21-114 Calculus for Architecture
SOLUTION OF TEST #2

Problem 1. (20 points)

1. **(5 points)** Differentiate the function $f(x) = x^4 - 2x^2 + 3$.
2. **(15 points)** Find all points on the curve $y = f(x) = x^4 - 2x^2 + 3$ where the tangent line is horizontal.

Solution:

1. $f'(x) = 4x^3 - 4x$
2. The tangent line is horizontal at the points x where $f'(x) = 0$. We have

$$\begin{aligned} f'(x) = 0 &\implies 4x(x^2 - 1) \\ &\implies x = 0 \quad \text{or} \quad x = \pm 1 \end{aligned}$$

- If $x = 0$ then $f(0) = 3$
- If $x = 1$ then $f(1) = 1 - 2 + 3 = 2$
- If $x = -1$ then $f(-1) = 1 - 2 + 3 = 2$

Hence, the points where the tangent line is horizontal are $(0, 3)$, $(1, 2)$ and $(-1, 2)$.

Problem 2. (15 points) Evaluate the derivatives of the following functions:

1. **(5 points)** $f(x) = (x^3 + x - 1)^{10}$

   Solution:

   $f'(x) = 10(3x^2 + 1)(x^3 + x - 1)^9$

2. **(5 points)** $g(x) = \frac{\sqrt{x}}{1+\sqrt{x}}$

   Solution:

$$\begin{aligned} g'(x) &= \frac{\frac{1}{2\sqrt{x}}(1+\sqrt{x}) - \frac{1}{2\sqrt{x}}\sqrt{x}}{(1+\sqrt{x})^2} \\ &= \frac{\frac{1}{2\sqrt{x}} + \frac{1}{2} - \frac{1}{2}}{(1+\sqrt{x})^2} \\ &= \frac{1}{2\sqrt{x}(1+\sqrt{x})^2} \end{aligned}$$

3. **(5 points)** $h(x) = x^2 \sin x$

   Solution:

   $h'(x) = 2x \sin x + x^2 \cos x$

Problem 3. (20 points)

1. **(15 points)** Find the slope of the tangent line to the curve

$$y = 2\tan x + \cos(2x)$$

   at $x = \frac{\pi}{4}$

2. **(5 points)** Write the equation of this tangent line.

Solution:

1. The slope of the tangent line is $\frac{dy}{dx}|_{x=\frac{\pi}{4}}$. First, the derivative is

$$\frac{dy}{dx} = 2\sec^2 x - 2\sin 2x$$

Then the slope is

$$\begin{aligned}\frac{dy}{dx}\Big|_{x=\frac{\pi}{4}} &= 2\sec^2\frac{\pi}{4} - 2\sin\frac{\pi}{2} \\ &= \frac{2}{\cos^2\frac{\pi}{4}} - 2.1 \\ &= \frac{2}{(\frac{\sqrt{2}}{2})^2} - 2 \\ &= 4 - 2 = 2\end{aligned}$$

2. The equation of this tangent line is

$$y - f(\frac{\pi}{4}) = 2(x - \frac{\pi}{4})$$

Since

$$f(\frac{\pi}{4}) = 2\tan\frac{\pi}{4} + \cos\frac{\pi}{2} = 2 \cdot 1 + 0 = 2$$

then the equation of the tangent line is

$$y - 2 = 2(x - \frac{\pi}{4}), \quad \text{that is} \quad y = 2x - \frac{\pi}{2} + 2$$

Problem 4. (20 points) Consider the function

$$f(x) = 2x^3 - 3x^2 - 12x + 1$$

1. **(10 points)** Find the intervals of increase or decrease of $f(x)$. Justify your answer.
2. **(10 points)** Find the local maximum and minimum of $f(x)$. Justify your answer.

Solution:

1. We need to find the intervals where the derivative of $f(x)$ is positive or negative.
The derivative of $f(x)$ is

$$f'(x) = 6x^2 - 6x - 12$$

The roots of the derivatives are

$$\begin{aligned} f'(x) = 0 &\implies 6(x^2 - x - 2) = 0 \\ &\implies (x-2)(x+1) = 0 \\ &\implies x = 2 \quad \text{or} \quad x = -1 \end{aligned}$$

- On the intervals $(-\infty, -1)$ and $(2, \infty)$, we have that $f'(x) > 0$. So $f(x)$ is increasing on these intervals.
- On the interval $(-1, 2)$, we have that $f'(x) < 0$. So $f(x)$ is decreasing on this interval.

2. The candidates for the local maximum and minimum of $f(x)$ are the points where $f'(x) = 0$, which are $x = -1$ and $x = 2$.

- At $x = -1$, we have that $f'(x)$ changes from a positive sign to a negative sign. So $f(x)$ has a local maximum at $x = -1$.
- At $x = 2$, we have that $f'(x)$ changes from a negative sign to a positive sign. So $f(x)$ has a local minimum at $x = 2$.

Problem 5. (25 points) Consider the function

$$f(x) = \frac{1}{4}x^2 + \cos x + 1, \quad 0 \leq x \leq 2\pi$$

1. **(15 points)** Find the intervals of concavity of $f(x)$. Justify your answer.
2. **(10 points)** Find the inflection points of $f(x)$. Justify your answer.

Solution:

1. We need to find the intervals where the second derivative of $f(x)$ is positive or negative. First, we compute the second derivative $f''(x)$ of $f(x)$.

$$f'(x) = \frac{1}{2}x - \sin x$$

and

$$f''(x) = \frac{1}{2} - \cos x$$

Next, we find the roots of $f''(x)$.

$$\begin{aligned} f''(x) = 0 &\implies \cos x = \frac{1}{2} \\ &\implies x = \frac{\pi}{3} \quad \text{or} \quad x = \frac{5\pi}{3} \end{aligned}$$

So

- On the intervals $[0, \frac{\pi}{3})$ and $(\frac{5\pi}{3}, 2\pi]$, we have that $\cos x > \frac{1}{2}$, that is $f''(x) < 0$. Hence $f(x)$ is concave downward on these intervals.
- On the interval $(\frac{\pi}{3}, \frac{5\pi}{3})$, we have that $\cos x < \frac{1}{2}$, that is $f''(x) > 0$. Hence $f(x)$ is concave upward on this interval.

2. At $x = \frac{\pi}{3}$ and $x = \frac{5\pi}{3}$, we have that $f''(x)$ changes signs. So the inflection points occur at $x = \frac{\pi}{3}$ and $x = \frac{5\pi}{3}$.

- If $x = \frac{\pi}{3}$, then $f(\frac{\pi}{3}) = \frac{1}{4}\frac{\pi^2}{9} + \frac{1}{2} + 1 = \frac{\pi^2}{36} + \frac{3}{2}$
- If $x = \frac{5\pi}{3}$, then $f(\frac{5\pi}{3}) = \frac{1}{4}\frac{25\pi^2}{9} + \frac{1}{2} + 1 = \frac{25\pi^2}{36} + \frac{3}{2}$

Therefore, the inflection points are $(\frac{\pi}{3}, \frac{\pi^2}{36} + \frac{3}{2})$ and $(\frac{5\pi}{3}, \frac{25\pi^2}{36} + \frac{3}{2})$.