Complexes of graphs with bounded independence number

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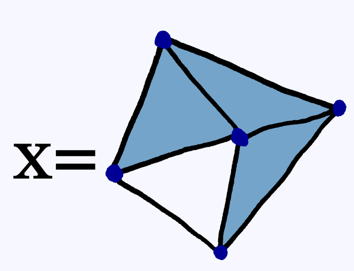
Abstract. Let $G=(V,E)$ be a graph and n a positive integer. Let $I_n(G)$ be the simplicial complex whose simplices are the subsets of V that do not contain an independent set of size n in G. We study the collapsibility numbers of the complexes $I_n(G)$ for various classes of graphs, focusing on the class of graphs with maximum degree bounded by Δ.

d-Collapsibility

Let X be an abstract simplicial complex on vertex set V. Let $\sigma \in X$ such that $|\sigma| \leq d$ and σ is contained in a unique maximal face $\tau \in X$. The operation of removing σ and all the faces containing it from X is called an elementary d-collapse. X is d-collapsible if there is a sequence of elementary d-collapses from X to the void complex $\emptyset$.

The collapsibility number of X, denoted by $C(X)$, is the minimal d such that X is d-collapsible.

Example.



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Upper bounds on collapsibility numbers

For $v \in V$, let

$$X \setminus v = \{\sigma \in X : v \notin \sigma\}, \quad \mathrm{lk}(X,v) = \{\sigma \in X : v \notin \sigma, \sigma \cup \{v\} \in X\}.$$

Our starting point is the following basic bound, due to Tancer:

Lemma 1 (Tancer [1]). *Let $v \in V$. Then,*

$$C(X) \leq \max\{C(X \setminus v), C(\mathrm{lk}(X,v)) + 1\}.$$

By inductive application of **Lemma 1**, we obtain several useful bounds on $C(X)$ **(Click here for details)**. In particular, we obtain the following result:

A *missing face* of X is a set $\tau \subset V$ such that $\tau \notin X$, but $\sigma \in X$ for any $\sigma \subsetneq \tau$.

Proposition 2. *Let X be a simplicial complex on vertex set V. If all the missing faces of X are of dimension at most d, then*

$$C(X) \leq \left\lfloor \frac{d|V|}{d+1} \right\rfloor.$$

Moreover, equality $C(X) = \frac{d|V|}{d+1}$ is obtained if and only if the set of missing faces of X consists of $\frac{|V|}{d+1}$ disjoint sets of size $d+1$.

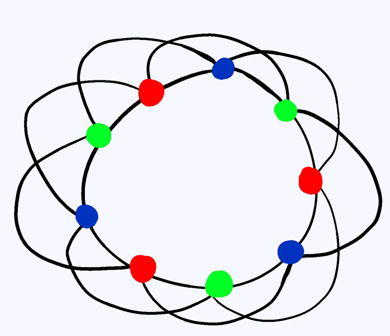
Rainbow independent sets

Let G be a graph, and let $\mathcal{F} = \{A_1, \ldots, A_m\}$ be a family of (not necessarily distinct) independent sets in G. An independent set A of size $n \leq m$ in G is called a rainbow independent set with respect to $\mathcal{F}$ if it can be written as $A = \{a_{i_1}, \ldots, a_{i_n}\}$, where $1 \leq i_1 < i_2 < \cdots < i_n \leq m$ and $a_{i_j} \in A_{i_j}$ for each $1 \leq j \leq n$.

For a positive integer n, let $f_G(n)$ be the minimum integer t such that every family of t independent sets of size n in G has a rainbow independent set of size n.

The parameters $f_G(n)$ were introduced by Aharoni, Briggs, Kim and Kim in [2].

Example.



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Rainbow sets and collapsibility

Let $G=(V,E)$ be a simple graph. For every integer $n \geq 1$, we define the simplicial complex

$$I_n(G) = \{U \subset V : U \text{ does not contain an independent set of size } n \text{ in } G\}.$$

By a standard application of Kalai and Meshulam's Colorful Helly Theorem for d-collapsible complexes ([3, Theorem 2.1]), the following bound is obtained:

Proposition 3. $f_G(n) \leq C(I_n(G)) + 1.$

Main results

Our main results are the following upper bounds on the collapsibility numbers of $I_n(G)$, for different families of graphs:

Theorem 4. *Let $G=(V,E)$ be a chordal graph. Then $C(I_n(G)) \leq n-1$. Moreover, if $\alpha(G) \geq n$, then $C(I_n(G)) = n-1$.*

Proposition 5. *Let G be a k-colorable graph. Then $C(I_n(G)) \leq k(n-1)$.*

Theorem 6. *Let $G=(V,E)$ be a graph with maximum degree at most Δ. Then $C(I_n(G)) \leq \Delta(n-1)$.*

The bound in **Theorem 6** is tight only for $\Delta \leq 2$. In the case $n \leq 3$ we can prove the following tight bounds, for general Δ:

Theorem 7. *Let $G=(V,E)$ be a graph with maximum degree at most Δ. Then*

$$C(I_2(G)) \leq \left\lceil \frac{\Delta+1}{2} \right\rceil.$$

Theorem 8. *Let $G=(V,E)$ be a graph with maximum degree at most Δ. Then*

$$C(I_3(G)) \leq \begin{cases} \Delta+2 & \text{if } \Delta \text{ is even,} \\ \Delta+1 & \text{if } \Delta \text{ is odd.} \end{cases}$$

Combining these bounds with **Proposition 3**, we recover several of the bounds for $f_G(n)$ first proved by Aharoni et al. in [2]. The following bound is new:

Theorem 9 (Click here for more details). *Let G be a claw-free graph with maximum degree at most Δ. Then*

$$f_G(n) \leq \left\lfloor \left(\frac{\Delta}{2}+1\right)(n-1) \right\rfloor + 1.$$

Some conjectures and a counterexample

The following conjecture was proposed in [2]:

Conjecture 10 (Aharoni, Briggs, Kim, Kim [2]). *Let G be a graph with maximum degree at most Δ, and let n be a positive integer. Then*

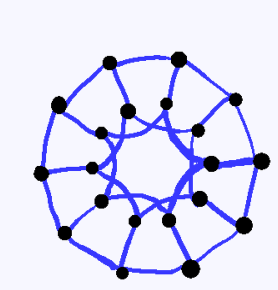
$$f_G(n) \leq \left\lceil \frac{\Delta+1}{2} \right\rceil (n-1) + 1.$$

It is natural to ask whether the following extension of **Conjecture 10** holds:

Question 11 (Aharoni). *Let G be a graph with maximum degree at most Δ, and let n be a positive integer. Does the following bound hold?*

$$C(I_n(G)) \leq \left\lceil \frac{\Delta+1}{2} \right\rceil (n-1).$$

Theorems 6,7 and 8 settle the question affirmatively in the special cases where $\Delta \leq 2$ or $n \leq 3$. Unfortunately, the bound in **Question 11** does not hold in general. We found a family of counterexamples to the case $\Delta = 3$. The proof is topological; it follows by bounding the Leray number, a homological variant of the collapsibility number, of our complexes.



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References

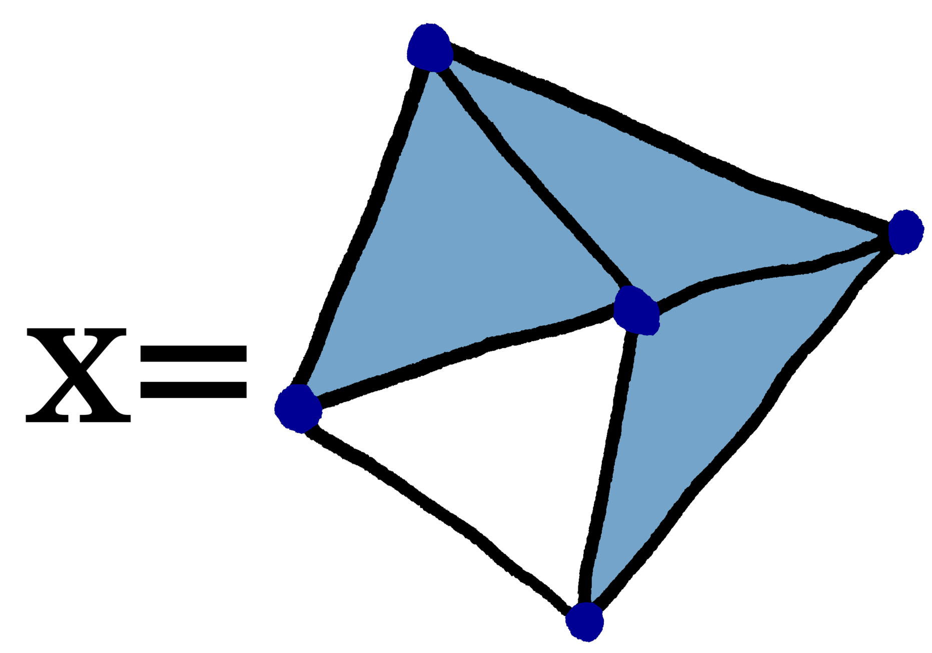
[1] Martin Tancer. Strong d-collapsibility. *Contributions to Discrete Mathematics*, 6(2):32–35, 2011.

[2] Ron Aharoni, Joseph Briggs, Jinha Kim and Minki Kim. Rainbow independent sets in certain classes of graphs. preprint, https://arxiv.org/abs/1909.13143, 2019.

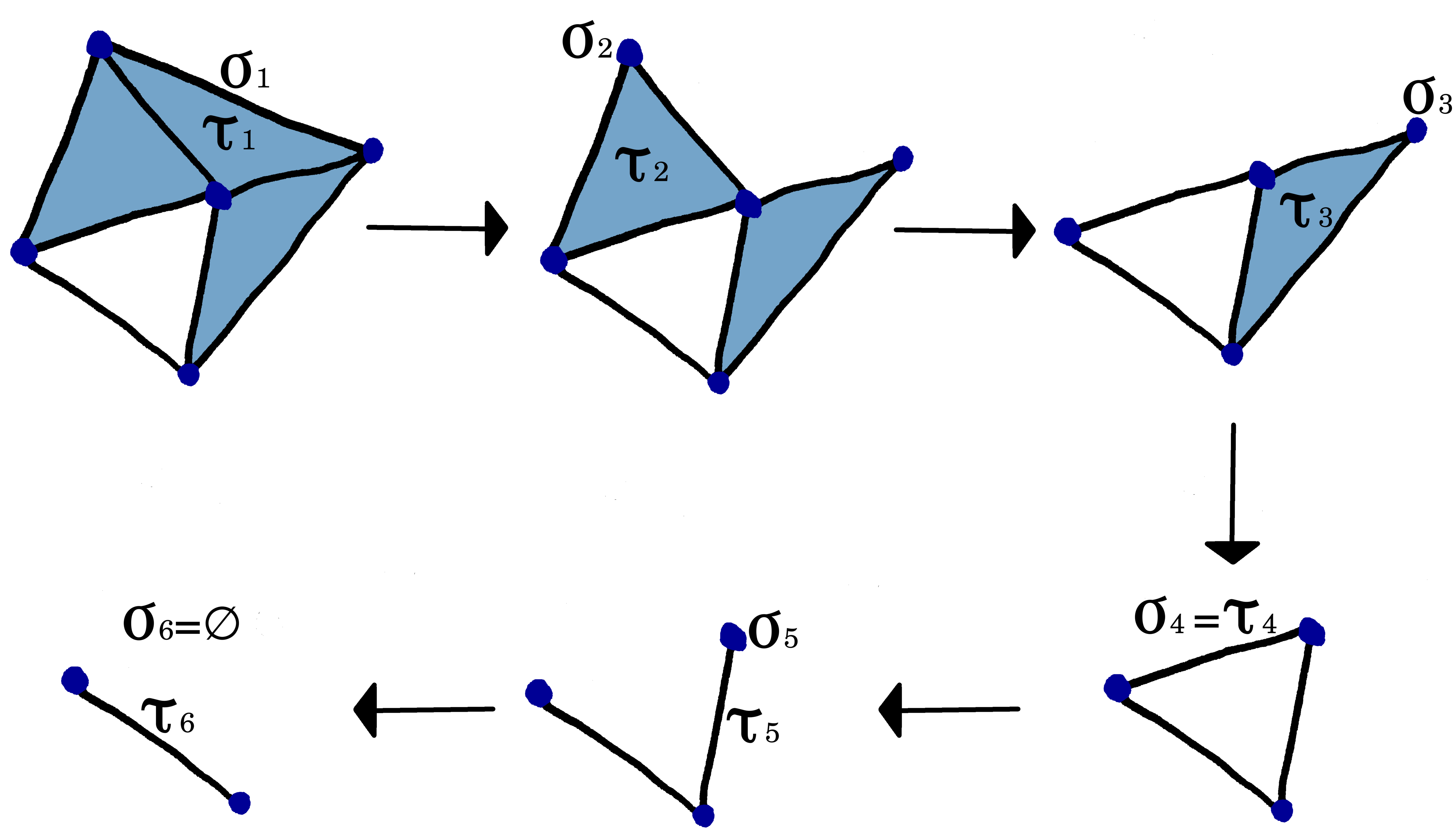
[3] Gil Kalai and Roy Meshulam. A topological colorful Helly theorem. *Adv. Math.*, 191(2):305–311, 2005.

d-Collapsibility - An example

Let X be the 2-dimensional complex:



X is not 1-collapsible: any vertex in X is contained in at least 2 different maximal faces. Hence, not even a single elementary 1-collapse can be performed on X.
On the other hand, X is 2-collapsible:



So, $C(X) = 2$.

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More upper bounds on collapsibility

First, we recall some definitions. For $U \subset V$, let

$$X[U] = \{\sigma \in X : \sigma \subset U\}.$$

For $\tau \in X$, let

$$\text{lk}(X, \tau) = \{\sigma \in X : \sigma \cap \tau = \emptyset, \sigma \cup \tau \in X\}.$$

Let $v \in V$. The complex X is called a cone over the vertex v if v is contained in every maximal face of X. The following bounds are the main technical tools used for our results on the collapsibility of the complexes $I_n(G)$:

Lemma 12: *Let $\sigma = \{v_1, \ldots, v_k\} \in X$. For $0 \leq i \leq k-1$, let $\sigma_i = \{v_j : 1 \leq j \leq i\}$. Let $d \geq k$. If for all $0 \leq i \leq k-1$,*

$$C(\text{lk}(X \setminus v_{i+1}, \sigma_i)) \leq d - i,$$

and

$$C(\text{lk}(X, \sigma)) \leq d - k,$$

then $C(X) \leq d$.

Lemma 13: *Let $B \subset V$, and let $<$ be a linear order on the vertices of B. Let $\mathcal{P} = \mathcal{P}(X, B)$ be the family of partitions (B_1, B_2) of B satisfying:*

- *$B_2 \in X$.*
- *For any $v \in B_2$, the complex $\text{lk}(X[V \setminus \{u \in B_1 : u < v\}], \{u \in B_2 : u < v\})$ is not a cone over v.*

If

$$C(\text{lk}(X[V \setminus B_1], B_2)) \leq d - |B_2|$$

for every $(B_1, B_2) \in \mathcal{P}$, then $C(X) \leq d$.

Both bounds follow by simple inductive applications of **Lemma 1** .

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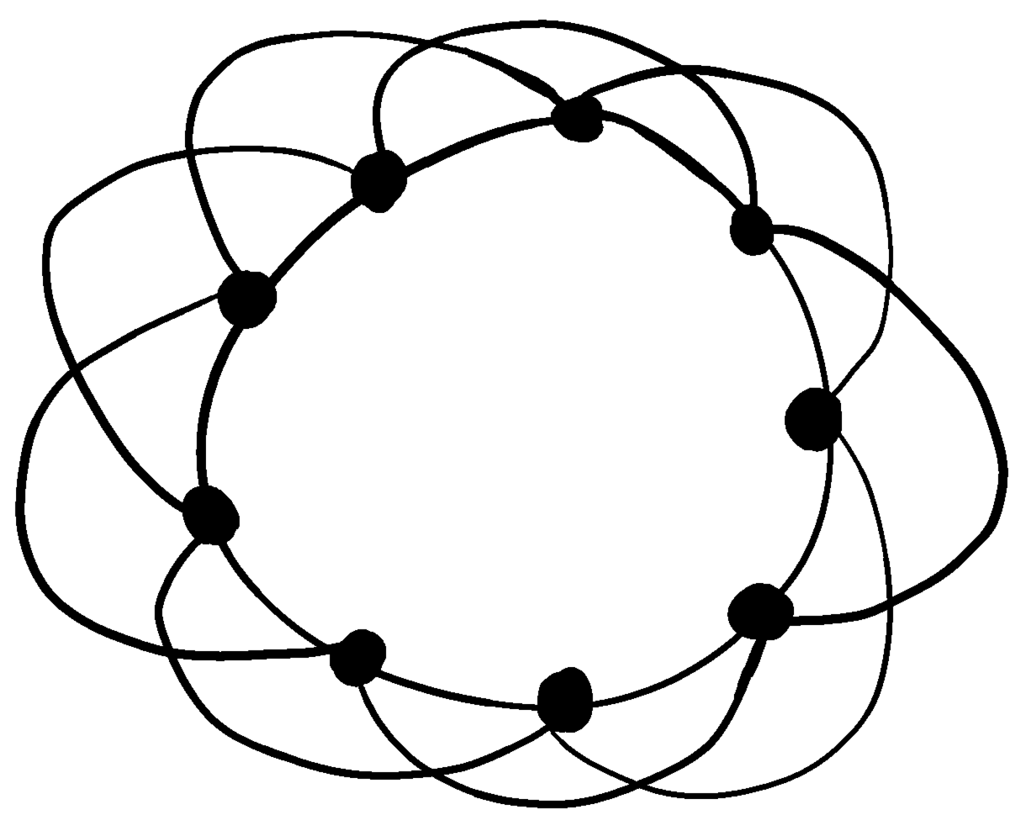
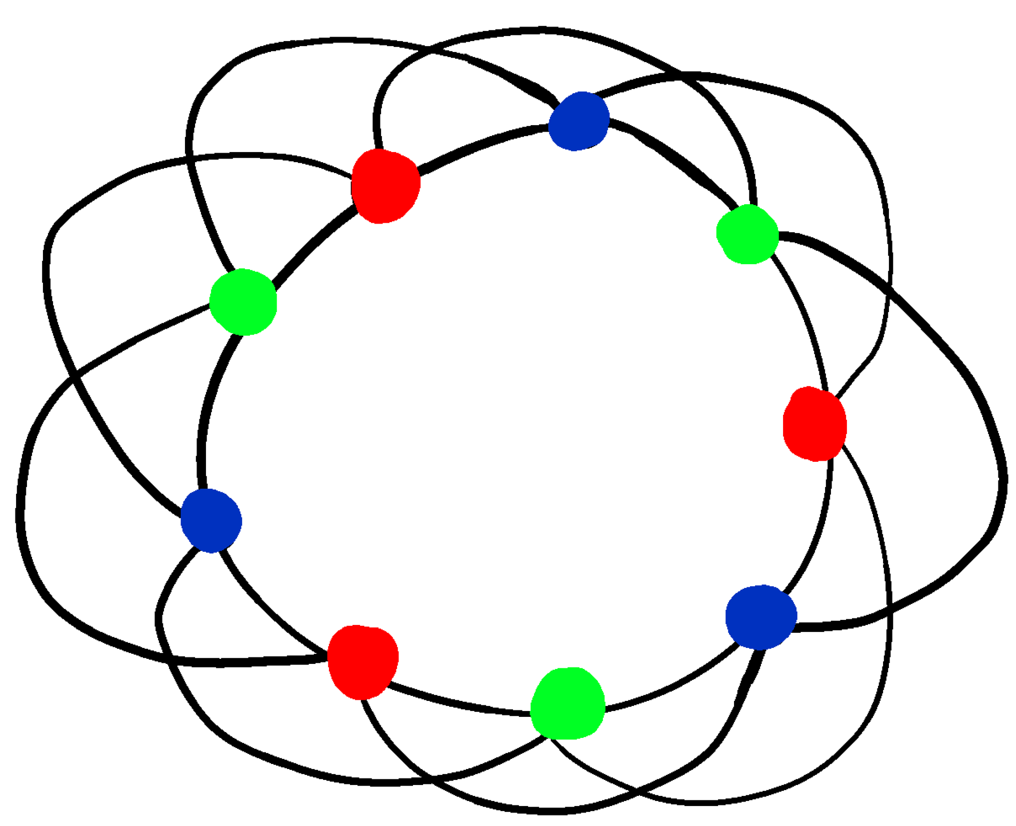
Rainbow independent sets - An example

Let $n = 3$. Let G be the following graph:



Let I_1 , I_2 , I_3 be the independent sets of size 3 in G:



Look at the family:

$$\mathcal{F} = \{I_1, I_1, I_2, I_2, I_3, I_3\}.$$

$\mathcal{F}$ does not have a rainbow independent set of size 3: Any rainbow set of $\mathcal{F}$ contains at most 2 vertices from each color class. Hence, $f_G(3) > 6$.
On the other hand, any collection of 7 independent sets of size 3 contains a rainbow independent set of size 3 (since any such collection must contain at least 3 copies of one of the independent sets I_1 , I_2 , or I_3). So,

$$f_G(3) = 7.$$

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A graph G is called claw-free if it does not contain the complete bipartite graph $K_{1,3}$ as an induced subgraph. The following is the main application of our results to the rainbow independent set problem:

Theorem 9. *Let G be a claw-free graph with maximum degree at most Δ. Then*

$$f_G(n) \leq \left\lfloor \left(\frac{\Delta}{2}+1\right)(n-1) \right\rfloor + 1.$$

The proof of **Theorem 9** relies on bounding the collapsibility numbers of certain subcomplexes of $I_n(G)$:

Proposition 14. *Let G be a claw-free graph with maximum degree at most Δ, and let $n \geq 1$ be an integer. Let A be an independent set of size $n-1$ in G. Then,*

$$C(\mathrm{lk}(I_n(G), A)) \leq \left\lfloor \frac{(n-1)\Delta}{2} \right\rfloor.$$

Examples in [2] show that, for even Δ, the bound in **Theorem 9** is tight. Proving a tight bound for the odd Δ case, and deciding whether such bounds hold also for general bounded degree graphs, are open questions (see **Conjecture 10**).

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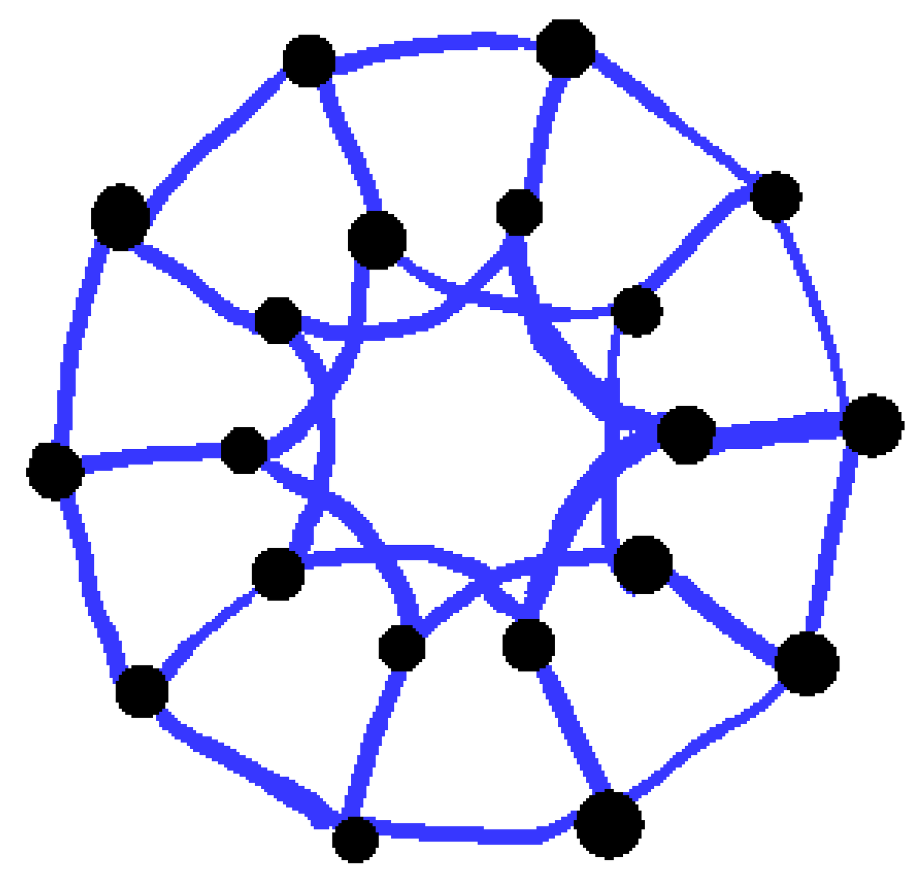
Let X be a simplicial complex. For $i \geq -1$, let $\tilde{H}_i(X)$ be the i-th reduced homology group of X with real coefficients. We say that X is d-Leray if for any induced subcomplex Y of X, $\tilde{H}_i(Y) = 0$ for all $i \geq d$. The Leray number of X, denoted by $L(X)$, is the minimum integer d such that X is d-Leray.
The Leray number of X is a lower bound for its collapsibility number: $C(X) \geq L(X)$.

Let G be the dodecahedral graph. We can represent G as a generalized Petersen graph, as follows:



The graph G is 3-regular (i.e. the degree of every vertex in G is 3). The maximum size of an independent set in G is 8.

Let $n = 8$. Applying standard topological tools (the Nerve Theorem and Alexander duality), we can compute the homology groups of the complex $I_8(G)$:

Proposition 15. *Let G be the dodecahedral graph. Then,*

$$\tilde{H}_i(I_8(G)) = \begin{cases} \mathbb{R}^4 & \text{if } i = 15, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $L(I_8(G)) \geq 16$.

We obtain $C(I_8(G)) \geq L(I_8(G)) \geq 16 > 2 \cdot (8-1) = 14$. Therefore, $I_8(G)$ does not satisfy the bound in **Question 11** . However, it is not hard to check that $f_G(8) \leq 11$. So, G does not contradict **Conjecture 10** .

The next result allows us to construct more examples of complexes that do not satisfy the bound in **Question 11** :

Theorem 16. *Let G be the disjoint union of the graphs $G_1, \ldots, G_m$. For $1 \leq i \leq m$, let t_i be the maximum size of an independent set in G_i and let $\ell_i = L(I_{t_i}(G_i))$. Let $t = \Sigma_{i=1}^m t_i$ be the maximum size of an independent set in G, and $\ell = L(I_t(G))$. Then,*

$$\ell = \sum_{i=1}^{m} \ell_i + m - 1.$$

Combining **Theorem 16** with **Proposition 15** , we obtain:

Corollary 17. *Let G_k be the union of k disjoint copies of the dodecahedral graph. Then*

$$L(I_{8k}(G_k)) \geq 17k - 1.$$

Note that the graphs G_k are 3-regular, and $\frac{L(I_{8k}(G_k))}{8k-1} \geq \frac{17k-1}{8k-1} > 2\frac{1}{8} > 2$. Thus, the complexes $I_{8k}(G_k)$ do not satisfy the bound in **Question 11** .

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