Representability and boxicity of simplicial complexes

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Technion – Israel Institute of Technology

Bar-Ilan University Combinatorics Seminar
December 2020

Boxicity

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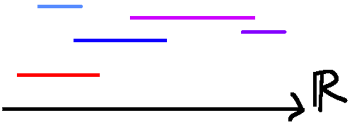

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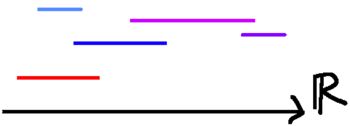

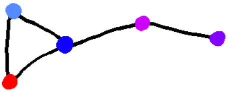

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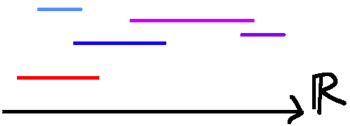

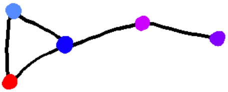

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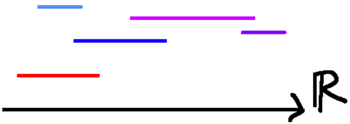

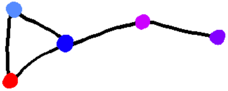

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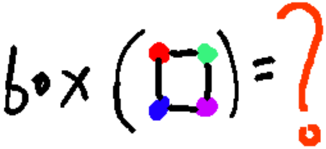

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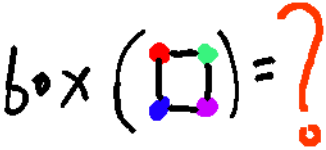

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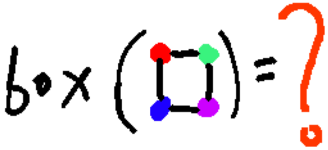

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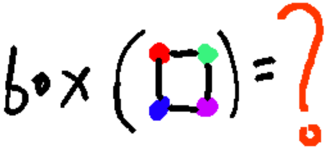

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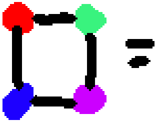

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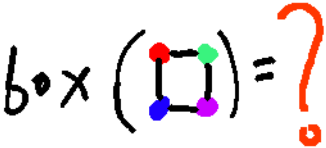

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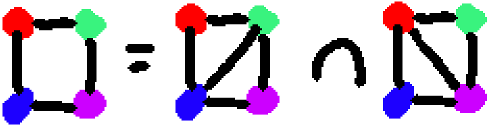

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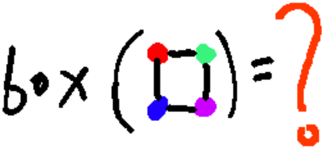

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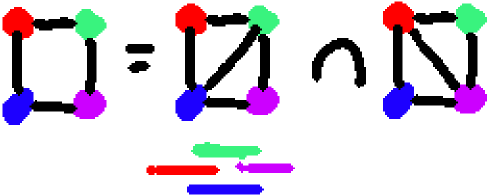

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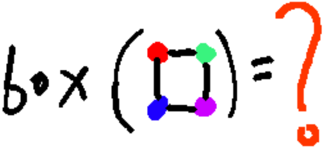

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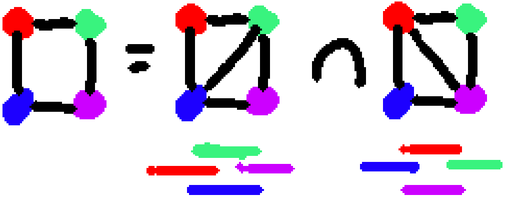

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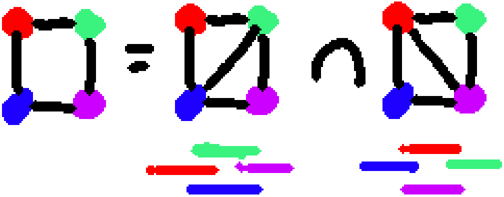

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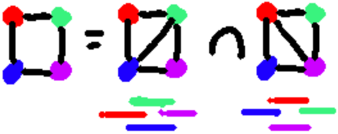

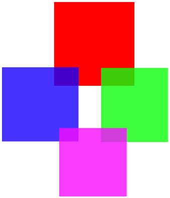

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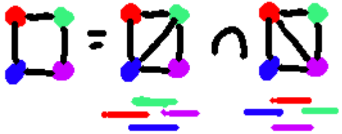

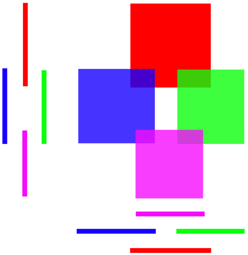

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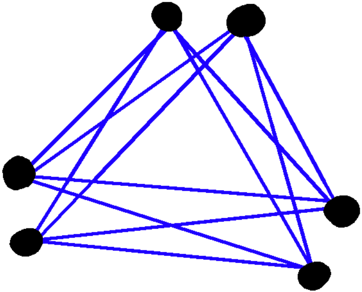

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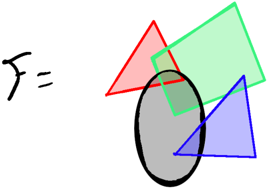

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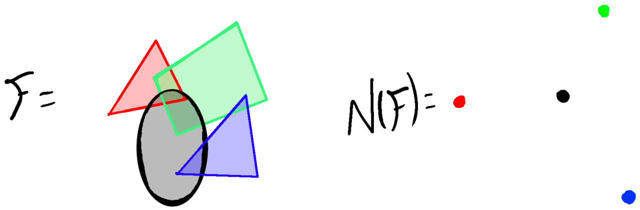

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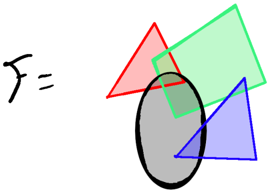

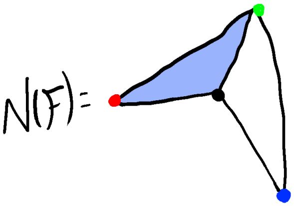

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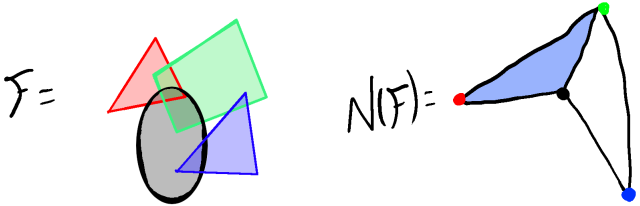

d-Representable complex = nerve of a family of convex sets in $\mathbb{R}^d$.

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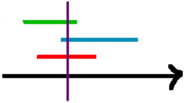

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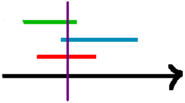

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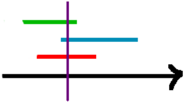

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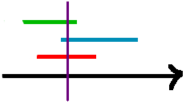

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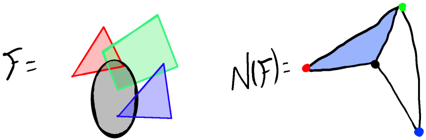

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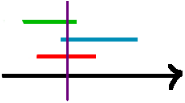

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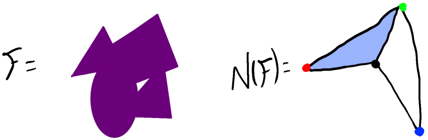

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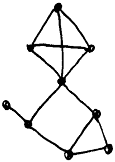

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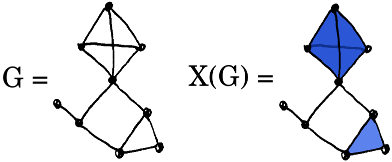

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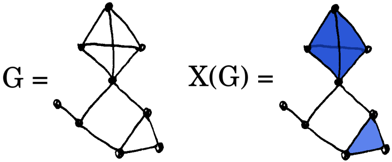

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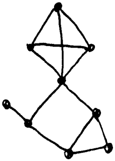
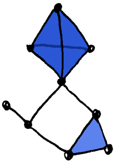
G =  X(G) = 

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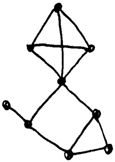
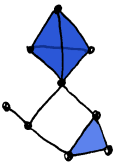
G =  X(G) = 

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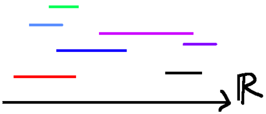

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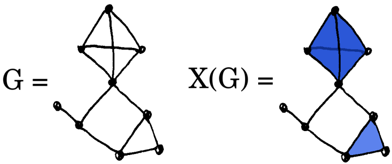

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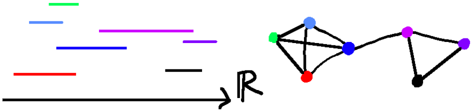

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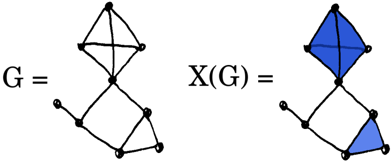

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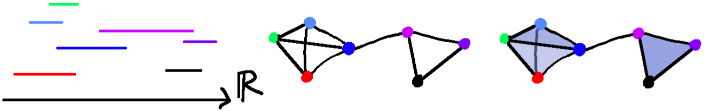

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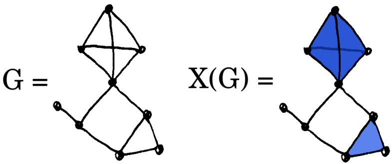

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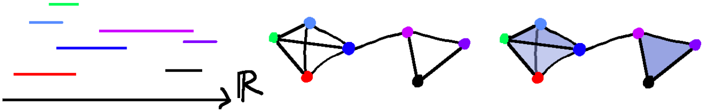

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Boxicity in terms of representability

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of graph $G = (V, E)$:
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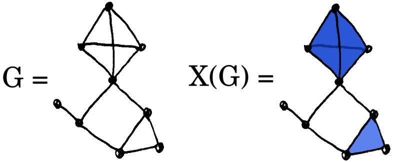

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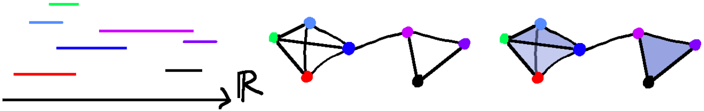

- $\text{box}(G) \leq k$ if and only if $X(G)$ can be written as the intersection of k 1-representable complexes. (Follows from fact that $G = G_1 \cap \cdots \cap G_k$ iff $X(G) = X(G_1) \cap \cdots \cap X(G_k)$).

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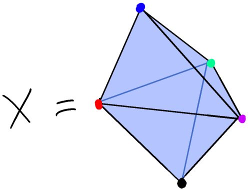

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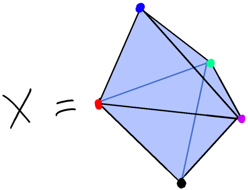

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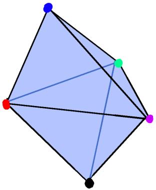
$X =$ 

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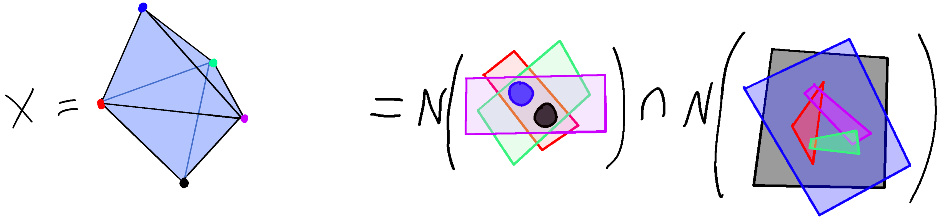

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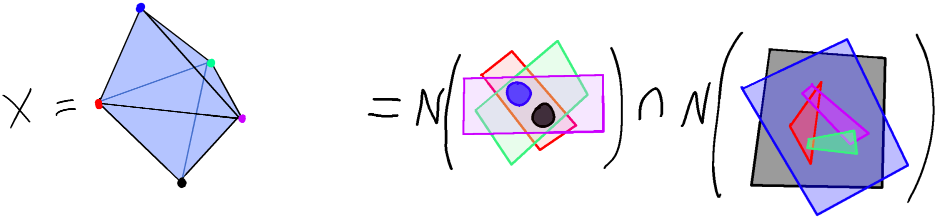

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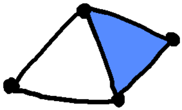

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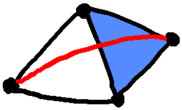

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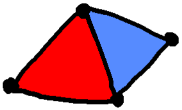

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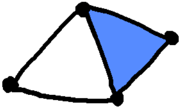

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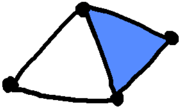

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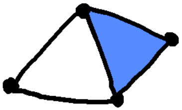

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• X is the clique complex of a graph $\iff$ $h(X) = 1$ (missing faces are the edges of the complement graph of G).

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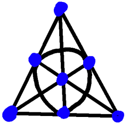

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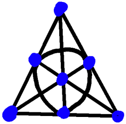

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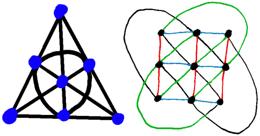

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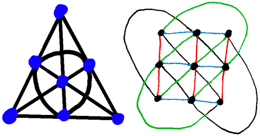

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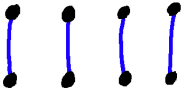

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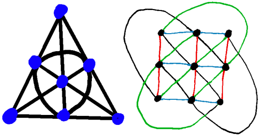

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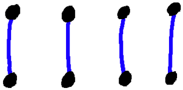

- Keevash ('14): For infinitely many values of n, Steiner (t, k, n)-systems exist.

Previously known results

Theorem (Witsenhausen '80):

Let X be a simplicial complex with n vertices satisfying $h(X) = d$. Then

$$\mathrm{box}_d(X) \leq \left\lfloor \frac{1}{2}\binom{n}{d} \right\rfloor .$$

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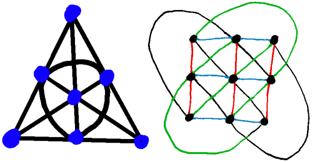

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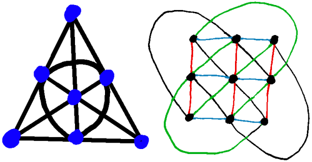

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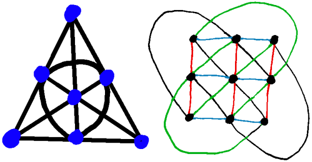

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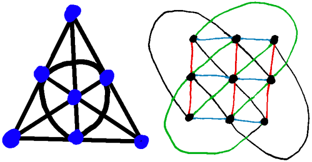

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- For $d \geq 2$, this improves previous bounds due to Witsenhausen.

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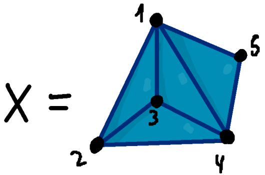

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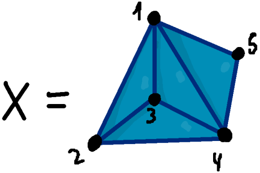

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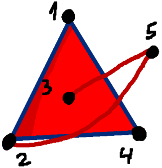

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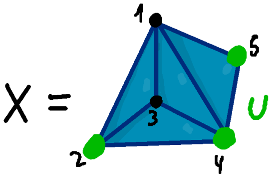
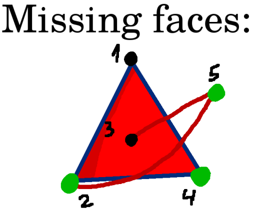


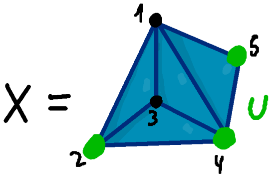


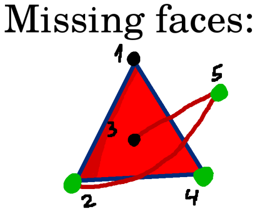
A bound on representability

A main ingredient in the proof of the bound $\mathrm{box}_d(X) \leq \left\lfloor \frac{1}{d+1}\binom{n}{d} \right\rfloor$ is the following result:

Theorem (L. '20):

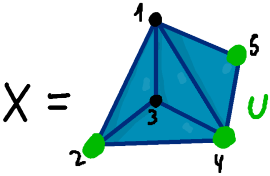
Let X be a simplicial complex on vertex set V. Let $U \subset V$ such that $U \notin X$, and for every missing face τ of X, $|\tau \setminus U| \leq 1$. Then, X is $(|U| - 1)$-representable.



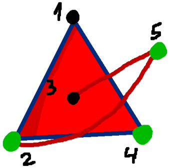


$\implies$ X is 2-representable.

A bound on representability– Example



Missing faces:



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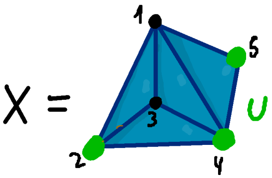

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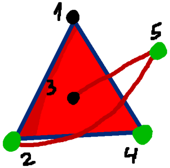

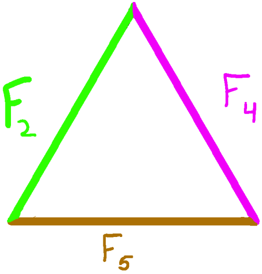

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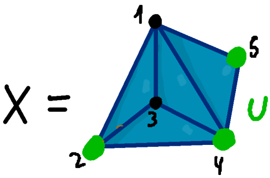

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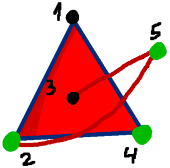

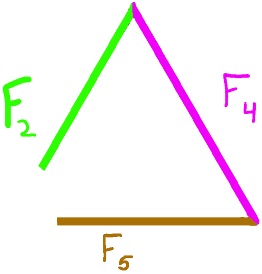

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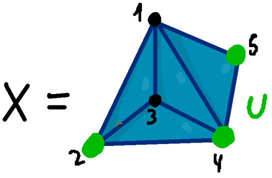

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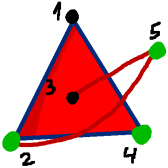

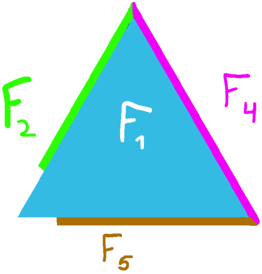

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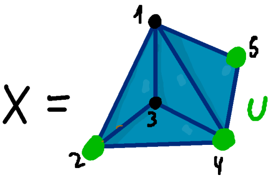

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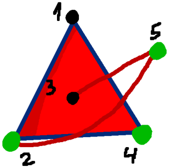

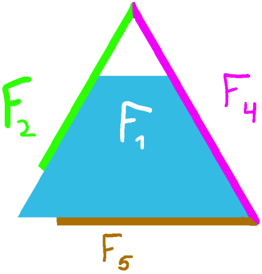

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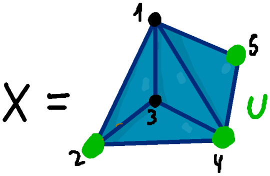

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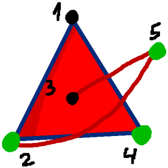

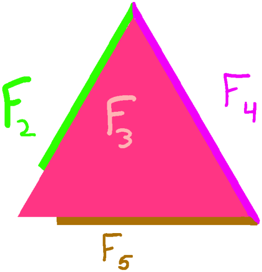

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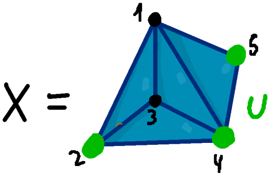

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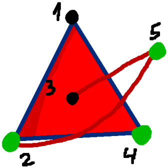

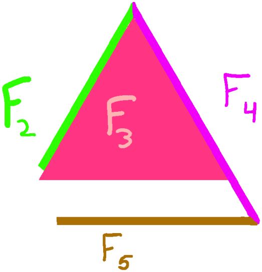

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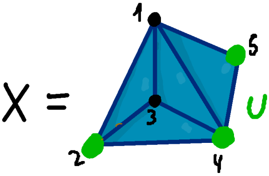

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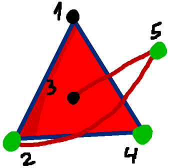

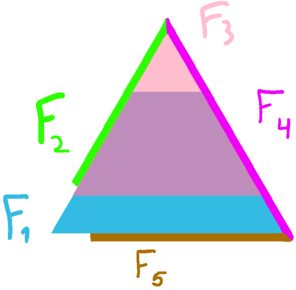

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Theorem (Wegner '67):

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Theorem (Wegner '67):

Let X be a simplicial complex with n vertices. Then X is $(n-1)$-representable.
Moreover, if X is not the boundary of an $(n-1)$-dimensional simplex, then it is $(n-2)$-representable.

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$\implies$ $\text{box}_d(X) = \frac{1}{d+1}\binom{n}{d}$.

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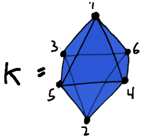

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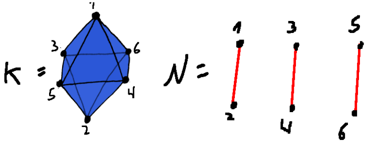

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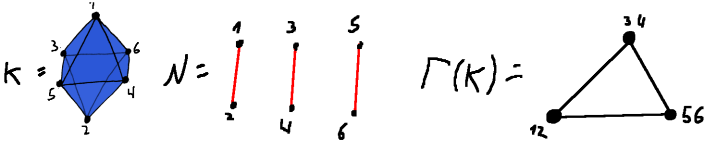

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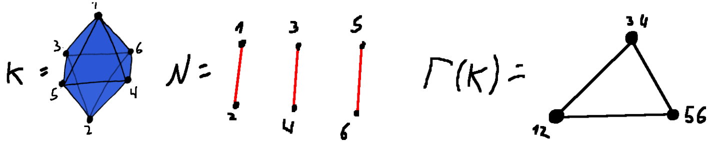

Theorem (Björner, Butler, Matveev '97):

If K is not the complete complex on W, then for all $j \geq 0$

$$H_j(K) \cong H_{|W|-j-3}(\Gamma(K)).$$

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$$|\tau_1 \cup \tau_2| - 3 = |\tau_1| + |\tau_2| - |\tau_1 \cap \tau_2| - 3 \geq (d+1) + (d+1) - (d-1) - 3 = d.$$

A contradiction to $H_j(Y) = 0$ for all $j \geq d$.

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What is the correct bound if $h(X) \leq d$ for some $d \geq 2$?

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Conjecture:

Let X be a simplicial complex on n vertices, with $h(X) \leq d$. Then

$$\mathrm{rep}(X) \leq \left\lfloor \frac{dn}{d+1} \right\rfloor .$$

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Moreover, $\mathrm{rep}(X) = \frac{dn}{d+1}$ if and only if the missing faces of X consist of $\frac{n}{d+1}$ pairwise disjoint sets of size $d+1$.

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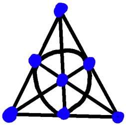

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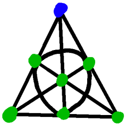

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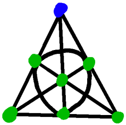

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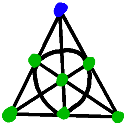

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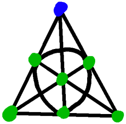

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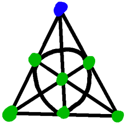

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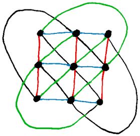

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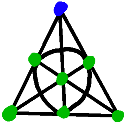

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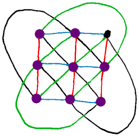

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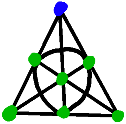

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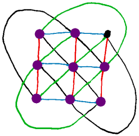

$$\implies \text{rep}(X) \leq 7$$

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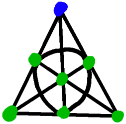

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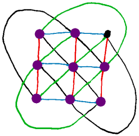

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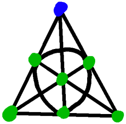

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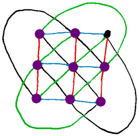

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Does rep$(X) \leq 5$ hold?

Thank you!