Effects of Periodic Homogenization in Phase Transition Problems

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joint work with Irene Fonseca and Riccardo Cristoferi

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Cahn–Hilliard, 1958

$\triangleright$ Equilibrium behavior of a fluid with two stable phases may be described by the Gibbs free energy per unit volume

$$E_\varepsilon(u) := \int_\Omega \left[W(u) + \varepsilon^2|\nabla u|^2\right] dx$$

where $\varepsilon > 0$ is a small parameter and $W : \mathbb{R} \to [0, +\infty)$ is a double well potential.

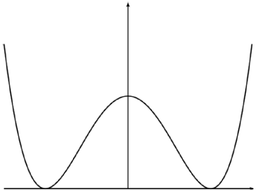

Figure: Example of double well potential $W(p) = (p^2 - 1)^2$.

Modica–Mortola, 1977

Asymptotic behavior of minimizers to E_ε described via Γ-convergence. Scaling by ε^{-1} yields

$$\varepsilon^{-1}E_\varepsilon \xrightarrow{\Gamma} E,$$

$$E(u) := \begin{cases} c_W \ P(A_0;\Omega) & u \in BV(\Omega;\{a,b\}) \\ +\infty & u \in L^1(\Omega)\setminus BV(\Omega;\{a,b\}) \end{cases}$$

where

$$A_0 = \{u(x) = a\},\ c_W = 2\int_a^b \sqrt{W(s)}ds.$$

Periodic Heterogeneity

We consider fluids which exhibit some periodic heterogeneity at small scales, i.e.

$$F_\varepsilon(u) := \int_\Omega \left[\frac{1}{\varepsilon} W\left(\frac{x}{\delta(\varepsilon)}, u\right) + \varepsilon |\nabla u|^2\right] dx$$

where

$$W(x,p) = 0 \iff p \in \{a,b\},$$

$$W(\cdot, p) \text{ is } Q\text{-periodic for every } p,$$

and

$$\delta(\varepsilon) \to 0 \text{ as } \varepsilon \to 0.$$

Goal: Identify Γ-limit F_ε.

Ansini, Braides, Piat (2003): W homogeneous, regularization $f\left(\frac{x}{\delta}, \nabla u\right)$

Scaling regime $\delta(\varepsilon) = \varepsilon$

Theorem (Cristoferi, Fonseca, H., Popovici)

Let $\delta(\varepsilon) = \varepsilon$. *Then* $F_\varepsilon \xrightarrow{\Gamma} F$,

$$F(u) := \begin{cases} \int_{\partial^* A_0} \sigma(\nu) d\mathcal{H}^{N-1} & u \in BV(\Omega; \{a, b\}) \\ +\infty & \textit{else} \end{cases}$$

where

$$A_0 := \{u(x) = a\}, \ \nu \textit{ is the outward normal to } A_0,$$

and

$$\sigma(\nu) := \lim_{T \to \infty} \inf_{u \in \mathcal{A}_{\nu,T}} \left\{ \frac{1}{T^{N-1}} \int_{TQ_\nu} \left[W(y, u(y)) + |\nabla u(y)|^2 \right] dy \right\}$$

Cell Problem

$$\sigma(\nu) = \lim_{T\to\infty} \inf_{u\in\mathcal{A}_{\nu,T}} \left\{ \frac{1}{T^{N-1}} \int_{TQ_\nu} \left[W(y,u(y)) + |\nabla u(y)|^2 \right] dy \right\}$$

where

$$\mathcal{A}_{\nu,T} := \left\{ u \in H^1(TQ_\nu;\mathbb{R}^d) : u(x) = (\rho_T * u_0)(x\cdot\nu) \text{ on } \partial TQ_\nu \right\}$$

$$u_0(t) := \begin{cases} b & \text{if } t > 0 \\ a & \text{if } t < 0 \end{cases}$$

$$\rho_T(x) := T^N \rho(Tx),\ \ \rho \in C_c^\infty(\mathbb{R}) \text{ with } \int_{\mathbb{R}} \rho = 1.$$

Outline of Proof

The Γ-limit Cookbook:

- Compactness: Bounded energy $\rightarrow BV$ structure
  - Reduction to classical MM technique
  - Lax growth conditions on $W(x,\cdot)$
  - Only need measurability of $W(\cdot,p)$.
- Γ-liminf: "Lower-semicontinuity" result using blow-up techniques
  - "Blow up" at points in jump set
  - De Giorgi's slicing method $\rightarrow$ prescribe boundary conditions from σ
  - Compare with optimal profiles given by σ
- Γ-limsup: Recovery sequences
  - Blow-Up Method
  - Recovery sequences for polyhedral sets with $\nu \in \mathbb{Q}^N \cap \mathbb{S}^{N-1}$
  - Density result and upper semicontinuity of σ

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Compactness

Growth condition: There exists $R > 0$ such, for a.e. $x \in Q$, $|p| > R$ implies

$$W(x,p) \geq \sup_{|q| \leq R} W(x,q) \tag{1}$$

Allows for truncation:

$$w(x) := \begin{cases} u(x) & |u(x)| \leq R \\ \frac{u(x)}{|u(x)|} R & |u(x)| > R \end{cases}$$

Since also

$$|\nabla w| \leq |\nabla u|,$$

we have

$$F_\varepsilon(w) \leq F_\varepsilon(u).$$

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▷ L^∞ control!

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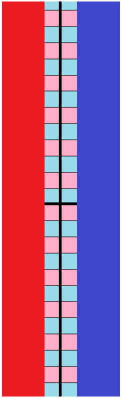
Easy Case: Transition Layer aligned with Principal Axes

If $\nu \in \{e_1, \ldots, e_N\}$, create recovery sequence by tiling optimal profiles from definition of σ.



Pick $T_k \subset \mathbb{N}$ and u_k s.t.

$$\sigma(e_N) = \lim_{k\to\infty} \frac{1}{T_k^{N-1}} \int_{T_k Q} W(y, u_k(y)) + |\nabla u_k(y)|^2 dy,$$

$$v_k(x) := u_k(T_k x), \text{ extended by } Q'\text{-periodicity},$$

$$v_{k,\varepsilon,r}(x) := \begin{cases} u_0(x) & |x_N| \geq \frac{\varepsilon T_k}{2r} \\ v_k\left(\frac{rx}{\varepsilon T_k}\right) & |x_N| < \frac{\varepsilon T_k}{2r} \end{cases}$$

$$u_{k,\varepsilon,r}(x) := v_{k,\varepsilon,r}\left(\frac{x}{r}\right) \to u \text{ in } L^1(rQ)$$

Transition Layer aligned with Principal Axes, cont.

Blow up:

$$\lim_{r\to 0}\frac{F(u;rQ)}{r^{N-1}} \leq \lim_{r\to 0}\lim_{\varepsilon\to 0}\frac{1}{r^{N-1}}\int_{rQ}\left[\frac{1}{\varepsilon}W(x,u_{k,\varepsilon,r})+\varepsilon|\nabla u_{k,\varepsilon,r}|^2\right]dx$$

$$= \lim_{r\to 0}\lim_{\varepsilon\to 0}\int_{Q'}\int_{-\varepsilon T_k/2r}^{\varepsilon T_k/2r}\left[\frac{r}{\varepsilon}W\left(\frac{r}{\varepsilon}y, v_k\left(\frac{ry}{\varepsilon T_k}\right)\right)\right.$$

$$\left. + \frac{r}{\varepsilon T_k^2}\left|\nabla v_k\left(\frac{ry}{\varepsilon T_k}\right)\right|^2\right]dy$$

$$= \lim_{r\to 0}\lim_{\varepsilon\to 0}\int_{Q'}\int_{-1/2}^{1/2}\left[T_k W\left(\left(T_k\frac{rz'}{\varepsilon T_k}, T_k z_N, v_k\left(\frac{rz'}{\varepsilon T_k}, z_N\right)\right)\right)\right.$$

$$\left. + \frac{1}{T_k}\left|\nabla v_k\left(\frac{rz'}{\varepsilon T_k}, z_N\right)\right|^2\right]dz$$

Transition Layer aligned with Principal Axes, cont.

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Since W and v_k are Q'-periodic and $T_k \in \mathbb{N}$, we can use the Riemann Lebesgue Lemma:

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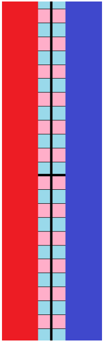
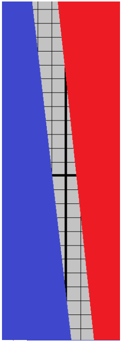
Other Transition Directions?



(a)
Aligned



(b)
Misaligned

Figure: Since W is Q-periodic, can tile along principal axes. What if the transition layer is not aligned?

Q-periodic implies $\lambda_\nu Q_\nu$-periodic

Key observation: Periodic microstructure in principal directions $\rightarrow$ periodicity in other directions.

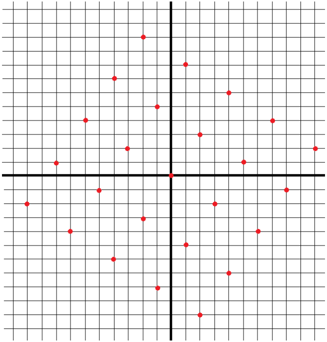


Figure: Integer lattice contains copies of itself, rotated and scaled

▷ W is $\lambda_\nu Q_\nu$-periodic for $\nu \in \mathbb{Q}^N \cap \mathbb{S}^{N-1}$: Dense!

Orthonormal Bases in $\mathbb{Q}^N$

Important: Every face of Q_ν has rational normal.

Need an orthonormal basis using rational vectors:

Theorem (Witt, '37)

Any isometry between two subspaces F_1 and F_2 of a finite-dimensional vector space V defined over a field $\mathbb{K}$ of characteristic different from 2 and provided with a metric structure induced from a nondegenerate symmetric or skew-symmetric bilinear form $B[\cdot,\cdot]$ may be extended to a metric automorphism of the entire space V.

In particular:

$$V = \mathbb{Q}^N, \; F_1 := \mathrm{span}_{\mathbb{Q}}(e_N), \; F_2 := \mathrm{span}_{\mathbb{Q}}(\nu), \; B[x,y] := x \cdot y$$

Then, the mapping $e_N \mapsto \nu$ extends to an isometry!

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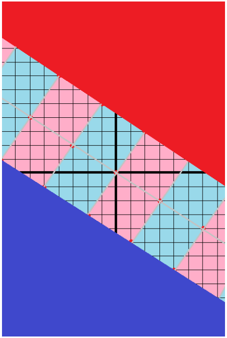
Transition Layer aligned with $\nu \in \mathbb{Q}^N \cap \mathbb{S}^{N-1}$

Same periodic tiling technique: Use $T_k \in \lambda_\nu \mathbb{N}$.



$\triangleright$ Blow up method $\rightarrow$ Recovery sequences for polyhedral sets A_0 with $\nu \in \mathbb{Q}^N \cap \mathbb{S}^{N-1}$.

Recovery sequences for arbitrary $u \in BV(\Omega; \{a, b\})$

- For $u \in BV(\Omega; \{a, b\})$, we can find $u^{(n)} \in BV(\Omega; \{a, b\})$ such that $A_0^{(n)}$ are polyhedral,
$$u^{(n)} \to u \text{ in } L^1$$
$$|Du^{(n)}|(\Omega) \to |Du|(\Omega).$$
Since $\mathbb{Q}^N \cap \mathbb{S}^{N-1}$ dense, can require $\nu^{(n)} \in \mathbb{Q}^N \cap \mathbb{S}^{N-1}$.
- Since σ upper-semicontinuous, by a theorem of Reshetnyak,
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- Find recovery sequences $u_\varepsilon^{(n)}$ for the $u^{(n)}$ so
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Since $\mathbb{Q}^N \cap \mathbb{S}^{N-1}$ dense, can require $\nu^{(n)} \in \mathbb{Q}^N \cap \mathbb{S}^{N-1}$.
- Since σ upper-semicontinuous, by a theorem of Reshetnyak,
$$\int_{\partial^* A_0} \sigma(\nu) d\mathcal{H}^{n-1} \leq \limsup_{n \to \infty} \int_{\partial^* A_0^{(n)}} \sigma\left(\nu^{(n)}\right) d\mathcal{H}^{n-1}$$
- Find recovery sequences $u_\varepsilon^{(n)}$ for the $u^{(n)}$ so
$$\int_{\partial^* A_0^{(n)}} \sigma\left(\nu^{(n)}\right) d\mathcal{H}^{n-1} \leq \limsup_{\varepsilon \to 0^+} F_\varepsilon\left(u_\varepsilon^{(n)}\right)$$
- Diagonalize!

Future problems

Current direction:

- Other scaling regimes $\delta(\varepsilon)$, forthcoming

Some interesting future directions:

- Problem of multiple wells.
- More general regularization terms, i.e. $|\nabla u|^2 \to f(x, u, \nabla u)$.
- Solid-solid phase transitions: $W\left(\frac{x}{\delta(\varepsilon)}, \nabla u(x)\right)$

Note: Solid-sold phase transitions without homogenization:

$$W(F) \approx |F|^p, \text{ Conti, Fonseca, Leoni, '02.}$$

$$W(F) \approx \text{dist}^p(F, SO(N)A \cup SO(N)B)$$

only studied for N=2 (Conti–Schweizer, '06)

Thank you for your attention!

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