum_{\tau\in X(k-1)\cap \mathrm{lk}(X,u)}\deg_X(\tau)\phi(u\tau)^2\\ &- 2\sum_{\eta\in X(k-2)}\sum_{vw\in \mathrm{lk}(X,\eta)}\sum_{u\in \mathrm{lk}(X,v\eta)\cap \mathrm{lk}(X,w\eta)}\phi(vu\eta)\phi(wu\eta)\\ &= \sum_{\sigma\in X(k)}\sum_{\tau\in\sigma(k-1)}\deg_X(\tau)\phi(\sigma)^2\\ &- 2\sum_{\eta\in X(k-2)}\sum_{vw\in \mathrm{lk}(X,\eta)}\sum_{u\in \mathrm{lk}(X,v\eta)\cap \mathrm{lk}(X,w\eta)}\phi(vu\eta)\phi(wu\eta).\end{aligned}$$

Let $\eta \in X(k-2)$, $vw \in \mathrm{lk}(X,\eta)$, and $u \in \mathrm{lk}(X, v\eta) \cap \mathrm{lk}(X, w\eta)$. We split into two different cases: $u \in \mathrm{lk}(X, vw\eta)$ or $u \notin \mathrm{lk}(X, vw\eta)$. Assume that $u \in \mathrm{lk}(X, vw\eta)$, and let $\tau = u\eta$. Then we have $vw \in \mathrm{lk}(X,\tau)$. This defines a map

$$\left\{(\eta, vw, u) : {}^{\eta\in X(k-2),}_{vw\in \mathrm{lk}(X,\eta), u\in \mathrm{lk}(X,vw\eta)}\right\} \to \left\{(\tau, vw) : {}^{\tau\in X(k-1),}_{vw\in \mathrm{lk}(X,\tau)}\right\}.$$

Each pair (τ, vw) has a preimage of size k (these are the tuples $(\tau \setminus u, vw, u)$ for each $u \in \tau$). Therefore we obtain

$$\begin{aligned}\sum_{u\in V}\|d_{k-1}\phi_u\|^2 &= \sum_{\sigma\in X(k)}\sum_{\tau\in\sigma(k-1)}\deg_X(\tau)\phi(\sigma)^2\\ &- 2\sum_{\eta\in X(k-2)}\sum_{vw\in \mathrm{lk}(X,\eta)}\sum_{u\in \mathrm{lk}(X,vw\eta)}\phi(vu\eta)\phi(wu\eta)\\ &- 2\sum_{\eta\in X(k-2)}\sum_{vw\in \mathrm{lk}(X,\eta)}\sum_{\substack{u\in \mathrm{lk}(X,v\eta)\cap \mathrm{lk}(X,w\eta)\\ u\notin \mathrm{lk}(X,vw\eta)}}\phi(vu\eta)\phi(wu\eta)\\ &= \sum_{\sigma\in X(k)}\sum_{\tau\in\sigma(k-1)}\deg_X(\tau)\phi(\sigma)^2 - 2k\sum_{\tau\in X(k-1)}\sum_{vw\in \mathrm{lk}(X,\tau)}\phi(v\tau)\phi(w\tau)\\ &- 2\sum_{\eta\in X(k-2)}\sum_{vw\in \mathrm{lk}(X,\eta)}\sum_{\substack{u\in \mathrm{lk}(X,v\eta)\cap \mathrm{lk}(X,w\eta)\\ u\notin \mathrm{lk}(X,vw\eta)}}\phi(vu\eta)\phi(wu\eta).\end{aligned}$$

□

Remark. If X is a clique complex and $u \in \mathrm{lk}(X, v\eta) \cap \mathrm{lk}(X, w\eta)$ for $\eta \in X(k-2)$ and $vw \in \mathrm{lk}(X,\eta)$, then all the 1-dimensional faces of the simplex $uvw\eta$ belong to X, therefore $uvw\eta \in X$ (i.e. $u \in \mathrm{lk}(X, vw\eta)$). Therefore in this case the last term of the previous equation vanishes (see [3, Claim 3.2]).

Claim 3.10 (see [3, Claim 3.3])**.** *For* $\phi \in C^k(X)$

$$\sum_{u\in V} \left\|d_{k-2}^*\phi_u\right\|^2 = k\left\|d_{k-1}^*\phi\right\|^2.$$

Proof.

$$\left\|d_{k-1}^*\phi\right\|^2 = \sum_{\tau\in X(k-1)} d_{k-1}^*\phi(\tau)^2 = \sum_{\tau\in X(k-1)}\left(\sum_{v\in\mathrm{lk}(X,\tau)}\phi(v\tau)\right)^2.$$

Similarly,

$$\begin{aligned}\sum_{u\in V}\left\|d_{k-2}^*\phi_u\right\|^2 &= \sum_{u\in V}\sum_{\eta\in X(k-2)}\left(\sum_{v\in\mathrm{lk}(X,\eta)}\phi_u(v\eta)\right)^2\\ &= \sum_{\eta\in X(k-2)}\sum_{u\in\mathrm{lk}(X,\eta)}\left(\sum_{v\in\mathrm{lk}(X,u\eta)}\phi(uv\eta)\right)^2\\ &= k\sum_{\tau\in X(k-1)}\left(\sum_{v\in\mathrm{lk}(X,\tau)}\phi(v\tau)\right)^2 = k\left\|d_{k-1}^*\phi\right\|^2.\end{aligned}$$

□

Let $A_k : C^k(X)\to C^k(X)$ be the linear transformation whose matrix representation in the standard basis is

$$[A_k]_{\sigma,\tau} = \begin{cases}(m(\sigma\cup\tau)-2)\cdot(\sigma:\sigma\cap\tau)\cdot(\tau:\sigma\cap\tau) & \text{if } {|\sigma\cap\tau|=k, \atop \sigma\cup\tau\notin X(k+1)},\\ 0 & \text{otherwise.}\end{cases}$$

Claim 3.11. *For* $\phi\in C^k(X)$

$$\langle A_k\phi,\phi\rangle = 2\sum_{\eta\in X(k-2)}\sum_{vw\in\mathrm{lk}(X,\eta)}\sum_{\substack{u\in\mathrm{lk}(X,v\eta)\cap\mathrm{lk}(X,w\eta),\\ u\notin\mathrm{lk}(X,vw\eta)}}\phi(vu\eta)\phi(wu\eta).$$

Proof.

$$\begin{aligned}\langle A_k\phi,\phi\rangle &= \sum_{\tau\in X(k)}\sum_{\substack{\sigma\in X(k),\\ |\sigma\cap\tau|=k,\\ \sigma\cup\tau\notin X(k+1)}}(m(\sigma\cup\tau)-2)(\sigma:\sigma\cap\tau)(\tau:\sigma\cap\tau)\phi(\tau)\phi(\sigma)\\ &= \sum_{\theta\in X(k-1)}\sum_{v\in\mathrm{lk}(X,\theta)}\sum_{\substack{w\in\mathrm{lk}(X,\theta),\\ vw\theta\notin X(k+1)}}(m(vw\theta)-2)(v\theta:\theta)(w\theta:\theta)\phi(v\theta)\phi(w\theta)\\ &= \sum_{\theta\in X(k-1)}\sum_{v\in\mathrm{lk}(X,\theta)}\sum_{\substack{w\in\mathrm{lk}(X,\theta),\\ vw\theta\notin X(k+1)}}(m(vw\theta)-2)\phi(v\theta)\phi(w\theta).\end{aligned}$$

Let $<$ be an order on the vertices of X. Look at the map

$$\left\{(\eta, u, vw) : \substack{\eta\in X(k-2),\, vw\in \mathrm{lk}(X,\eta),\\ u\in \mathrm{lk}(X,v\eta)\cap \mathrm{lk}(X,w\eta),\\ u\notin \mathrm{lk}(X,vw\eta)}\right\} \to \left\{(\theta, v, w) : \substack{\theta\in X(k-1),\\ v,w\in \mathrm{lk}(X,\theta),\, v<w,\\ vw\theta\notin X(k+1)}\right\}$$

defined by $(\eta, u, vw) \mapsto (u\eta, v, w)$. Note that for any (η, u, vw) in the domain, we have $u, v, w \in \mathrm{Mis}\,(uvw\eta)$. Let (θ, v, w) in the codomain, and let $u \in \mathrm{Mis}\,(vw\theta) \setminus \{v, w\}$ and $\eta = \theta \setminus \{u\}$. Then $vw \in \mathrm{lk}(X, \eta)$ (since $vw\eta$ doesn't contain u, therefore can't contain any missing face). Similarly, $u \in \mathrm{lk}(X, v\eta) \cap \mathrm{lk}(X, w\eta)$, but $u \notin \mathrm{lk}(X, vw\eta)$ (otherwise $uvw\eta = vw\theta \in X(k + 1)$). Therefore (η, u, vw) is in the preimage of (θ, v, w). Hence (θ, v, w) has a preimage of size $m\,(vw\theta) - 2$. So we have

$$\begin{aligned}
\langle A_k\phi, \phi\rangle &= \sum_{\theta\in X(k-1)} \sum_{v\in \mathrm{lk}(X,\theta)} \sum_{\substack{w\in \mathrm{lk}(X,\theta),\\ vw\theta\notin X(k+1)}} (m\,(vw\theta) - 2)\phi(v\theta)\phi(w\theta)\\
&= 2 \sum_{\theta\in X(k-1)} \sum_{v\in \mathrm{lk}(X,\theta)} \sum_{\substack{w\in \mathrm{lk}(X,\theta),\\ v<w,\\ vw\theta\notin X(k+1)}} (m\,(vw\theta) - 2)\phi(v\theta)\phi(w\theta)\\
&= 2 \sum_{\eta\in X(k-2)} \sum_{vw\in \mathrm{lk}(X,\eta)} \sum_{\substack{u\in \mathrm{lk}(X,v\eta)\cap \mathrm{lk}(X,w\eta),\\ u\notin \mathrm{lk}(X,vw\eta)}} \phi(vu\eta)\phi(wu\eta).
\end{aligned}$$

□

Proof of Proposition 3.5. Let $\phi \in C^k(X)$. By Claim 3.11 we have

$$\begin{aligned}
\langle B_k\phi, \phi\rangle = \sum_{\sigma\in X(k)} &\left(k \deg_X(\sigma) - \sum_{\tau\in\sigma(k-1)} \deg_X(\tau)\right) \phi(\sigma)^2\\
&+ 2 \sum_{\eta\in X(k-2)} \sum_{vw\in \mathrm{lk}(X,\eta)} \sum_{\substack{u\in \mathrm{lk}(X,v\eta)\cap \mathrm{lk}(X,w\eta)\\ u\notin \mathrm{lk}(X,vw\eta)}} \phi(vu\eta)\phi(wu\eta).
\end{aligned}$$

By Claims 3.8 and 3.9 we obtain

$$\begin{aligned}
k\left\|d_k\phi\right\|^2 &= \sum_{u\in V} \left\|d_{k-1}\phi_u\right\|^2 + \sum_{\sigma\in X(k)} \left(k \deg_X(\sigma) - \sum_{\tau\in\sigma(k-1)} \deg_X(\tau)\right) \phi(\sigma)^2\\
&+ 2 \sum_{\eta\in X(k-2)} \sum_{vw\in \mathrm{lk}(X,\eta)} \sum_{\substack{u\in \mathrm{lk}(X,v\eta)\cap \mathrm{lk}(X,w\eta)\\ u\notin \mathrm{lk}(X,vw\eta)}} \phi(vu\eta)\phi(wu\eta)\\
&= \sum_{u\in V} \left\|d_{k-1}\phi_u\right\|^2 + \langle B_k\phi, \phi\rangle\,.
\end{aligned}$$

Then by the previous equation and Claim 3.10, we obtain

$$\begin{aligned}
k \langle L_k\phi, \phi\rangle &= k \left\langle d_k^* d_k \phi + d_{k-1} d_{k-1}^* \phi, \phi \right\rangle = k \left\| d_k \phi \right\|^2 + k \left\| d_{k-1}^* \phi \right\|^2 \\
&= \sum_{u\in V} \left\| d_{k-1}\phi_u \right\|^2 + \langle B_k \phi, \phi\rangle + \sum_{u\in V} \left\| d_{k-2}^* \phi_u \right\|^2 \\
&= \sum_{u\in V} \langle L_{k-1}\phi_u, \phi_u\rangle + \langle B_k\phi, \phi\rangle .
\end{aligned}$$

Substracting $(d-1)\langle L_k\phi, \phi\rangle$ from both sides of the equation we get

$$\begin{aligned}
(k-d+1)\langle L_k\phi, \phi\rangle &= \sum_{u\in V} \langle L_{k-1}\phi_u, \phi_u\rangle - \langle ((d-1)L_k - B_k)\phi, \phi\rangle \\
&= \sum_{u\in V} \langle L_{k-1}\phi_u, \phi_u\rangle - \langle R_k \phi, \phi\rangle .
\end{aligned}$$

□

For the proof of Proposition 3.6 we will need the next result, which follows from the definition of B_k and Claim 2.10.

Claim 3.12. *The matrix representation of R_k in the standard basis is*

$$[R_k]_{\sigma,\tau} = \begin{cases} \sum_{\eta\in\sigma(k-1)} \deg_X(\eta) - (k-d+1)\deg_X(\sigma) + (d-1)(k+1) & \text{if } \sigma=\tau, \\ (d+1-m(\sigma\cup\tau))\cdot(\sigma:\sigma\cap\tau)\cdot(\tau:\sigma\cap\tau) & \text{if } {|\sigma\cap\tau|=k, \atop \sigma\cup\tau\notin X(k+1)}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof of Proposition 3.6. Let K_r be the $(k+1)$-dimensional simplicial complex on vertex set V, with full k-skeleton, whose $(k+1)$-dimensional faces are the simplices $\eta \in \binom{V}{k+2}$ such that $m(\eta) = r$. By Claim 2.10, we have

$$\begin{aligned}
\left[L_k^+(K_r)\right]_{\sigma,\tau} &= \begin{cases} \deg_{K_r}(\sigma) & \text{if } \sigma=\tau, \\ -(\sigma:\sigma\cap\tau)\cdot(\tau:\sigma\cap\tau) & \text{if } |\sigma\cap\tau|=k, m(\sigma\cup\tau)=r, \\ 0 & \text{otherwise.} \end{cases} \\
&= \begin{cases} |\{v\in V : m(v\sigma)=r\}| & \text{if } \sigma=\tau, \\ -(\sigma:\sigma\cap\tau)\cdot(\tau:\sigma\cap\tau) & \text{if } |\sigma\cap\tau|=k, m(\sigma\cup\tau)=r, \\ 0 & \text{otherwise.} \end{cases} \qquad (3.3)
\end{aligned}$$

Denote by $M_{k,r}$ the principal submatrix of $\left[L_k^+(K_r)\right]$ obtained by keeping only the rows and columns corresponding to simplices in $X(k)$. $M_{k,r}$ is a positive semidefinite matrix (as a principal submatrix of a positive semidefinite matrix).

Define a new matrix

$$M_k = [R_k] + \sum_{r=2}^{d} (d+1-r) M_{k,r}.$$

Since $M_{k,r}$ is positive semidefinite it follows that $\lambda_{\max}(-M_{k,r}) \leq 0$ for all $2 \leq r \leq d$ and therefore

$$\lambda_k = \lambda_{\max}([R_k]) \leq \lambda_{\max}(M_k) + \sum_{r=2}^{d} (d+1-r) \lambda_{\max}(-M_{k,r}) \leq \lambda_{\max}(M_k). \tag{3.4}$$

By equation (3.3), Lemma 3.1 and Claim 3.12 we see that the matrix M_k is diagonal, and

$$(M_k)_{\sigma,\sigma} = \sum_{\eta \in \sigma(k-1)} \deg_X(\eta) - (k-d+1) \deg_X(\sigma) + (d-1)(k+1) + \sum_{r=2}^{d} (d+1-r) \cdot |\{ v \in V : m(v\sigma) = r \}|.$$

Let $\sigma \in X(k)$. We can write

$$\deg_X(\sigma) = |\{ v \in V : v \in \mathrm{lk}(X, \sigma) \}|$$

and

$$k+1 = |\{ v \in V : v \in \sigma \}|,$$

and by Lemma 3.3

$$\sum_{\eta \in \sigma(k-1)} \deg_X(\eta) = |\{ v \in V : v \in \sigma \}| + (k+1) \cdot |\{ v \in V : v \in \mathrm{lk}(X, \sigma) \}| + \sum_{r=2}^{d+1} (r-1) \cdot |\{ v \in V : m(v\sigma) = r \}|.$$

Hence,

$$(M_k)_{\sigma,\sigma} = d \cdot |\{ v \in V : v \in \sigma \}| + d \cdot |\{ v \in V : v \in \mathrm{lk}(X, \sigma) \}| + \sum_{r=2}^{d+1} d \cdot |\{ v \in V : m(v\sigma) = r \}| \leq d \cdot |V| = dn.$$

Therefore $\lambda_{\max}(M_k) \leq dn$, so by inequality (3.4): $\lambda_k \leq dn$. □

Chapter 4

Vector domination and geometric Hall type theorems

In this chapter we show some applications of our main results. In Section 4.1 we prove a new lower bound on the connectivity of a simplicial complex X, depending on its vector domination number $\Gamma(X)$. Then in Section 4.2 we apply this bound to find sufficient conditions for the existence of colorful sets in general position in a matroid.

4.1 Vector domination

In this section we study the vector domination number $\Gamma(X)$ of a simplicial complex X, leading up to the proof of Theorem 1.6 that provides an upper bound on $\Gamma(X)$ in terms of the homological connectivity of X.

First we prove Proposition 1.5, relating $\Gamma(X)$ to the total domination number $\tilde{\gamma}(X)$.

Proposition 1.5. Let X be a simplicial complex with all its missing faces of dimension equal to d. Then

$$\Gamma(X) \leq \binom{\tilde{\gamma}(X)}{d}.$$

Proof. Let S be a totally dominating set in X. Let $\sigma \in S(X) = \binom{V}{d-1}$. Let f_σ be the characteristic vector of $S \setminus \sigma$. Define $\alpha_\sigma = \frac{1}{d} f_\sigma$ if $\sigma \subset S$, and $\alpha_\sigma = 0$ otherwise. Then for every vector representation P of X and every $w \in V$ we have

$$\sum_{\sigma \in S(X)} \sum_{v \in V} \alpha_\sigma(v) P_\sigma(v) \cdot P_\sigma(w) = \sum_{\sigma \in \binom{S}{d-1}} \sum_{v \in S \setminus \sigma} \frac{1}{d} P_\sigma(v) \cdot P_\sigma(w).$$

S is totally dominating, therefore there is some $\tau \subset S$ such that $\tau \in X$ but $w\tau \notin X$. Since all the missing faces are of dimension d we must have $|\tau| \geq d$, and by taking a subset if necessary we may assume $|\tau| = d$. For every $\sigma \in \binom{\tau}{d-1}$, let u be the unique vertex in $\tau \setminus \sigma$. Then $wu\sigma = w\tau$ is a missing face of X, thus $P_\sigma(u) \cdot P_\sigma(w) \geq 1$. Hence

$$\sum_{\sigma \in \binom{S}{d-1}} \sum_{v \in S \setminus \sigma} \frac{1}{d} P_\sigma(v) \cdot P_\sigma(w) \geq \sum_{\sigma \in \binom{\tau}{d-1}} \frac{1}{d} = d \cdot \frac{1}{d} = 1.$$

So $\sum_{\sigma \in S(X)} \alpha_\sigma P_\sigma P_\sigma^T \geq 1$, therefore $\{\alpha_\sigma\}_{\sigma \in S(X)}$ is dominating for P. So we have

$$|P| \leq \sum_{\sigma \in S(X)} \sum_{v \in V} \alpha_\sigma(v) = \sum_{\sigma \in \binom{S}{d-1}} \sum_{v \in S \setminus \sigma} \frac{1}{d} = \binom{|S|}{d-1} \frac{|S| - d + 1}{d} = \binom{|S|}{d}.$$

Therefore $\Gamma(X) \leq \binom{\tilde{\gamma}(X)}{d}$. □

Let X be a simplicial complex on vertex set V. For each $i \in J_X$, let X_i be the complex on the same vertex set, whose missing faces are $\mathcal{M}_X(i)$. Note that X_i has full $(i-1)$-dimensional skeleton and $X = \bigcap_{i \in J_X} X_i$.

We want to bound the spectral gaps of X by the spectral gaps of the complexes X_i. We will need the following lemma:

Lemma 4.1. *Let $A_1, \ldots, A_m$ be simplicial complexes on vertex set V, where $|V| = n$. Then for all $k \geq 0$*

$$\mu_k(\cap_{i=1}^m A_i) \geq \sum_{i=1}^m \mu_k(A_i) - (m-1)n.$$

Proof. We argue by induction on m. For $m = 1$ the statement is trivial. Assume $m = 2$. For any complex C on vertex set V containing $A_1 \cap A_2$, denote by $\tilde{L}_k(C)$ the principal submatrix of $[L_k(C)]$ obtained by keeping only the rows and columns corresponding to simplices of $A_1 \cap A_2$.

We have $\lambda_{\min}(\tilde{L}_k(C)) \geq \mu_k(C)$ and $\lambda_{\max}(\tilde{L}_k(C)) \leq \lambda_{\max}(L_k(C)) \leq n$ (by Lemma 2.11).

We will check by Claim 2.10 that

$$[L_k(A_1 \cap A_2)] = \tilde{L}_k(A_1) + \tilde{L}_k(A_2) - \tilde{L}_k(A_1 \cup A_2).$$

Let $\sigma, \tau \in (A_1 \cap A_2)(k)$. We consider the following cases:

- If $\sigma = \tau$ then

$$\begin{aligned}
[L_k(A_1 \cap A_2)]_{\sigma,\sigma} &= \deg_{A_1 \cap A_2}(\sigma) + k + 1 \\
&= |\{\eta \in (A_1 \cap A_2)(k+1) : \sigma \subset \eta\}| + k + 1 \\
&= |\{\eta \in A_1(k+1) : \sigma \subset \eta\}| + |\{\eta \in A_2(k+1) : \sigma \subset \eta\}| \\
&\quad - |\{\eta \in (A_1 \cup A_2)(k+1) : \sigma \subset \eta\}| + k + 1 \\
&= [L_k(A_1)]_{\sigma,\sigma} + [L_k(A_2)]_{\sigma,\sigma} - [L_k(A_1 \cup A_2)]_{\sigma,\sigma}.
\end{aligned}$$

- If $|\sigma \cap \tau| < k$ then

$$[L_k(A_1 \cap A_2)]_{\sigma,\tau} = [L_k(A_1)]_{\sigma,\tau} = [L_k(A_2)]_{\sigma,\tau} = [L_k(A_1 \cup A_2)]_{\sigma,\tau} = 0,$$

  so in particular

$$[L_k(A_1 \cap A_2)]_{\sigma,\tau} = [L_k(A_1)]_{\sigma,\tau} + [L_k(A_2)]_{\sigma,\tau} - [L_k(A_1 \cup A_2)]_{\sigma,\tau}.$$

- If $|\sigma \cap \tau| = k$ and $\sigma \cup \tau \in (A_1 \cap A_2)(k+1)$, then again

$$[L_k(A_1 \cap A_2)]_{\sigma,\tau} = [L_k(A_1)]_{\sigma,\tau} = [L_k(A_2)]_{\sigma,\tau} = [L_k(A_1 \cup A_2)]_{\sigma,\tau} = 0,$$

  so

$$[L_k(A_1 \cap A_2)]_{\sigma,\tau} = [L_k(A_1)]_{\sigma,\tau} + [L_k(A_2)]_{\sigma,\tau} - [L_k(A_1 \cup A_2)]_{\sigma,\tau}.$$

- If $|\sigma \cap \tau| = k$ and $\sigma \cup \tau \notin (A_1 \cap A_2)(k+1)$, then either $\sigma \cup \tau \in (A_1 \cup A_2)(k+1)$ or $\sigma \cup \tau \notin (A_1 \cup A_2)(k+1)$.

  In the first case we can assume without loss of generality that $\sigma \cup \tau \in A_1(k+1)$ but $\sigma \cup \tau \notin A_2(k+1)$. Then

$$\begin{aligned}&[L_k(A_1)]_{\sigma,\tau} + [L_k(A_2)]_{\sigma,\tau} - [L_k(A_1 \cup A_2)]_{\sigma,\tau} \\ &\qquad = 0 + (\sigma : \sigma \cap \tau) \cdot (\tau : \sigma \cap \tau) - 0 = [L_k(A_1 \cap A_2)]_{\sigma,\tau}.\end{aligned}$$

  In the second case we must have $\sigma \cup \tau \notin A_1(k+1)$ and $\sigma \cup \tau \notin A_2(k+1)$, therefore

$$\begin{aligned}&[L_k(A_1 \cap A_2)]_{\sigma,\tau} = [L_k(A_1)]_{\sigma,\tau} \\ &\qquad = [L_k(A_2)]_{\sigma,\tau} = [L_k(A_1 \cup A_2)]_{\sigma,\tau} = (\sigma : \sigma \cap \tau) \cdot (\tau : \sigma \cap \tau),\end{aligned}$$

  so

$$[L_k(A_1 \cap A_2)]_{\sigma,\tau} = [L_k(A_1)]_{\sigma,\tau} + [L_k(A_2)]_{\sigma,\tau} - [L_k(A_1 \cup A_2)]_{\sigma,\tau}.$$

Hence $[L_k(A_1 \cap A_2)] = \tilde{L}_k(A_1) + \tilde{L}_k(A_2) - \tilde{L}_k(A_1 \cup A_2)$, therefore

$$\begin{aligned}\mu_k(A_1 \cap A_2) &\geq \lambda_{\min}(\tilde{L}_k(A_1)) + \lambda_{\min}(\tilde{L}_k(A_2)) - \lambda_{\max}(\tilde{L}_k(A_1 \cup A_2)) \\ &\geq \mu_k(A_1) + \mu_k(A_2) - n.\end{aligned}$$

For $m > 2$ we obtain by the case $m = 2$ and the induction hypothesis:

$$\begin{aligned}\mu_k(\cap_{i=1}^m A_i) &\geq \mu_k(A_1) + \mu_k(\cap_{i=2}^m A_i) - n \\ &\geq \mu_k(A_1) + \left(\sum_{i=2}^m \mu_k(A_i) - (m-2)n\right) - n = \sum_{i=1}^m \mu_k(A_i) - (m-1)n.\end{aligned}$$

□

For $i \in J_X$ let Y_i be the i-dimensional complex on vertex set V with full $(i-1)$-dimensional skeleton whose i-dimensional faces are the sets in $\mathcal{M}_X(i)$. Denote the maximal eigenvalue of $L^+_{i-1}(Y_i)$ by $\lambda^i_{\max}(X)$.

Claim 4.2. *For all $i \in J_X$*

$$\mu_{i-1}(X_i) = n - \lambda^i_{\max}(X).$$

Proof. By Lemma 2.12 we have $L^+_{i-1}(Y_i) = nI - L_{i-1}(X_i)$. So every eigenvector of $L_{i-1}(X_i)$ with eigenvalue λ is an eigenvector of of $L^+_{i-1}(Y_i)$ with eigenvalue $n-\lambda$. In particular, $n - \mu_{i-1}(X_i)$ is the largest eigenvalue of $L^+_{i-1}(Y_i)$. □

Claim 4.3. *For $k \geq 0$,*

$$\mu_k(X) \geq n - \sum_{i \in J_X} \binom{k+1}{i} \lambda^i_{\max}(X).$$

Proof. By Lemma 4.1 we obtain

$$\mu_k(X) = \mu_k(\cap_{i \in J_X} X_i) \geq \sum_{i \in J_X} \mu_k(X_i) - (|J_X| - 1)n.$$

Applying Claim 3.7 to each of the complexes X_i (note that $h(X_i) = i$ and X_i has full $(i-1)$-dimensional skeleton) we get

$$\mu_k(X) \geq \sum_{i \in J_X} \left[\binom{k+1}{i} \mu_{i-1}(X_i) - \left(\binom{k+1}{i} - 1 \right) n \right] - (|J_X| - 1)n.$$

Then by Claim 4.2

$$\begin{aligned}\mu_k(X) &\geq \sum_{i \in J_X} \left[\binom{k+1}{i} (n - \lambda^i_{\max}(X)) - \left(\binom{k+1}{i} - 1 \right) n \right] - (|J_X| - 1)n \\ &= n - \sum_{i \in J_X} \binom{k+1}{i} \lambda^i_{\max}(X).\end{aligned}$$

□

Claim 4.4.

$$\sum_{i \in J_X} \binom{\eta(X)}{i} \lambda^i_{\max}(X) \geq n.$$

Proof. Let k be the integer such that

$$\sum_{i \in J_X} \binom{k-1}{i} \lambda^i_{\max}(X) < n \leq \sum_{i \in J_X} \binom{k}{i} \lambda^i_{\max}(X).$$

Let $j \leq k-2$. By Claim 4.3,

$$\mu_j(X) \geq n - \sum_{i \in J_X} \binom{j+1}{i} \lambda_{\max}^i(X) > 0,$$

therefore by Corollary 2.17 we have $\tilde{H}_j(X;\mathbb{R}) = 0$. So $\eta(X) \geq k$, thus

$$\sum_{i \in J_X} \binom{\eta(X)}{i} \lambda_{\max}^i(X) \geq \sum_{i \in J_X} \binom{k}{i} \lambda_{\max}^i(X) \geq n.$$

□

Claim 4.5. *Let $i \in J_X$. Then for $\phi \in C^{i-1}(Y_i)$,*

$$\left\langle L_{i-1}^{+}(Y_i)\phi, \phi \right\rangle \leq \sum_{\sigma \in \binom{V}{i-1}} \sum_{vw \in \mathrm{lk}(Y_i,\sigma)} (\phi(v\sigma) - \phi(w\sigma))^2.$$

Proof.

$$\begin{aligned}
&\sum_{\sigma \in Y_i(i-2)} \sum_{vw \in \mathrm{lk}(Y_i,\sigma)} (\phi(v\sigma) - \phi(w\sigma))^2 \\
&\quad = \sum_{\sigma \in Y_i(i-2)} \sum_{v \in \mathrm{lk}(Y_i,\sigma)} \deg_{Y_i}(v\sigma)\phi(v\sigma)^2 - 2 \sum_{\sigma \in Y_i(i-2)} \sum_{vw \in \mathrm{lk}(Y_i,\sigma)} \phi(v\sigma)\phi(w\sigma) \\
&\quad = i \cdot \sum_{\eta \in Y_i(i-1)} \deg_{Y_i}(\eta)\phi(\eta)^2 - 2 \sum_{\sigma \in Y_i(i-2)} \sum_{vw \in \mathrm{lk}(Y_i,\sigma)} \phi(v\sigma)\phi(w\sigma).
\end{aligned}$$

By Claim 3.8 we have

$$\begin{aligned}
\left\langle L_{i-1}^{+}(Y_i)\phi, \phi \right\rangle &= \|d_{i-1}\phi\|^2 \\
&= \sum_{\eta \in Y_i(i-1)} \deg_{Y_i}(\eta)\phi(\eta)^2 - 2 \sum_{\sigma \in Y_i(i-2)} \sum_{vw \in \mathrm{lk}(Y_i,\sigma)} \phi(v\sigma)\phi(w\sigma).
\end{aligned}$$

Hence

$$\begin{aligned}
\left\langle L_{i-1}^{+}(Y_i)\phi, \phi \right\rangle &= \sum_{\sigma \in Y_i(i-2)} \sum_{vw \in \mathrm{lk}(Y_i,\sigma)} (\phi(v\sigma) - \phi(w\sigma))^2 \\
&\quad - (i-1) \cdot \sum_{\eta \in Y_i(i-1)} \deg_{Y_i}(\eta)\phi(\eta)^2 \leq \sum_{\sigma \in Y_i(i-2)} \sum_{vw \in \mathrm{lk}(Y_i,\sigma)} (\phi(v\sigma) - \phi(w\sigma))^2.
\end{aligned}$$

Y_i has full $(i-1)$-dimensional skeleton, therefore $Y_i(i-2) = \binom{V}{i-1}$. Thus

$$\left\langle L_{i-1}^{+}(Y_i)\phi, \phi \right\rangle \leq \sum_{\sigma \in \binom{V}{i-1}} \sum_{vw \in \mathrm{lk}(Y_i,\sigma)} (\phi(v\sigma) - \phi(w\sigma))^2$$

□

Claim 4.6. *Let P be a vector representation of X. Then for all $i \in J_X$*

$$\lambda^i_{\max}(X) \leq i \cdot \max_{\sigma \in \binom{V}{i-1},\, v \in V} \left(P_\sigma(v) \cdot \sum_{w \in V} P_\sigma(w) \right).$$

Proof. Let $\phi \in C^{i-1}(Y_i)$. For $\sigma \in Y_i(i-2) = \binom{V}{i-1}$ and $v, w \in V \setminus \sigma$, $v \neq w$, we have, by the definition of P, $P_\sigma(v) \cdot P_\sigma(w) \geq 1$ if $vw \in \mathrm{lk}(Y_i, \sigma)$, and $P_\sigma(v) \cdot P_\sigma(w) \geq 0$ otherwise. Therefore we obtain

$$\begin{aligned}
&\sum_{\sigma \in \binom{V}{i-1}} \sum_{vw \in \mathrm{lk}(Y_i,\sigma)} (\phi(v\sigma) - \phi(w\sigma))^2 \\
&\leq \frac{1}{2} \sum_{\sigma \in \binom{V}{i-1}} \sum_{v,w \in V \setminus \sigma} (\phi(v\sigma) - \phi(w\sigma))^2 P_\sigma(v) \cdot P_\sigma(w) \\
&= \sum_{\sigma \in \binom{V}{i-1}} \sum_{v \in V \setminus \sigma} \phi(v\sigma)^2 P_\sigma(v) \cdot \sum_{w \in V \setminus \sigma} P_\sigma(w) - \sum_{\sigma \in \binom{V}{i-1}} \left\| \sum_{v \in V \setminus \sigma} \phi(v\sigma) P_\sigma(v) \right\|^2 \\
&\leq \sum_{\sigma \in \binom{V}{i-1}} \sum_{v \in V \setminus \sigma} \phi(v\sigma)^2 P_\sigma(v) \cdot \sum_{w \in V \setminus \sigma} P_\sigma(w) \\
&\leq \left(\sum_{\sigma \in \binom{V}{i-1}} \sum_{v \in V \setminus \sigma} \phi(v\sigma)^2 \right) \cdot \max_{\sigma \in \binom{V}{i-1},\, v \in V \setminus \sigma} P_\sigma(v) \cdot \sum_{w \in V \setminus \sigma} P_\sigma(w). \qquad (4.1)
\end{aligned}$$

Since Y_i has full $(i-1)$-dimensional skeleton, we have

$$\sum_{\sigma \in \binom{V}{i-1}} \sum_{v \in V \setminus \sigma} \phi(v\sigma)^2 = \sum_{\sigma \in Y_i(i-2)} \sum_{v \in \mathrm{lk}(Y_i,\sigma)} \phi(v\sigma)^2 = i \sum_{\eta \in Y_i(i-1)} \phi(\eta)^2 = i \left\| \phi \right\|^2. \qquad (4.2)$$

Combining (4.1),(4.2) and Claim 4.5 we obtain

$$\begin{aligned}
\left\langle L^+_{i-1}(Y_i)\, \phi, \phi \right\rangle &\leq \sum_{\sigma \in \binom{V}{i-1}} \sum_{vw \in \mathrm{lk}(Y_i,\sigma)} (\phi(v\sigma) - \phi(w\sigma))^2 \\
&\leq i \left\| \phi \right\|^2 \cdot \max_{\sigma \in \binom{V}{i-1},\, v \in V \setminus \sigma} P_\sigma(v) \cdot \sum_{w \in V \setminus \sigma} P_\sigma(w) \\
&\leq i \left\| \phi \right\|^2 \cdot \max_{\sigma \in \binom{V}{i-1},\, v \in V} P_\sigma(v) \cdot \sum_{w \in V} P_\sigma(w).
\end{aligned}$$

Thus

$$\lambda^i_{\max}(X) = \max_{0 \neq \phi \in C^{i-1}(Y_i)} \frac{\left\langle L^+_{i-1} \phi, \phi \right\rangle}{\|\phi\|^2} \leq i \cdot \max_{\sigma \in \binom{V}{i-1}, v \in V} \left(P_\sigma(v) \cdot \sum_{w \in V} P_\sigma(w) \right).$$

□

Lemma 4.7. *Let P be a vector representation of X. Then*

$$|P| = \max \{ \alpha \cdot \boldsymbol{1} : \quad \alpha \geq 0,\, \alpha P_\sigma P_\sigma^T \leq \boldsymbol{1} \quad \forall \sigma \in S(X) \} .$$

Proof. Let $\sigma_1, \ldots, \sigma_m$ be all the sets in $S(X)$. For each $i \in [m]$ let $A_i = P_{\sigma_i} P_{\sigma_i}^T \in \mathbb{R}^{|V| \times |V|}$. Note that $A_i = A_i^T$. Define the matrix

$$A = (A_1 | A_2 | \cdots | A_m)^T \in \mathbb{R}^{(m|V|) \times |V|}.$$

Let $x \in \mathbb{R}^{m|V|}$. Write $x = (\alpha_{\sigma_1} | \alpha_{\sigma_2} | \cdots | \alpha_{\sigma_m})$, where $\alpha_{\sigma_i} \in \mathbb{R}^{|V|}$ for each $i \in [m]$. We have

$$xA = \sum_{i=1}^{m} \alpha_{\sigma_i} A_i = \sum_{\sigma \in S(X)} \alpha_\sigma P_\sigma P_\sigma^T ,$$

therefore

$$\begin{aligned} |P| &= \min \{ \sum_{\alpha \in S(X)} \alpha_\sigma \cdot \mathbf{1} : \, \alpha_\sigma \geq 0 \; \forall \sigma \in S(X), \sum_{\sigma \in S(X)} \alpha_\sigma P_\sigma P_\sigma^T \geq \mathbf{1} \} \\ &= \min \{ x \cdot \mathbf{1} : \, x \geq 0,\, xA \geq \mathbf{1} \} . \end{aligned}$$

By linear programming duality

$$|P| = \max \{ y \cdot \mathbf{1} : y \geq 0,\, yA^T \leq \mathbf{1} \} .$$

But $yA^T = (yA_1 | yA_2 | \cdots | yA_m)$, so $yA^T \leq \mathbf{1}$ if and only if $yA_i \leq \mathbf{1}$ for all $i \in [m]$. Therefore

$$|P| = \max \{ y \cdot \mathbf{1} : y \geq 0,\, yP_\sigma P_\sigma^T \leq \mathbf{1} \quad \forall \sigma \in S(X) \} .$$

□

Let $\mathbb{Z}_+$ denote the positive integers, and $\mathbb{Q}_+$ the positive rationals. Let $a \in \mathbb{Z}_+^V$ and

$$V_a = \{ (v, i) : \, v \in V,\, 1 \leq i \leq a(v) \} .$$

Define the projection $\pi : V_a \to V$ by $\pi((v, i)) = v$, and let

$$X_a = \pi^{-1}(X) = \{ \sigma \subset V_a : \, \pi(\sigma) \in X \} .$$

The missing faces of X_a are the sets $\sigma \subset V_a$ such that $|\pi(\sigma)| = |\sigma|$ and $\pi(\sigma)$ is a missing face of X.

Claim 4.8. *For all $k \geq -1$*

$$\tilde{H}^k (X; \mathbb{R}) = \tilde{H}^k (X_a; \mathbb{R}) .$$

Proof. We argue by induction on $N = |V_a|$. If $N = n$ then π is an isomorphism of simplicial complexes, and the claim is trivial.

Assume $N > n$. Then there must be some $v \in V$ such that $|\pi^{-1}(v)| = t \geq 2$. Note that $V_a \setminus \{(v,t)\} = V_{a'}$, where $a'(u) = a(u)$ for all $u \in V$ other than v, and $a'(v) = a(v) - 1$. Therefore $X_a[V_a \setminus \{(v,t)\}] = X_a[V_{a'}] = X_{a'}$. So by the induction hypothesis we have for all $k \geq -1$

$$\tilde{H}^k\left(X_a[V_a \setminus \{(v,t)\}];\mathbb{R}\right) = \tilde{H}^k\left(X_{a'};\mathbb{R}\right) = \tilde{H}^k\left(X;\mathbb{R}\right).$$

In addition, each maximal face σ of $\mathrm{lk}(X_a,(v,t))$ contains the vertex $(v,1)$ (since for every $\tau \in X_a$, if $(v,t) \in \tau$ then $\pi(\tau \cup \{(v,1)\}) = \pi(\tau) \in X$, hence $\tau \cup \{(v,1)\} \in X_a$). So $\mathrm{lk}(X_a,(v,t))$ is a cone over the vertex $(v,1)$, therefore $\tilde{H}_j\left(\mathrm{lk}(X_a,(v,t));\mathbb{R}\right) = 0$ for all $j \geq -1$.

By Proposition 2.8 we obtain for all $k \geq -1$

$$\tilde{H}^k\left(X_a;\mathbb{R}\right) = \tilde{H}^k\left(X_a[V_a \setminus \{(v,t)\}];\mathbb{R}\right) = \tilde{H}^k\left(X;\mathbb{R}\right).$$

$\square$

Remark. It can be shown that π induces a homotopy equivalence between X_a and X (see [16, Lemma 2.14]).

Theorem 1.6.

$$\sum_{r \in J_X} r\binom{\eta(X)}{r} \geq \Gamma(X).$$

Proof. Let $P = \{P_\sigma\}_{\sigma \in S(X)}$ be a vector representation of X. Let $\alpha \in \mathbb{Q}_+^V$ such that $\alpha P_\sigma P_\sigma^T \leq \mathbf{1}$ for all $\sigma \in S(X)$. Write $\alpha = a/k$ where $k \in \mathbb{Z}_+$ and $a \in \mathbb{Z}_+^V$. Denote $N = |V_a| = \sum_{v \in V} a(v)$. For $\sigma \in S(X_a)$ and $(v,j) \in V_a$ define

$$Q_\sigma((v,j)) = \begin{cases} P_{\pi(\sigma)}(v) & \text{if } |\pi(\sigma)| = |\sigma|, \\ 0 & \text{otherwise.} \end{cases}$$

$Q = \{\, Q_\sigma : \sigma \in S(X_a) \,\}$ is a vector representation of X_a: Let $\sigma \in S(X_a)$ of size $r-1$, and let $\tilde{v} = (v,i), \tilde{u} = (u,j) \in V_a$ such that $\tilde{u}\tilde{v}\sigma \in \mathcal{M}_{X_a}(r)$. Then $\pi(\tilde{u}\tilde{v}\sigma) = uv\pi(\sigma) \in \mathcal{M}_X(r)$. In particular $|\pi(\sigma)| = |\sigma|$, therefore, since P is a representation of X,

$$Q_\sigma(\tilde{v}) \cdot Q_\sigma(\tilde{u}) = P_{\pi(\sigma)}(v) \cdot P_{\pi(\sigma)}(u) \geq 1.$$

Let $r \in J_X$. By Claim 4.6

$$\lambda^r_{\max}(X_a) \le r \cdot \max_{\sigma \in \binom{V_a}{r-1}, (v,j) \in V_a} \left(Q_\sigma((v,j)) \cdot \sum_{(w,k) \in V_a} Q_\sigma((w,k)) \right)$$
$$= r \cdot \max_{\tau \in \binom{V}{r-1}, v \in V} \left(P_\tau(v) \cdot \sum_{w \in V} a(w) P_\tau(w) \right) \le r \cdot k.$$

By Claim 4.4 we obtain

$$\sum_{r \in J_{X_a}} \binom{\eta(X_a)}{r} r \cdot k \ge \sum_{r \in J_{X_a}} \binom{\eta(X_a)}{r} \lambda^r_{\max}(X_a) \ge N.$$

Therefore

$$\alpha \cdot \mathbf{1} = \frac{1}{k} \sum_{v \in V} a(v) = \frac{N}{k} \le \sum_{r \in J_{X_a}} r \binom{\eta(X_a)}{r} = \sum_{r \in J_X} r \binom{\eta(X)}{r},$$

the last equality following from Claim 4.8. Thus by Lemma 4.7

$$|P| = \max \{ \alpha \cdot \mathbf{1} : \alpha \ge 0, \alpha P_\sigma P_\sigma^T \le \mathbf{1} \, \forall \sigma \in S(X) \}$$
$$= \sup \{ \alpha \cdot \mathbf{1} : \alpha \in \mathbb{Q}^V_+, \alpha P_\sigma P_\sigma^T \le \mathbf{1} \, \forall \sigma \in S(X) \} \le \sum_{r \in J_X} r \binom{\eta(X)}{r},$$

therefore $\Gamma(X) \le \sum_{r \in J_X} r \binom{\eta(X)}{r}$. □

For the proof of Theorem 1.7 we need the following Hall-type condition for the existence of colorful simplices, which appears in [4, 17], and more explicitly in [18]:

Proposition 4.9. *Let Z be a simplicial complex on vertex set $W = \uplus_{i=1}^m W_i$. If for all $\emptyset \ne I \subset [m]$*

$$\eta(Z[\uplus_{i \in I} W_i]) \ge |I|$$

then Z contains a colorful simplex.

Theorem 1.7. *If for every $\emptyset \ne I \subset [m]$*

$$\Gamma(X[\uplus_{i \in I} V_i]) > \sum_{r \in J_{X[\uplus_{i \in I} V_i]}} r \binom{|I|-1}{r},$$

then X has a colorful simplex.

Proof. Let $\emptyset \ne I \subset [m]$. By Theorem 1.6 we have

$$\sum_{r \in J_{X[\uplus_{i \in I} V_i]}} r \binom{\eta(X[\uplus_{i \in I} V_i]))}{r} \ge \Gamma(X[\cup_{i \in I} V_i]) > \sum_{r \in J_{X[\uplus_{i \in I} V_i]}} r \binom{|I|-1}{r},$$

therefore

$$\eta(X[\dot\cup_{i\in I}V_i])) > |I|-1.$$

Thus by Proposition 4.9 X has a colorful simplex. □

4.2 Colorful sets in general position

In this section we apply our results to prove Theorems 1.10 and 1.11.

First we prove Lemma 1.9:

Lemma 1.9. $\varphi^*_M(S) \geq \varphi_M(S)$.

Proof. Let f be the characteristic function of a set $S' \subset S$ in general position. Let $1 \leq k \leq d$, and let F be a flat of M of rank k and $\sigma \subset F \cap S$ of size $k-1$. Then

$$\sum_{\substack{v\in S,\\ \mathrm{cl}(v\sigma)=F}} f(v) = \big|\{\, v \in S' : \mathrm{cl}(v\sigma) = F \,\}\big| \leq |S' \cap F| \leq k \leq d,$$

so f is in fractional general position. Therefore

$$\varphi^*_M(S) \geq \varphi_M(S).$$

□

Let M be a matroid of rank $d+1$ on vertex set V. Let $\tilde{M}$ be the simplicial complex on vertex set V whose simplices are the subsets $S \subset V$ in general position with respect to M. The missing faces of $\tilde{M}$ are the dependent sets $S \subset V$ with $|S| \leq d+1$ such that any $|S|-1$ points in S are independent in M.

Claim 4.10. *For $U \subset V$,*

$$\varphi^*_M(U) \leq d \cdot \Gamma(\tilde{M}[U]).$$

Proof. We construct a vector representation of the complex $\tilde{M}[U]$. Let $1 \leq r \leq d$ and let $\mathcal{F}_r$ be the set of flats of M of rank r.

Let $\sigma \in S(\tilde{M}[U])$ with $|\sigma| = r-1$, and let $v \in U$. Define $P_\sigma(v) \in \mathbb{R}^{\mathcal{F}_r}$ by

$$P_\sigma(v)(F) = \begin{cases} 1 & \text{if } \mathrm{cl}(v\sigma) = F, \\ 0 & \text{otherwise.} \end{cases}$$

For $v, w \in U$, if $vw\sigma$ is a missing face of $\tilde{M}[U]$ of dimension r then $vw\sigma$ lies in a flat of rank r, which is spanned by any r points in $vw\sigma$. In particular $\mathrm{cl}(v\sigma) = \mathrm{cl}(w\sigma) \in \mathcal{F}_r$, therefore

$$P_\sigma(v) \cdot P_\sigma(w) = 1.$$

Hence P is a vector representation of $\tilde{M}[U]$.

Let $f : U \to \mathbb{R}_{\geq 0}$ be a function in fractional general position with $\sum_{v\in U} f(v) = \varphi^*_M(U)$. Define $\alpha \in \mathbb{R}^U$ by $\alpha(v) = f(v)/d$.

Let $w \in U$, and let $F = \mathrm{cl}(w\sigma)$. If $F \notin \mathcal{F}_r$ then $P_\sigma(w) = 0$, therefore $\sum_{v\in U} \alpha(v)P_\sigma(v)\cdot P_\sigma(w) = 0 \leq 1$. If $F \in \mathcal{F}_r$ then

$$\sum_{v\in U} \alpha(v)P_\sigma(v)\cdot P_\sigma(w) = \sum_{\substack{v\in U,\\ \mathrm{cl}(v\sigma)=F}} \alpha(v) = \frac{1}{d}\sum_{\substack{v\in U,\\ \mathrm{cl}(v\sigma)=F}} f(v) \leq 1.$$

So $\alpha P_\sigma P_\sigma^T \leq \mathbf{1}$ for each $\sigma \in S(\tilde{M}[U])$, therefore by Lemma 4.7

$$\Gamma(\tilde{M}[U]) \geq |P| \geq \alpha \cdot \mathbf{1} = \frac{\varphi^*_M(U)}{d}.$$

□

Theorem 1.10. *If for every $\emptyset \neq I \subset [m]$*

$$\varphi^*_M(\uplus_{i\in I} V_i) > d\sum_{r=1}^{d} r\binom{|I|-1}{r},$$

then V contains a colorful subset in general position.

Proof. Let $\emptyset \neq I \subset [m]$. By Claim 4.10

$$\Gamma(\tilde{M}[\uplus_{i\in I} V_i]) \geq \frac{\varphi^*_M(\uplus_{i\in I} V_i)}{d} > \sum_{r=1}^{d} r\binom{|I|-1}{r}.$$

Thus by Theorem 1.7 there is a colorful simplex of $\tilde{M}$, i.e. a colorful subset of V in general position. □

Theorem 1.11. *If for every $\emptyset \neq I \subset [m]$*

$$\varphi_M(\uplus_{i\in I} V_i) > \begin{cases} |I|-1 & \text{if } |I| \leq d+1,\\ d\sum_{r=1}^{d} r\binom{|I|-1}{r} & \text{if } |I| \geq d+2,\end{cases}$$

then V contains a colorful subset in general position.

Proof. Let $\emptyset \neq I \subset [m]$. Assume $|I| \leq d+1$. The d-dimensional skeleton of $\tilde{M}[\uplus_{i\in I} V_i]$ is $M[\uplus_{i\in I} V_i]$, therefore for all $0 \leq k \leq d-1$

$$\tilde{H}^k\left(\tilde{M}[\uplus_{i\in I} V_i];\mathbb{R}\right) = \tilde{H}^k\left(M[\uplus_{i\in I} V_i];\mathbb{R}\right).$$

$M[\uplus_{i\in I} V_i]$ are matroids, therefore by Theorem 2.25 $\tilde{H}^k\left(M[\uplus_{i\in I} V_i];\mathbb{R}\right) = 0$ for $0 \leq k \leq \rho(\uplus_{i\in I} V_i) - 2$. Hence $\eta(\tilde{M}[\uplus_{i\in I} V_i]) \geq \rho(\uplus_{i\in I} V_i)$. But

$$\rho(\uplus_{i\in I} V_i) = \min\left\{\, d+1, \varphi_M(\uplus_{i\in I} V_i)\,\right\},$$

so if $\varphi_M(\uplus_{i\in I}V_i) > |I|-1$, then $\eta(\tilde{M}[\uplus_{i\in I}V_i]) > |I|-1$.

Assume now that $|I| \geq d+2$. If $\varphi_M(\uplus_{i\in I}V_i) > d\sum_{r=1}^{d} r\binom{|I|-1}{r}$ then, by Lemma 1.9, $\varphi^*_M(\uplus_{i\in I}V_i) > d\sum_{r=1}^{d} r\binom{|I|-1}{r}$, and therefore by Theorem 1.6 and Claim 4.10

$$\sum_{r=1}^{d} r\binom{\eta(\tilde{M}[\uplus_{i\in I}V_i])}{r} \geq \Gamma(\tilde{M}[\uplus_{i\in I}V_i]) \geq \frac{\varphi^*_M(\uplus_{i\in I}V_i)}{d} > \sum_{r=1}^{d} r\binom{|I|-1}{r},$$

so $\eta(\tilde{M}[\uplus_{i\in I}V_i]) > |I|-1$. Therefore by Proposition 4.9 there is a colorful subset of V in general position. □

The following claim shows that Theorem 1.11 implies Theorem 1.8:

Claim 4.11. *Let $k \geq d$. Then*

$$\sum_{r=1}^{d} r\binom{k-1}{r} \leq \binom{2k-2}{d}.$$

Proof. By Vandermonde's identity

$$\binom{2k-2}{d} = \sum_{r=0}^{d}\binom{k-1}{d-r}\binom{k-1}{r}.$$

Thus

$$\begin{aligned}
&\binom{2k-2}{d} - \sum_{r=1}^{d} r\binom{k-1}{r} = \sum_{r=0}^{d}\left(\binom{k-1}{d-r} - r\right)\binom{k-1}{r} \\
&= \sum_{r=0}^{d-1}\left(\binom{k-1}{d-r} - r\right)\binom{k-1}{r} - (d-1)\binom{k-1}{d} \\
&\geq \sum_{r=0}^{d-1}\left(\binom{k-1}{d-r} - r\right)\binom{k-1}{r} - d\binom{k-1}{d} \\
&= \sum_{r=0}^{d-1}\left(\binom{k-1}{d-r}\binom{k-1}{r} - r\binom{k-1}{r} - \binom{k-1}{d}\right).
\end{aligned}$$

Therefore it is enough to show that for each $0 \leq r \leq d-1$

$$\binom{k-1}{d-r}\binom{k-1}{r} \geq r\binom{k-1}{r} + \binom{k-1}{d}. \tag{4.3}$$

For $r = 0$ the inequality trivially holds, therefore we may assume $r \geq 1$. Let

$$\mathcal{F} = \{\, (A,B) : A, B \subset [k-1], |A| = d-r, |B| = r \,\}$$

and

$$\mathcal{H} = \{\, (c,C) : C \subset [k-1], |C| = r, c \in C \,\} \uplus \{\, D : D \subset [k-1], |D| = d \,\}.$$

We have $|\mathcal{F}| = \binom{k-1}{d-r}\binom{k-1}{r}$ and $|\mathcal{H}| = r\binom{k-1}{r} + \binom{k-1}{d}$. We will show that there exists an injection $\phi : \mathcal{H} \to \mathcal{F}$, and thus inequality (4.3) holds.

For $(c, C) \in \mathcal{H}$, take $d - r - 1$ elements in $[k-1] \setminus C$ (there are such elements since $|[k-1] \setminus C| = k - 1 - r \geq d - r - 1$.) Define A to be the union of these elements and the singleton $\{c\}$. Define $\phi(c, C) = (A, C)$.

For $D \in \mathcal{H}$, let A, B be any partition of D with sizes $|A| = d - r$, $|B| = r$, and define $\phi(D) = (A, B)$.

Now we show that ϕ is injective: Let $(A, B) \in \phi(\mathcal{H})$. If $A \cap B = \emptyset$ then $\phi^{-1}(A, B) = A \cup B$. If the sets intersect, let c be the only element of $A \cap B$, and then $\phi^{-1}(A, B) = (c, B)$. $\square$

Chapter 5

Minimal degrees and spectral gaps

In this chapter we prove Theorem 1.13 and its corollary Proposition 1.12. We also present examples showing that the inequalities in Theorem 1.13 are tight. In the case of clique complexes we characterize all the extremal examples.

Theorem 1.13. *Let X be a simplicial complex with $h(X) = d$ on vertex set V, where $|V| = n$. Let $k \geq 0$ and let $\delta_k(X)$ denote the minimal degree of a simplex in $X(k)$. Then*

$$\mu_k(X) \geq (d+1)(\delta_k(X) + k + 1) - dn.$$

Proof. By Claim 2.10, for $\sigma \in X(k)$

$$[L_k(X)]_{\sigma,\sigma} = \deg_X(\sigma) + k + 1,$$

and

$$\begin{aligned}\sum_{\substack{\eta\in X(k),\\ \eta\neq\sigma}} \left|[L_k(X)]_{\sigma,\eta}\right| &= |\{\, \eta \in X(k) : |\sigma\cap\eta| = k,\, \sigma\cup\eta \notin X(k+1)\,\}| \\ &= \sum_{\tau\in\sigma(k-1)} |\{\, v \in V\setminus\sigma :\, v \in \mathrm{lk}(X,\tau),\, v \notin \mathrm{lk}(X,\sigma)\,\}| \\ &= \sum_{\tau\in\sigma(k-1)} (\deg_X(\tau) - 1 - \deg_X(\sigma)) \\ &= \sum_{\tau\in\sigma(k-1)} \deg_X(\tau) - (k+1)(\deg_X(\sigma)+1).\end{aligned}$$

By Geršgorin's Theorem (Theorem 2.4) we obtain

$$\begin{aligned}\mu_k(X) &\geq \min_{\sigma\in X(k)}\left([L_k(X)]_{\sigma,\sigma} - \sum_{\eta\neq\sigma}\left|[L_k(X)]_{\sigma,\eta}\right|\right)\\ &= \min_{\sigma\in X(k)}\left(\deg_X(\sigma)+k+1-\sum_{\tau\in\sigma(k-1)}\deg_X(\tau)+(k+1)(\deg_X(\sigma)+1)\right)\\ &= \min_{\sigma\in X(k)}\left((k+2)\deg_X(\sigma)+2(k+1)-\sum_{\tau\in\sigma(k-1)}\deg_X(\tau)\right)\end{aligned} \tag{5.1}$$

Recall that by Lemma 3.4 we have

$$\sum_{\tau\in\sigma(k-1)}\deg_X(\tau)-(k-d+1)\deg_X(\sigma)\leq dn-(d-1)(k+1). \tag{5.2}$$

Combining (5.1) and (5.2) we obtain

$$\begin{aligned}\mu_k(X) &\geq \min_{\sigma\in X(k)}\left((d+1)(\deg_X(\sigma)+k+1)-dn\right)\\ &= (d+1)(\delta_k(X)+k+1)-dn,\end{aligned} \tag{5.3}$$

as wanted. □

We obtain Proposition 1.12 as a corollary:

Proposition 1.12. *Let X be a simplicial complex on n vertices with $h(X)=d$. Then $\tilde{H}^k(X;\mathbb{R})=0$ for all $k>\frac{d}{d+1}n-1$.*

Proof. Let $k>\frac{d}{d+1}n-1$. By Theorem 1.13 we have

$$\mu_k(X)\geq(d+1)(k+1)-dn>(d+1)\frac{d}{d+1}n-dn=0.$$

So by the simplicial Hodge theorem (Corollary 2.17), $\tilde{H}^k(X;\mathbb{R})=0$. □

The following example shows that the inequality in Theorem 1.13 is tight:

Example. Let $X_1,\dots,X_t$ be simplicial complexes isomorphic to $\Delta_d^{(d-1)}$, i.e. the boundary of the d-dimensional simplex. Let Y be isomorphic to Δ_{r-1}, the complete complex on r vertices. Let $X=X_1*X_2*\cdots*X_t*Y$. Let $n=(d+1)t+r$, the number of vertices of X. Note that all the missing faces of X are of dimension d, and X is of dimension $dt+r-1=n-t-1$. By Theorem 2.13

$$\mu_k(X)=\min\left\{\mu_{i_1}(X_1)+\cdots+\mu_{i_t}(X_t)+\mu_j(Y):\ \substack{-1\leq i_1,\dots,i_t\leq d-1,\\ -1\leq j\leq r-1,\\ i_1+\cdots+i_t+j=k-t}\right\}.$$

By Claim 2.18 we have

$$\mu_j(X_i) = \begin{cases} d+1 & \text{if } -1 \le j \le d-2, \\ 0 & \text{if } \quad\quad j = d-1, \end{cases}$$

and $\mu_j(Y) = r$ for all $-1 \le j \le r-1$. Therefore $\mu_k(X) = n - m(d+1)$, where m is the maximal number of indices in $i_1, \dots, i_t$ that can be chosen to be equal to $d-1$. That is, m is the maximal integer between 0 and t such that there exist $-1 \le i_1, \dots, i_{t-m} \le d-1$ and $-1 \le j \le r-1$ satisfying

$$m(d-1) + i_1 + \cdots + i_{t-m} + j = k - t.$$

We obtain

$$m = \begin{cases} \lfloor \frac{k+1}{d} \rfloor & \text{if } -1 \le k \le dt-1, \\ t & \text{if } \quad dt \le k \le dt+r-1. \end{cases}$$

So

$$\mu_k(X) = \begin{cases} (d+1)\left(t - \lfloor \frac{k+1}{d} \rfloor\right) + r & \text{if } -1 \le k \le dt-1, \\ r & \text{if } \quad dt \le k \le dt+r-1. \end{cases}$$

Let's consider now the degrees of simplices in X: For each $\sigma \in X(k)$, $\deg_X(\sigma) = n - (k+1) - s(\sigma)$, where

$$s(\sigma) = |\{ i \in [t] : |\sigma \cap V(X_i)| = d \}|.$$

So the minimal degree of a simplex in $X(k)$ is

$$\delta_k(X) = \begin{cases} n - (k+1) - \lfloor \frac{k+1}{d} \rfloor & \text{if } -1 \le k \le dt-1, \\ n - (k+1) - t & \text{if } \quad dt \le k \le dt+r-1. \end{cases}$$

Therefore $\delta_k(X) = n - (k+1) - m$, thus

$$(d+1)(\delta_k(X) + k + 1) - dn = n - (d+1)m = \mu_k(X).$$

Hence X achieves equality in the inequality of Theorem 1.13 in all dimensions.

Now we look at the case $d = 1$. If X is a clique complex with n vertices, then by Theorem 1.13 we have $\mu_k(X) \ge 2(k+1+\delta_k(X)) - n$ for all $k \ge 0$, and we found a family of examples achieving equality in all dimensions. In particular, at the top dimension we get $\mu_k(X) = 2(k+1) - n$. The next proposition shows that these are the only extremal examples:

Proposition 5.1. *Let X be a clique complex on a vertex set V of size n, such that*

$\mu_k(X) = 2(k+1) - n$ *for some* $k \geq 0$. *Then*

$$X \cong \left(\Delta_1^{(0)}\right)^{*(n-k-1)} * \Delta_{2(k+1)-n-1},$$

Proof. By (5.1) and (5.3) we have

$$\begin{aligned} 2(k+1) - n &= \mu_k(X) \\ &\geq \min_{\sigma \in X(k)} \left((k+2)\deg_X(\sigma) + 2(k+1) - \sum_{\tau \in \sigma(k-1)} \deg_X(\tau) \right) \\ &\geq \min_{\sigma \in X(k)} \left(2(\deg_X(\sigma) + k + 1) - n\right) \\ &\geq 2(k+1) - n. \end{aligned}$$

So all the inequalities are actually equalities, therefore there exists some $\sigma \in X(k)$ such that

$$\sum_{\tau \in \sigma(k-1)} \deg_X(\tau) - k \deg_X(\sigma) = n$$

and $\deg_X(\sigma) = 0$. By Lemma 3.3 we have

$$\begin{aligned} n = \sum_{\tau \in \sigma(k-1)} \deg_X(\tau) = k + 1 + (k+1)\deg_X(\sigma) + |\{v \in V \setminus \sigma :\, m\,(v\sigma) = 2\}| \\ = |\sigma| + |\{v \in V \setminus \sigma :\, m\,(v\sigma) = 2\}|. \end{aligned}$$

So $|\{v \in V \setminus \sigma :\, m\,(v\sigma) = 2\}| = n - |\sigma| = |V \setminus \sigma|$, hence for every vertex $v \in V \setminus \sigma$ there is exactly one vertex $u \in \sigma$ such that $uv \notin X(1)$.

Let $t = n - k - 1$. Denote the vertices in $V \setminus \sigma$ by $v_1, \ldots, v_t$. For each v_i denote by u_i the unique vertex in σ such that $u_i v_i \notin X(1)$.

Let $U = \sigma \setminus \{\, u_1, \ldots, u_t \,\}$ and $r = |U|$. Each vertex in U is connected in the graph $X(1)$ to any other vertex. Therefore, since X is a clique complex, we have $X = X[U] * Y$, where $Y = X[V \setminus U]$.

But $X[U] \cong \Delta_{r-1}$, therefore $\mu_i(X[U]) = r$ for all $-1 \leq i \leq r-1$, so by Theorem 2.13:

$$\begin{aligned} \mu_k(X) &= \min_{i+j=k-1} \mu_i(X[U]) + \mu_j(Y) \\ &= r + \min\{\mu_j(Y) :\, \max\{-1, k-r\} \leq j \leq \min\{k, \dim(Y)\}\} \geq r. \quad (5.4) \end{aligned}$$

If the vertices $u_1, \ldots, u_t$ are not all distinct, then we have

$$r > |\sigma| - t = k + 1 - (n - k - 1) = 2(k+1) - n = \mu_k(X),$$

a contradiction to (5.4). Therefore $u_1, \ldots, u_t$ are all different vertices, so $r = \mu_k(X)$.

This implies that the inequality in 5.4 is an equality. So there exists some $j \geq k-r$ such that $\mu_j(Y)=0$. But by Theorem 1.13 we have for all $j > k-r$:

$$\begin{aligned}\mu_j(Y) \geq 2(j+1) - (n-r) = 2(j+1) - 2(k-r+1)\\ > 2(k-r+1) - 2(k-r+1) = 0.\end{aligned}$$

Hence we must have $\mu_{k-r}(Y)=0$. Therefore $\tilde{H}^{k-r}(Y;\mathbb{R}) \neq 0$, so by Theorem 2.9 we obtain

$$f_1(Y) \geq 4\binom{k-r+1}{2}.$$

But we already have $t = k-r+1$ edges $u_i v_i \notin Y(1)$. Therefore

$$\begin{aligned}f_1(Y) \leq \binom{n-r}{2} - (k-r+1) = \binom{2(k-r+1)}{2} - (k-r+1)\\ = 2(k-r+1)(k-r) = 4\binom{k-r+1}{2}.\end{aligned}$$

So $f_1(Y) = 4\binom{k-r+1}{2}$, therefore the edges $u_1v_1, \ldots, u_tv_t$ are the only missing faces of Y. Thus

$$Y = \{u_1, v_1\} * \{u_2, v_2\} * \cdots * \{u_t, v_t\} \cong \left(\Delta_1^{(0)}\right)^{*(n-k-1)}.$$

Hence, $X \cong \left(\Delta_1^{(0)}\right)^{*(n-k-1)} * \Delta_{r-1}$. □

Chapter 6

Some families of complexes without large missing faces

In this chapter we introduce some families of generalized flag complexes arising from different finite geometries. We prove some basic facts about this complexes and show the results of computer calculations for some examples. Based on these results we then make some conjectures about the spectral gaps and cohomology groups of these complexes.

6.1 The class of $\mathcal{L}(d, r, m)$ complexes

Let d, r, m be integers, such that $d \geq 1$, $r \geq d + 1$. Let $H = (V, E)$ be an r-uniform hypergraph such that any two of its edges intersect in at most m vertices. Let $\mathcal{M}$ be the collection of all subsets of size $d + 1$ of the edges in E, and let X be the simplicial complex on vertex set V whose set of missing faces is $\mathcal{M}$.

We will call H the *underlying hypergraph* of X. Since all the missing faces of X are of dimension d, we have $h(X) = d$.

Denote by $\mathcal{L}(d, r, m)$ the family of all the simplicial complexes constructed this way.

In this chapter we will first prove some simple facts about complexes in $\mathcal{L}(d, r, m)$, and then we will present some families of complexes that we believe have interesting spectral or homological properties. In particular we present a family of examples that we conjecture are extremal with respect to Theorem 1.3.

Theorem 6.1. *Let $X \in \mathcal{L}(d, r, m)$ for $m \leq d - 1$, and let $H = (V, E)$ be its underlying hypergraph, with $|V| = n$. Then*

$$\mathrm{Spec}_{d-1}(X) = \{\underbrace{n-r, n-r, \ldots, n-r}_{\substack{|E|\cdot\binom{r-1}{d} \\ \textit{times}}}, \underbrace{n, n, \ldots, n}_{\substack{\binom{n}{d}-|E|\cdot\binom{r-1}{d} \\ \textit{times}}}\}.$$

In particular $\mu_{d-1}(X) = n - r$.

Proof. Note first that every $\sigma \in X(d-1) = \binom{V}{d}$ is contained in at most one edge $e \in E$, since otherwise there would be two edges $e_1, e_2 \in E$ such that

$$|e_1 \cap e_2| \geq |\sigma| = d > d-1,$$

a contradiction to $m \leq d-1$.

For each $e \in E$, let

$$\mathcal{F}_e = \{\sigma \in X(d-1) : \sigma \subset e\} = \binom{e}{d}.$$

Define also

$$\mathcal{F}' = \{\sigma \in X(d-1) : \sigma \not\subset e \quad \forall e \in E\}.$$

So $\cup_{e \in E} \mathcal{F}_e \cup \mathcal{F}'$ is a partition of $X(d-1)$. We will show that $[L_{d-1}(X)]$ is a block diagonal matrix, with blocks corresponding to this partition:

- Let $\sigma \in \mathcal{F}'$. Then

  $$\deg_X(\sigma) = n - |\sigma| = n - d,$$

  since if there existed some $v \in V \setminus \sigma$ such that $v\sigma \notin X$, then $v\sigma$ would be a missing face of X, therefore $v\sigma \subset e$ for some $e \in E$, a contradiction to $\sigma \in \mathcal{F}'$. Therefore by Claim 2.10

  $$[L_{d-1}(X)]_{\sigma,\sigma} = n - d + (d-1) + 1 = n.$$

  Let $\tau \in X(d-1)$ such that $|\sigma \cap \tau| = d-1$. Then $\sigma \cup \tau \in X(d)$, since otherwise $\sigma \cup \tau$ would be a missing face of X, therefore $\sigma \cup \tau \subset e$ for some $e \in E$, again a contradiction to $\sigma \in \mathcal{F}'$. Therefore by Claim 2.10

  $$[L_{d-1}(X)]_{\sigma,\tau} = 0$$

  for any $\tau \neq \sigma$. So the faces in $\mathcal{F}'$ form a diagonal block $L' = nI$ in $[L_{d-1}(X)]$.

- Now let $e \in E$ and let $\sigma \in \mathcal{F}_e$. We have

  $$\deg_X(\sigma) = n - r,$$

  since for each $v \in e \setminus \sigma$ we have $v\sigma \subset e$, therefore $v\sigma \notin X$, and for each $v \notin e$, $v\sigma$ isn't contained in any edge of H (since otherwise σ would be contained in two different edges), therefore $v\sigma \in X$.

  Let $\tilde{e} \in E \setminus \{e\}$ and $\tau \in \mathcal{F}_{\tilde{e}}$ such that $|\sigma \cap \tau| = d-1$. Assume that $\sigma \cup \tau \notin X(d)$. Then $\sigma \cup \tau \subset e'$ for some $e' \in E$. But then $\sigma \subset \sigma \cup \tau \subset e'$, therefore $e' = e$ (since $\sigma \subset e$, and it can be contained in at most one edge). But similarly, $\tau \subset \sigma \cup \tau \subset e'$, therefore $e' = \tilde{e}$, a contradiction. So $\sigma \cup \tau \in X(d)$, therefore by Claim 2.10 $[L_{d-1}(X)]_{\sigma,\tau} = 0$. So the faces in $\mathcal{F}_e$ form a diagonal block L_e in $[L_{d-1}(X)]$.

Let $\tau \in \mathcal{F}_e$. Then $\sigma \cup \tau \subset e$, therefore by Claim 2.10

$$[L_{d-1}(X)]_{\sigma,\tau} = \begin{cases} n-r+d & \text{if } \sigma = \tau, \\ (\sigma : \sigma\cap\tau)(\tau : \sigma\cap\tau) & \text{if } |\sigma\cap\tau| = d-1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $L_e = (n-r)I + \left[L_{d-1}\left(\Delta_{r-1}^{(d-1)}\right)\right]$. By Claim 2.18, $L_{d-1}\left(\Delta_{r-1}^{(d-1)}\right)$ has eigenvalue 0 with multiplicity $\binom{r-1}{d}$, and all the rest of the eigenvalues are equal to r. Therefore the block L_e has eigenvalue $n-r$ with multiplicity $\binom{r-1}{d}$, and all the rest of the eigenvalues are equal to n. Since there are $|E|$ such blocks, we get

$$\mathrm{Spec}_{d-1}(X) = \{\underbrace{n-r, n-r, \ldots, n-r}_{\substack{|E|\cdot\binom{r-1}{d} \\ \text{times}}}, \underbrace{n, n, \ldots, n}_{\substack{\binom{n}{d}-|E|\cdot\binom{r-1}{d} \\ \text{times}}}\},$$

as wanted. □

We next specialize to the case $m = 1$. Here we give a necessary condition for a complex in $\mathcal{L}(d, r, 1)$ for achieving equality in Theorem 1.2:

Let $X \in \mathcal{L}(d, r, 1)$. We will call a simplex $\sigma \in X(k)$ *scattering* if

$$|\{v \in V \setminus \sigma : m(v\sigma) = 1\}| = 0.$$

Equivalently, σ is scattering if for any two edges e_1, e_2 of its underlying hypergraph such that $|e_1 \cap \sigma| = d$ and $|e_2 \cap \sigma| = d$, we have $(e_1 \cap e_2) \setminus \sigma = \emptyset$ (that is, the edges intersect inside σ, or not at all).

Denote by $\mathrm{Sc}_k(X)$ the set of all scattering faces of X of dimension k, and let $\mathrm{Sc}(X) = \cup_k \mathrm{Sc}_k(X)$.

Claim 6.2. $\mathrm{Sc}(X)$ *is a subcomplex of* X.

Proof. Let $\sigma \in \mathrm{Sc}(X)$, and let $\tau \subset \sigma$. Let e_1, e_2 be two edges of the underlying hypergraph such that $|e_1 \cap \tau| = d$ and $|e_2 \cap \tau| = d$, and assume that there is a point $u \in (e_1 \cap e_2) \setminus \tau$. But $|e_1 \cap \sigma| \geq d$ and $|e_2 \cap \sigma| \geq d$, therefore since σ is scattering, $u \in \sigma$. But then we obtain $|e_1 \cap \sigma| = d+1$, a contradiction to $\sigma \in X$. So $(e_1 \cap e_2) \setminus \tau = \emptyset$, therefore τ is scattering. □

Theorem 6.3. *Let* $X \in \mathcal{L}(d, r, 1)$, *and let* $H = (V, E)$ *be its underlying hypergraph, with* $|V| = n$. *Let* $k \geq d$ *and assume that*

$$(k-d+1)\mu_k(X) = (k+1)\mu_{k-1}(X) - dn,$$

i.e. there is equality in the inequality of Theorem 1.2 in dimension k. *Let* $\phi \in C^k(X)$ *be* *an eigenvector of $L_k(X)$ with eigenvalue $\mu_k(X)$. Then*

$$\{\sigma \in X(k) : \phi(\sigma) \neq 0\} \subset \mathrm{Sc}_k(X).$$

In particular, a necessary condition for X to achieve equality in Theorem 1.2 for some $k \geq d$ is the existence of scattering simplices of dimension k.

For the proof we will need the following Lemma:

Lemma 6.4. *Let $X \in \mathcal{L}(d, r, 1)$ with underlying hypergraph $H = (V, E)$, and let $\sigma \in X$. Then for all $v \notin \mathrm{lk}(X, \sigma)$*

$$m(v\sigma) \in \{1, d + 1\}.$$

Proof. Let $v \notin \mathrm{lk}(X, \sigma)$, and assume that $m(v\sigma) \leq d$, that is, $v\sigma$ contains at least two different missing faces of X. Every edge $e \in E$ contains at most $d + 1$ points of $v\sigma$ (otherwise it would contain $d + 1$ points of σ, a contradiction to $\sigma \in X$). Therefore every missing face of X contained in $v\sigma$ belongs to a different edge in E. Since every two edges intersect in at most one vertex and all the missing faces in $v\sigma$ must contain v, the intersection of all the missing faces must be $\{v\}$, therefore $m(v\sigma) = 1$. □

Proof of Theorem 6.3. By examining the proof of Theorem 1.2 we can see that equality can be achieved only if ϕ is an eigenvector of R_k with eigenvalue dn. Recall that by Claim 3.12 the matrix representation of R_k in the standard basis is

$$[R_k]_{\sigma,\tau} = \begin{cases} \sum_{\eta\in\sigma(k-1)} \deg_X(\eta) - (k-d+1)\deg_X(\sigma) + (d-1)(k+1) & \text{if } \sigma = \tau, \\ (d + 1 - m(\sigma \cup \tau)) \cdot (\sigma : \sigma \cap \tau) \cdot (\tau : \sigma \cap \tau) & \text{if } {|\sigma\cap\tau|=k, \atop \sigma\cup\tau\notin X(k+1),} \\ 0 & \text{otherwise.} \end{cases} \tag{6.1}$$

We will show that in our case $[R_k]$ is a diagonal matrix:

Let $\sigma \in X(k)$. By (6.1) and Lemma 3.3 we have

$$\begin{aligned} [R_k]_{\sigma,\sigma} &= \sum_{\eta\in\sigma(k-1)} \deg_X(\eta) - (k - d + 1)\deg_X(\sigma) + (d - 1)(k + 1) \\ &= d(k + 1) + d\deg_X(\sigma) + \sum_{r=2}^{d+1}(r - 1)|\{v \in V : m(v\sigma) = r\}| \\ &= dn - \sum_{r=1}^{d+1}(d - r + 1)|\{v \in V : m(v\sigma) = r\}|. \end{aligned}$$

By Lemma 6.4 we obtain

$$[R_k]_{\sigma,\sigma} = dn - d|\{v \in V : m(v\sigma) = 1\}|.$$

Now let $\sigma, \tau \in X(k)$ such that $|\sigma \cap \tau| = k$, $\sigma \cup \tau \notin X(k + 1)$.

k	0	1	2	3
$f_k(X)$	7	21	28	7
$\mu_k(X)$	7	4	0	4
$\tilde{H}_k(X;\mathbb{Z})$	0	0	$\mathbb{Z}^6$	0

Figure 6.1: The complex $C_p(2,2)$

k	0	1	2	3
$f_k(X)$	9	36	72	54
$\mu_k(X)$	9	6	0	0
$\tilde{H}_k(X;\mathbb{Z})$	0	0	$\mathbb{Z}$	$\mathbb{Z}^{11}$

Figure 6.2: The complex $C_a(2,3)$

k	0	1	2	3
$f_k(X)$	13	78	234	234
$\mu_k(X)$	13	9	1	0
$\tilde{H}_k(X;\mathbb{Z})$	0	0	$(\mathbb{Z}_3)^3$	$\mathbb{Z}^{66}$

Figure 6.3: The complex $C_p(2,3)$

By Lemma 3.1 we have $m(\sigma\cup\tau)\geq 2$, but since $\sigma\cup\tau=v\sigma$ for $\{v\}=\tau\setminus\sigma$, we obtain by Lemma 6.4 that $m(\sigma\cup\tau)=d+1$. So by (6.1) we obtain $[R_k]_{\sigma,\tau}=0$ for any $\sigma,\tau\in X(k)$ such that $\sigma\neq\tau$.

So the matrix form of R_k is

$$[R_k(X)]_{\sigma,\tau}=\begin{cases} dn-d|\{v\in V: m(v\sigma)=1\}| & \text{if } \sigma=\tau,\\ 0 & \text{otherwise.}\end{cases} \tag{6.2}$$

Therefore, for ϕ to be an eigenvector of R_k with eigenvalue dn, we must have $\{v\in V: m(v\sigma)=1\}=\emptyset$ whenever $\phi(\sigma)\neq 0$. In other words, the support of ϕ must belong to $\mathrm{Sc}_k(X)$. □

6.2 Complexes of caps in finite projective and affine spaces

Let $PG(n,q)$ be the projective space of dimension n over $\mathbb{F}_q$, and $AG(n,q)$ be the affine space of dimension n over $\mathbb{F}_q$. We will think of these spaces as hypergraphs, whose vertices are the points in the space, and the edges are the lines.

A cap in $PG(n,q)$ (or in $AG(n,q)$) is a set S of points such that any line ℓ in the space contains at most two points from S. Let $C_p(n,q)$ be the simplicial complex whose simplices are the caps in $PG(n,q)$, and $C_a(n,q)$ be the complex whose simplices are the caps in $AG(n,q)$.

Note that $C_p(n,q)$ is the complex in $\mathcal{L}(2,q+1,1)$ with underlying hypergraph $PG(n,q)$, and $C_a(n,q)$ is the complex in $\mathcal{L}(2,q,1)$ with underlying hypergraph $AG(n,q)$.

In Figures 6.1, 6.2, 6.3, 6.4, 6.5 and 6.6 we collect information about some examples of complexes of caps for different affine and projective planes. When we were not able to compute all the information about some complex, we fill the corresponding cells in the table with question marks. If the face number in dimension k is filled by a question

k	0	1	2	3	4	5
$f_k(X)$	16	120	480	840	288	48
$\mu_k(X)$	16	12	4	0	6	6
$\tilde{H}_k(X;\mathbb{Z})$	0	0	0	$\mathbb{Z}^{225}$	0	0

Figure 6.4: The complex $C_a(2,4)$

k	0	1	2	3	4	5
$f_k(X)$	21	210	1120	2520	1008	168
$\mu_k(X)$	21	16	6	0	6	6
$\tilde{H}_k(X;\mathbb{Z})$	0	0	0	$\mathbb{Z}^{750}$	0	0

Figure 6.5: The complex $C_p(2,4)$

mark, it means that we could compute the complex up to dimension $k-1$ only, and we don't know the actual dimension of the complex.

Note that all the examples with $q>2$ achieve equality in the inequality of Theorem 1.2 for $k=2$, but not for $k>2$.

Furthermore, we checked the spectral gaps of the 2-Laplacian for larger affine and projective planes, including the four non-isomorphic projective planes of order 9, and they also achieve the extremal values (see Figures 6.7, 6.8, 6.9, 6.10). Based on this we conjecture the following:

Conjecture 6.5. *Let Π be a projective plane of order $q \geq 3$. Let X be the complex of caps of Π, and let $n = q^2+q+1$ be the number of vertices of X. Then*

$$\mu_2(X) = n - 3(q+1) = q^2 - 2(q+1).$$

Let Π' be an affine plane of order $q \geq 3$. Let Y be the complex of caps of Π', and $n = q^2$ be the number of vertices of Y. Then

$$\mu_2(Y) = n - 3q = q^2 - 3q.$$

k	0	1	2	3	4	5
$f_k(X)$	25	300	2000	6500	6600	1000
$\mu_k(X)$	25	20	10	0	0	6

Figure 6.6: The complex $C_a(2,5)$

k	0	1	2	3	...
$f_k(X)$	49	1176	16464	127596	...
$\mu_k(X)$	49	42	28	9	?

Figure 6.7: The complex $C_a(2,7)$

k	0	1	2	3	...
$f_k(X)$	57	1596	26068	234612	...
$\mu_k(X)$	57	49	33	?	?

Figure 6.8: The complex $C_p(2,7)$

k	0	1	2	3	...
$f_k(X)$	73	2628	56064	686784	...
$\mu_k(X)$	73	64	46	?	?

Figure 6.9: The complex $C_p(2,8)$

k	0	1	2	3	...
$f_k(X)$	91	4095	110565	1769040	...
$\mu_k(X)$	91	81	61	?	?

Figure 6.10: The complex $C_p(2,9)$ (exactly the same spectral gaps were obtained also for the three non-Desarguesian projective planes of order 9)

k	0	1	2	3	4
$f_k(X)$	27	351	2808	14742	50544
$\mu_k(X)$	27	24	18	9	7.129507
k	5	6	7	8	-
$f_k(X)$	107406	126360	63180	2106	-
$\mu_k(X)$	0.914125	0	0	9	-

Figure 6.11: The complex $C_a(3,3)$

k	0	1	2	3	4
$f_k(X)$	81	3240	85203	1654614	?
$\mu_k(X)$	81	78	72	63	?

Figure 6.12: The complex $C_a(4,3)$

Remark. The face numbers of $C_p(2,q)$ in small dimensions are known (see [12]):

$$\begin{aligned}
f_0(C_p(2,q)) &= q^2+q+1\\
f_1(C_p(2,q)) &= \frac{1}{2}q(q+1)(q^2+q+1)\\
f_2(C_p(2,q)) &= \frac{1}{3!}q^3(q+1)(q^2+q+1)\\
f_3(C_p(2,q)) &= \frac{1}{4!}q^3(q^2-1)(q^3-1)\\
f_4(C_p(2,q)) &= \frac{1}{5!}q^3(q^2-1)(q^3-1)(q-2)(q-3)\\
f_5(C_p(2,q)) &= \frac{1}{6!}q^3(q^2-1)(q^3-1)(q-2)(q-3)(q^2-9q+21).
\end{aligned}$$

In order to find extremal examples for $k>2$, we looked at spaces of higher dimension. We found some examples: $C_a(3,3)$ achieves equality in the inequalities of Theorem 1.2 up to $k=3$. $C_a(4,3)$ also achieves equality up to $k=3$ (we believe it also achieves equality for $k=4$ but we weren't able to check it). $C_p(3,3)$ achieves equality up to $k=4$ (see Figures 6.11, 6.12 and 6.13). $X=C_p(3,3)$ is also extremal with respect to Theorem 1.3: $\mu_1(X)=36=(1-\binom{5}{2}^{-1})\cdot 40$, but $\tilde{H}^4(X;\mathbb{R})\neq 0$ (since $\mu_4(X)=0$).

The number of points in $PG(n,3)$ is $N=\frac{1}{2}(3^{n+1}-1)$, therefore by Theorem 6.1

k	0	1	2	3	4	5	6
$f_k(X)$	40	780	9360	72540	353808	1010880	?
$\mu_k(X)$	40	36	28	16	0	?	?

Figure 6.13: The complex $C_p(3,3)$

$\mu_1(C_p(n,3)) = \frac{1}{2}(3^{n+1}-1)-4$. For odd n this gives us $\mu_1(C_p(n,3)) = (1-\binom{k+1}{2}^{-1})N$, where $k = \frac{1}{2}(3^{(n+1)/2}-1)$. This fact and our computations in the case $n=3$ suggest the following conjecture:

Conjecture 6.6. *Let $n \geq 3$ be odd, and let $k = \frac{1}{2}(3^{(n+1)/2}-1)$. Then*

$$\tilde{H}^k\left(C_p(n,3);\mathbb{R}\right) \neq 0.$$

Conjecture 6.6 would imply that $X = C_p(n,3)$ (for odd values of n) are extremal examples for Theorem 1.2 and Theorem 1.3 for $k = \frac{1}{2}(3^{(n+1)/2}-1)$. In particular this implies (by Theorem 6.3) that there must be scattering faces of size $k+1$ in X. This brings us to the following conjecture:

Conjecture 6.7. *Let $n \geq 3$ be odd. Then the maximal size of a scattering simplex in $C_p(n,3)$ is $\frac{1}{2}(3^{(n+1)/2}+1)$.*

By computer calculation we were able to check Conjecture 6.7 for $n=3$. Moreover, for $n=5$, we were able to find by computer search a scattering simplex of size 14, but could not find larger scattering simplices.

Chapter 7

Concluding remarks

- In Chapter 3 we established the following connection between the spectral gaps of consecutive Laplacians of a complex X without missing faces of dimension larger than d:

  Theorem 1.2. *For $k \geq d$*

  $$(k-d+1)\mu_k(X) \geq (k+1)\mu_{k-1}(X) - dn. \tag{7.1}$$

  As a corollary we obtained the following result.

  Theorem 1.3. *If*

  $$\mu_{d-1}(X) > \left(1 - \binom{k+1}{d}^{-1}\right) n,$$

  then $\tilde{H}^j(X;\mathbb{R}) = 0$ for all $d-1 \leq j \leq k$.

  A natural question that arises is to what extent are our results sharp. In the case $d = 1$ (i.e. when X is a flag complex) the question was addressed by Aharoni, Berger and Meshulam in [3], where a family of complexes achieving equality in (7.1) in all dimensions was presented. These complexes are extremal also with respect to Theorem 1.3.

  In Chapter 6 we investigated certain families of simplicial complexes whose missing faces are all of dimension $d = 2$. We managed to find extremal examples for (7.1) in the cases $k = 2, 3, 4$, and for Theorem 1.3 in the cases $k = 2, 4$. One of the complexes investigated is the following:

  Let $C_p(n,3)$ be the complex of caps in $PG(n,3)$, i.e. the complex whose vertices are the points of the projective space of dimension n over $\mathbb{F}_3$, and whose missing faces are the sets of three points that lie in the same line in the space. Assume that n is odd, and let $k' = \frac{1}{2}(3^{(n+1)/2} - 1)$. We conjecture that $C_p(n,3)$ achieves equality in (7.1) for all $k \leq k'$. In particular this implies the following conjecture:

Conjecture 6.6. $\tilde{H}^{k'}(C_p(n,3);\mathbb{R}) \neq 0$.

By Theorem 6.1 we have $\mu_1(C_p(n,3)) = \left(1 - \binom{k'+1}{2}^{-1}\right) N$ (where N is the number of vertices of $C_p(n,3)$, i.e. the number of points in $PG(n,3)$), thus Conjecture 6.6 would imply that $C_p(n,3)$ is extremal with respect to Theorem 1.3. We were able to check this conjecture by computer for $n = 3$ (see Figure 6.13).

Other examples that we studied are the complexes $C_a(2,q)$ and $C_p(2,q)$ of caps in (respectively) the affine and projective planes over $\mathbb{F}_q$. We were able to check by computer for the first values of $q \geq 3$ that these complexes achieve equality in (7.1) for $k = 2$ (see Figures 6.2–6.10). We believe that this is true for all $q \geq 3$, maybe also for complexes of caps in non-Desarguesian planes:

Conjecture 6.5. *Let* Π *be a projective plane of order* $q \geq 3$*. Let* X *be the complex of caps of* Π*, and let* $n = q^2 + q + 1$ *be the number of vertices of* X*. Then*

$$\mu_2(X) = n - 3(q+1) = q^2 - 2(q+1).$$

Let Π' *be an affine plane of order* $q \geq 3$*. Let* Y *be the complex of caps of* Π'*, and* $n = q^2$ *be the number of vertices of* Y*. Then*

$$\mu_2(Y) = n - 3q = q^2 - 3q.$$

We could not find examples showing sharpness of (7.1) for $d \geq 3$. It would be interesting to find such an example even for the first case: $d = k = 3$. That is, finding a complex X on n vertices, whose missing faces are of dimension at most 3 (or preferably of dimension exactly 3), such that $\mu_3(X) = 4\mu_2(X) - 3n$. It would be even better to find such a complex satisfying $\mu_3(X) = 0$, that is, a complex with $\mu_2(X) = \frac{3}{4}n$ and $\tilde{H}^3(X;\mathbb{R}) \neq 0$ (since this would be an extremal example also for Theorem 1.3).

- In Chapter 5 we prove the following lower bound on the spectral gaps of a complex X:

Theorem 1.13. *Let* X *be a simplicial complex with* $h(X) = d$ *on vertex set* V*, where* $|V| = n$*. Let* $k \geq 0$ *and let* $\delta_k(X)$ *denote the minimal degree of a simplex in* $X(k)$*. Then*

$$\mu_k(X) \geq (d+1)(\delta_k(X) + k + 1) - dn. \tag{7.2}$$

We also showed that the inequality (7.2) is sharp: The complexes

$$X = \left(\Delta_d^{(d-1)}\right)^{*t} * \Delta_{r-1} \tag{7.3}$$

satisfy $\mu_k(X) = (d+1)(\delta_k(X) + k + 1) - dn$ for all $0 \leq k \leq \dim(X)$.

In the case of flag complexes ($d = 1$) we showed that all the complexes achieving equality in (7.2) in their top dimension must have such a form (Proposition 5.1). We don't know if this is the case also for $d \geq 2$. It may be interesting to try to find a complex Y of dimension k whose missing faces are all of dimension d, such that

$$\mu_k(Y) = (d+1)(k+1) - dn,$$

but Y is not isomorphic to a complex of the form (7.3).

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$$\varphi_M(\uplus_{i\in I} V_i) > \begin{cases} |I|-1 & ,|I| \le d+1 \text{ אם} \\ d\sum_{r=1}^{d} r\binom{|I|-1}{r} & ,|I| \ge d+2 \text{ אם} \end{cases}$$

אז יש ל־V תת קבוצה ססגונית במצב כללי.

העבודה מאורגנת באופן הבא: פרק 2 מכיל את חומר הרקע שדרוש לשאר הפרקים, כולל הקדמה לקוהומולוגיה סימפליציאלית ולאופרטורי הלפלסיאן על קומפלקסים סימפליציאלים. בנוסף כלול חומר רקע על ערכים עצמיים של מטריצות ועל התורה של מטרואידים.

בפרק 3 אנו מוכיחים את התוצאות המרכזיות שלנו על הפערים הספקטרליים של אופרטורי הלפלסיאן (משפטים 1.2 ו־1.3).

בפרק 4 אנו מציגים שימושים של משפט 1.2. החלק הראשון של הפרק עוסק בפרמטר $\Gamma(X)$ והקשר שלו לקשירות ההומולוגית של הקומפלקס X. בחלק השני של הפרק אנו משתמשים בקשר הזה כדי להוכיח את משפטים 1.10 ו־ 1.11, שמספקים תנאים מספיקים לקיום של אוסף נקודות ססגוני במצב כללי במטרואיד.

בפרק 5 אנו מוכיחים חסם תחתון נוסף על הפער הספקטרלי $\mu_k(X)$, כתלות במספר הקודקודים ובדרגה המינימלית של הפאות ממימד k.

בפרק 6 אנו מציגים כמה משפחות של קומפלקסים שמקורן בגיאומטריות סופיות שונות. חלק מהדוגמאות הללו מהוות דוגמאות קיצוניות למשפטים 1.2 ו־1.3.

פרק 7 מכיל הערות לסיכום, ומצוינות בו כמה מן השאלות הפתוחות וההשערות שנבעו מהמחקר.

תוצאה זו מכלילה את משפט 1.3 ב־[3], שמתייחס למקרה שבו X הוא קומפלקס הדגלים של גרף.

יהי M מטרואיד על קבוצת קודקודים V עם פונקצית דרגה ρ, ונניח ש־$\rho(V) = d + 1$. עבור $S \subset V$, נגדיר את הסגור שלו $\mathrm{cl}(S) = \{ v \in V : \rho(S) = \rho(S \cup \{v\}) \}$. תת קבוצה $F \subset V$ נקראת ישריה של M אם $F = \mathrm{cl}(F)$.

נגיד שקבוצה $S \subset V$ היא במצב כללי ביחס ל־M אם לכל $1 \leq k \leq d$ כל ישריה של M מדרגה k מכילה לכל היותר k נקודות של S. תנאי זה שקול לדרישה שכל $S' \subset S$ עם $|S'| \leq d+1$ היא קבוצה בלתי תלויה ב־M. עבור $S \subset V$ נסמן ב־$\varphi_M(S)$ את הגודל המקסימלי של תת קבוצה של S במצב כללי.

תהי $V_1, V_2, \ldots, V_m$ חלוקה של V. נגיד שקבוצה $S \subset V$ היא ססגונית אם $|S \cap V_i| = 1$ לכל $i \in [m]$.

התוצאה הבאה מופיעה ב־[13]:

משפט 1.8. אם לכל $\emptyset \neq I \subset [m]$

$$\varphi_M(\uplus_{i\in I} V_i) > \begin{cases} |I| - 1 & ,|I| \leq d+1 \text{ אם} \\ d\binom{2|I|-2}{d} & ,|I| \geq d+2 \text{ אם} \end{cases}$$

אז יש ל־V תת קבוצה ססגונית במצב כללי.

אנו מכלילים את התוצאה הנ"ל באופן הבא: יהי $S \subset V$. פונקציה $f : S \to \mathbb{R}_{\geq 0}$ היא במצב כללי שברי ביחס ל־M אם לכל $1 \leq k \leq d$ ולכל ישריה F של M מדרגה k ותת קבוצה $\sigma \subset F \cap S$ בגודל $k-1$ מתקיים

$$\sum_{\substack{v \in S, \\ \mathrm{cl}(v\sigma) = F}} f(v) \leq d.$$

נסמן ב־$\varphi^*_M(S)$ את המקסימום של $\sum_{v \in S} f(v)$ על פני כל הפונקציות על S במצב כללי שברי. מתקיים תמיד $\varphi^*_M(S) \geq \varphi_M(S)$. כמסקנה ממשפט 1.6 אנו מקבלים:

משפט 1.10. אם לכל $\emptyset \neq I \subset [m]$ מתקיים

$$\varphi^*_M(\uplus_{i\in I} V_i) > d \sum_{r=1}^{d} r \binom{|I|-1}{r},$$

אז יש ל־V תת קבוצה ססגונית במצב כללי.

בפרט אנו מקבלים את החיזוק הבא למשפט 1.8:

משפט 1.11. אם לכל $\emptyset \neq I \subset [m]$ מתקיים

ניתן לחשוב על שתי התוצאות הנ"ל כעל גרסאות גלובליות של שיטת גרלנד, אשר בצורתה המקורית מקשרת בין הפערים הספקטרלים של קומפלקס סימפליציאלי לבין הפערים הספקטרלים של הלינקים של הפאות שלו (ראה [10, 20]).

כמסקנה ממשפט 1.2 אנו מקבלים:

משפט 1.3. יהי X קומפלקס סימפליציאלי עם $h(X)=d$, על קבוצת קודקודים V בגודל n. אם

$$\mu_{d-1}(X) > \left(1-\binom{k+1}{d}^{-1}\right)n,$$

אז לכל $d-1\le j\le k$ מתקיים $\tilde{H}^j(X;\mathbb{R})=0$.

עבור $d=1$ משפטים 1.2 ו־ 1.3 הם הדוקים (ראה [3]). עבור $d=2$ הצלחנו למצוא דוגמאות קיצוניות עבור משפט 1.2 רק במקרה $k\le 4$, ועבור משפט 1.3 רק במקרה $k=2,4$ (ראה פרק 6).

יהי $\eta(X)=\mathrm{conn}_{\mathbb{R}}(X)+2$, כאשר $\mathrm{conn}_{\mathbb{R}}(X)=\min\{\,i:\tilde{H}^i(X;\mathbb{R})\neq 0\,\}-1$ היא הקשירות ההומולוגית של X מעל $\mathbb{R}$.

נשתמש במשפט 1.2 כדי לקבל חסם תחתון על $\eta(X)$. לשם כך נגדיר את מספר השליטה הוקטורי של X, אותו נסמן ב־$\Gamma(X)$:

יהי k מספר טבעי, ויהי $\mathcal{M}_X(k)$ אוסף הפאות החסרות של X ממימד k.

נגדיר $J_X=\{\,i\in\mathbb{N}:\mathcal{M}_X(i)\neq\emptyset\,\}$, ו־ $S(X)=\bigcup_{i\in J_X}\binom{V}{i-1}$.

לכל $\sigma\in S(X)$ נקבע $\ell=\ell(\sigma)\in\mathbb{N}$. הצגה וקטורית של X ביחס ל־ σ היא פונקציה $P_\sigma:V\to\mathbb{R}^\ell$ כך ש־ $P_\sigma(v)\cdot P_\sigma(w)\ge 1$ אם $vw\sigma\in\mathcal{M}_X(|\sigma|+1)$, ו־$P_\sigma(v)\cdot P_\sigma(w)\ge 0$ אחרת. נקרא לאוסף $P=\{\,P_\sigma:\sigma\in S(X)\,\}$ הצגה וקטורית של X.

לכל $\sigma\in S(X)$, יהי $\alpha_\sigma\in\mathbb{R}^V$ וקטור אי־שלילי. נגיד שהאוסף $\{\alpha_\sigma:\sigma\in S(X)\}$ הוא שולט עבור P אם לכל $w\in V$ מתקיים

$$\sum_{\sigma\in S(X)}\sum_{v\in V}\alpha_\sigma(v)P_\sigma(v)\cdot P_\sigma(w)\ge 1.$$

הערך של P הוא

$$|P|=\min\left\{\sum_{\sigma\in S(X)}\sum_{v\in V}\alpha_\sigma(v):\ \text{שולט עבור } P\ \ \{\alpha_\sigma\}_{\sigma\in S(X)}\right\}.$$

נגדיר את $\Gamma(X)$ כסופרימום של $|P|$ על פני כל ההצגות הוקטוריות P של X. אנו מוכיחים את אי־השיוויון הבא:

משפט 1.6.

$$\sum_{r\in J_X} r\binom{\eta(X)}{r}\ge\Gamma(X).$$

תקציר

יהי X קומפלס סימפליציאלי על קבוצת קודקודים V. סימפלקס $\sigma \subset V$ נקרא פאה חסרה של X אם $\sigma \notin X$ אבל $\tau \in X$ לכל $\tau \subsetneq \sigma$. קבוצת הפאות החסרות $\mathcal{M}_X$ של הקומפלקס X קובעת לחלוטין את X:

$$X = \{\, \tau \subset V : \sigma \in \mathcal{M}_X \text{ לכל } \sigma \not\subset \tau \,\}.$$

נסמן ב־$h(X)$ את המימד המקסימלי של פאה חסרה של X.

עבור $k \geq -1$ יהיה $C^k(X)$ מרחב הקו־שרשראות בעלות ערך ממשי על הקומפלקס X, ויהי $d_k : C^k(X) \to C^{k+1}(X)$ אופרטור הקו־שפה. עבור $k \geq 0$ נגדיר את הלפלסיאן ה־k מימדי של X להיות האופרטור

$$L_k(X) = d_{k-1}d_{k-1}^* + d_k^* d_k.$$

הלפלסיאן הוא אופרטור אי־שלילי מ־$C^k(X)$ לעצמו. הפער הספקטרלי ה־k מימדי של X, מסומן ב־$\mu_k(X)$, הוא הערך עצמי הקטן ביותר של $L_k(X)$.

יהי $G = (V, E)$ גרף על n קודקודים. קומפלקס הדגלים $X(G)$ הוא הקומפלקס הסימפליציאלי על קבוצת הקודקודים V שהסימפלקסים שלו הם הקליקות ב־G. נשים לב שקומפלקסי הדגלים של גרפים הם בדיוק הקומפלקסים המקיימים $h(X) = 1$. אכן, הפאות החסרות של $X(G)$ הן הצלעות של המשלים של G.

ב־[3], אהרוני, ברגר ומשולם מוכיחים את התוצאה הבאה:

משפט 1.1. יהי $G = (V, E)$ גרף, כאשר $|V| = n$, ויהי $X = X(G)$ קומפלקס הדגלים שלו. אז עבור $k \geq 1$ מתקיים

$$k\mu_k(X) \geq (k+1)\mu_{k-1}(X) - n.$$

התוצאה המרכזית שלנו היא ההכללה הבאה של משפט 1.1:

משפט 1.2. יהי X קומפלקס סימפליציאלי עם $h(X) = d$, על קבוצת קודקודים V בגודל n. אז עבור $k \geq d$ מתקיים

$$(k - d + 1)\mu_k(X) \geq (k+1)\mu_{k-1}(X) - dn.$$

המחקר בוצע בהנחייתו של פרופסור רועי משולם בפקולטה למתמטיקה.

אני מודה לפרופסור משולם על עזרתו ועידודו.

אני מודה לטכניון ולקרן המלגות לתארים מתקדמים ע"ש רות ופרופ' פינצ'י
ז"ל על התמיכה הכספית הנדיבה בהשתלמותי

פערים ספקטרלים של קומפלקסי דגלים מוכללים

חיבור על מחקר

לשם מילוי חלקי של הדרישות לקבלת התואר
מגיסטר למדעים במתמטיקה

אלן לאו

הוגש לסנט הטכניון — מכון טכנולוגי לישראל
כסלו התשע"ח   חיפה   דצמבר 2017

פערים ספקטרלים של
קומפלקסי דגלים מוכללים

אלן לאו