Collapsibility of simplicial complexes of graphs and hypergraphs

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Bar-Ilan Combinatorics Seminar
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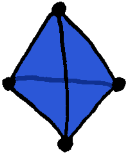
Collapsibility of X:

$$C(X) = \text{minimal } d \text{ such that } X \text{ is } d\text{-collapsible}.$$

Examples

Example 1:

$C(X) = 0 \iff X$ is a simplex



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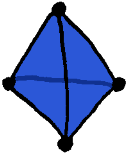

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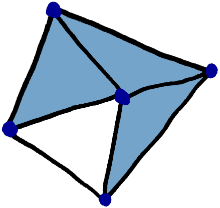

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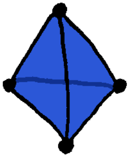

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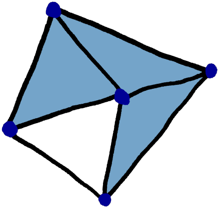
X= 

X is not 1-collapsible

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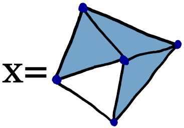
A 2-collapsing sequence for X:



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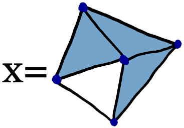

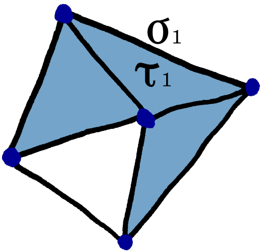

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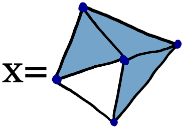

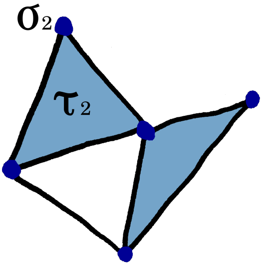

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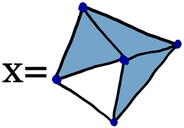

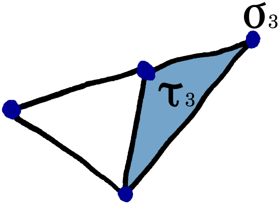

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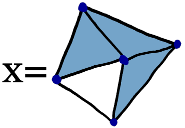

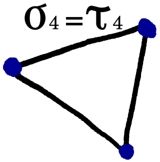

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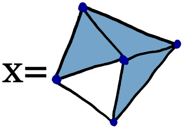

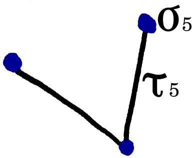

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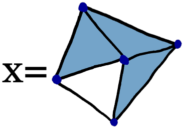

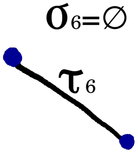

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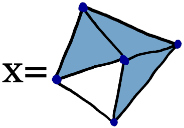

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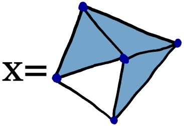

$C(X) = 2.$

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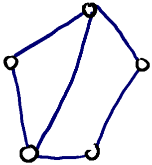

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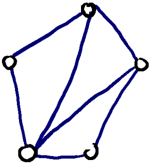

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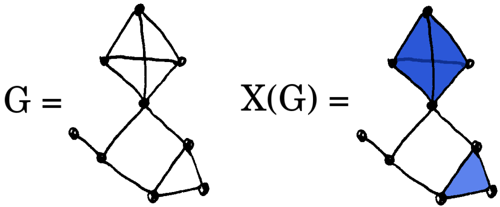

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A simplicial complex X is 1-collapsible if and only if $X = X(G)$ for some chordal graph G.

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The proof relies on the following fact:

Lemma (Lekkerkerker-Boland '62):

Any chordal graph contains a simplicial vertex (a vertex whose neighbors form a clique).

Some properties of d-collapsibility

Claim: [Wegner '75]

X is d-collapsible $\implies$ X is homotopy equivalent to a complex of dimension $< d$.

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Claim: [Wegner '75]

Every induced subcomplex of a d-collapsible complex is d-collapsible.

d-Collapsibility of nerves

Let $\mathcal{F} = \{F_1, \ldots, F_n\}$ be a family of sets.
The nerve of the family is the simplicial complex:

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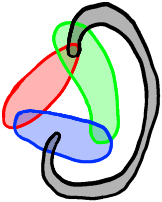

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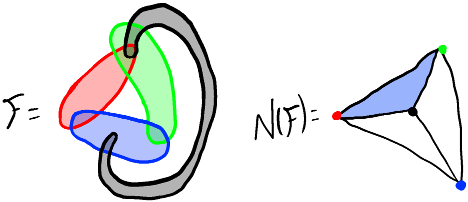

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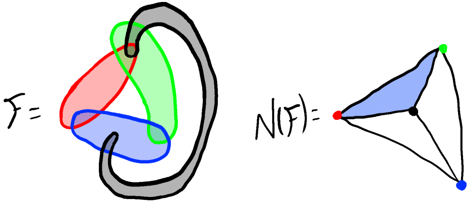

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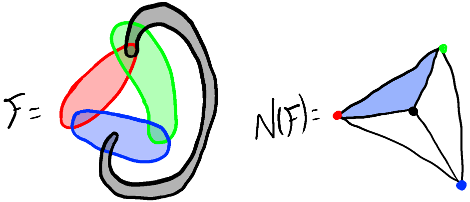

Theorem: [Wegner '75]

The nerve of a family of convex sets in $\mathbb{R}^d$ is d-collapsible.

Theorem: [Matoušek-Tancer '09]

The nerve of a family of finite sets of size $\leq d$ is d-collapsible.

Complex of hypergraphs with bounded covering number

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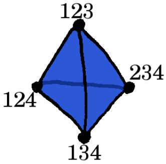

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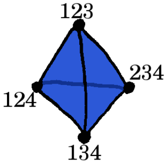
$\mathbf{Cov}_{\mathcal{H},1} =$ 

Remark: $\mathrm{Cov}_{\mathcal{H},1} = N(\mathcal{H})$.

Homology of $\mathrm{Cov}_{\mathcal{H},P}$

Question:

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Theorem (Jonsson '05):

Let K_n be the complete graph on n vertices, and let $p \leq 3$. Then

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Theorem (Matoušek-Tancer '09):

Let $\mathcal{H}$ be an r-uniform hypergraph. Then $\tilde{H}_i(\mathrm{Cov}_{\mathcal{H},1}) = 0$ for $i \geq r$.

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Theorem 1: [L '19]

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Corollary:

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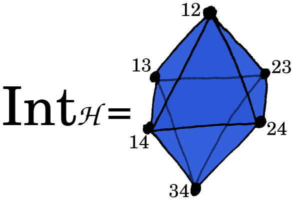

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Corollary:

$$\tilde{H}_i(\text{Int}_{\mathcal{H}}) = 0$$

for all $i \geq \frac{1}{2}\binom{2r}{r}$.

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Theorem: [Matoušek-Tancer '09, L '19]

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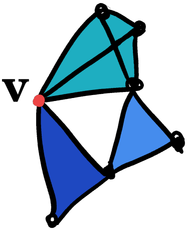

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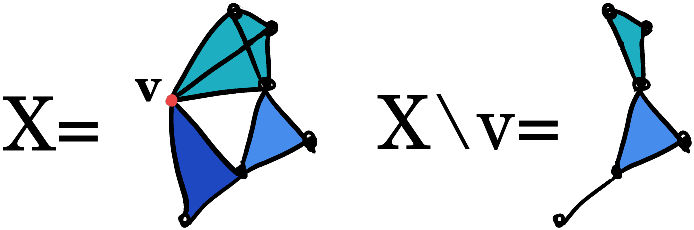

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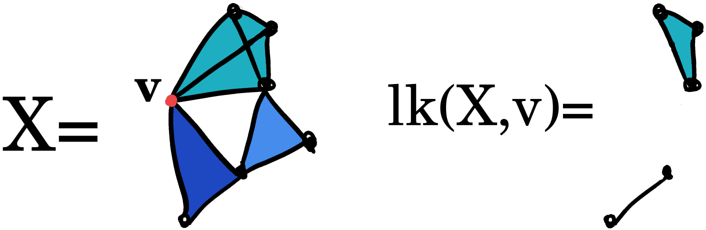

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Lemma:
If v is not contained in all maximal faces of X, then

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Skew-intersecting families of sets

Lemma: [Frankl '82, Kalai '84]

Let $\{A_1, \ldots, A_k\}$, $\{B_1, \ldots, B_k\}$ families of sets such that:

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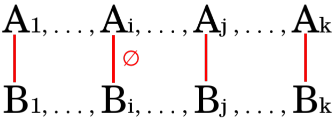

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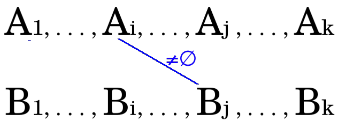

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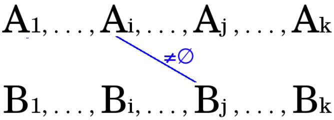

Then

$$k \leq \binom{r+p}{r}.$$

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$$A \in \mathcal{F}_i \iff A \cap C_i \neq \emptyset.$$

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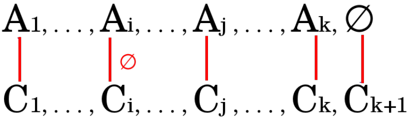

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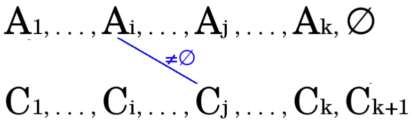

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That is, $\mathrm{Cov}_{\mathcal{H},p}$ is $\left(\binom{r+p}{r} - 1\right)$-collapsible.

□

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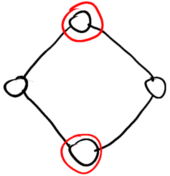

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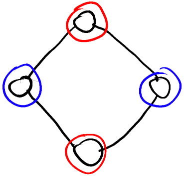

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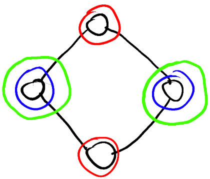

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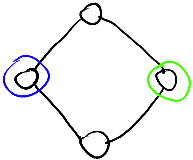

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Theorem (Aharoni–Briggs–Kim–Kim):

Let $G = (V, E)$ be a chordal graph, and let $n \geq 1$ be an integer. Then

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Conjecture (Aharoni–Briggs–Kim–Kim):

Let G be a graph with maximum degree at most Δ, and let n be a positive integer. Then

$$f_G(n) \leq \left\lceil \frac{\Delta + 1}{2} \right\rceil (n - 1) + 1.$$

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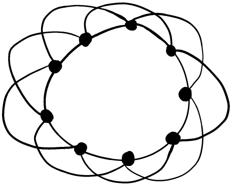

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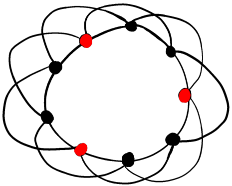

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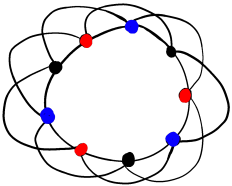

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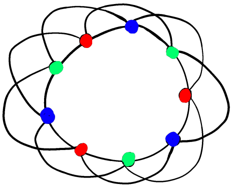

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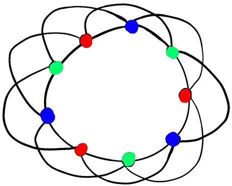

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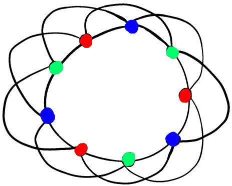

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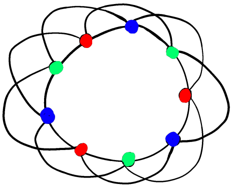

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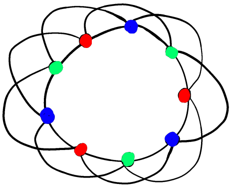

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In general, $f_{G_{\Delta,n}}(n) = \left(\frac{\Delta}{2}+1\right)(n-1)+1$.

Rainbow sets and collapsibility

$G = (V, E)$ a graph, $n \geq 1$ an integer. Define the simplicial complex:

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But I_j is an independent set of size n in G, a contradiction.

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Last bound is not tight for $\Delta \geq 3$.

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But it is not true in general!

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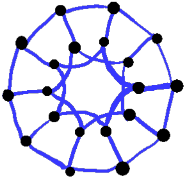

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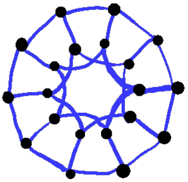

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On the other hand

$$f_G(8) \leq 11 < 2(n-1) + 1.$$

Thank you!