21-114 Calculus for Architecture
SOLUTION OF TEST #1

Problem 1. (15 points) Find the domain of definition of the function

$$f(x) = \frac{\sqrt{x+4}}{x^2+x-6}$$

Write your answer as a union of intervals.

Solution:

The function $f(x)$ is defined for $x + 4 \geq 0$ and $x^2 + x - 6 \neq 0$.

$$x + 4 \geq 0 \Longrightarrow x \geq -4$$

$x^2 + x - 6 = 0 \Longrightarrow (x+3)(x-2) = 0$, that is, $x = -3$ or $x = 2$.
So the domain of f contains all x such that $x \geq -4$ and $x \neq -3$ and $x \neq 2$.
Writing it as a union of intervals, we get

$$[-4, -3) \cup (-3, 2) \cup (2, \infty)$$

Problem 2. (15 points) Show that the equation

$$x^2 + y^2 + 6x - 4y + 4 = 0$$

represents a circle, and find the center and radius. Sketch this circle.

Solution:

We need to rewrite this equation in the form $(x - x_0)^2 + (y - y_0)^2 = r^2$ which is an equation of the circle of center (x_0, y_0) and radius r. We have

$$\begin{aligned} x^2 + y^2 + 6x - 4y + 4 &= 0 \\ (x^2 + 6x) + (y^2 - 4y) + 4 &= 0 \\ \left[(x+3)^2 - 9\right] + \left[(y-2)^2 - 4\right] + 4 &= 0 \\ (x+3)^2 + (y-2)^2 &= 9 = 3^2 \end{aligned}$$

Therefore the equation represents the circle of center $(-3, 2)$ and radius 3.

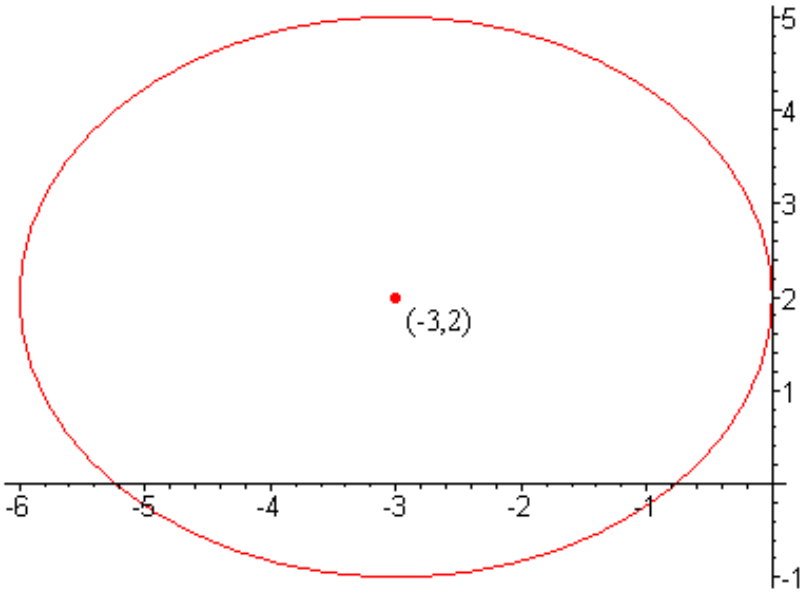


Figure 1: Sketch of the circle (note the axis labels).

Problem 3. (15 points) Sketch in the xy-plane the region

$$\{(x, y): \quad x - 1 \leq y < -x^2 + 2x + 1\}$$

Indicate clearly the region and its boundaries on your graphs. A boundary that is not included in the region should be represented in dashes.

Solution:

$y = x - 1$ is the line with slope 1 and y-intersect -1.
$y = -x^2 + 2x + 1$ is the parabola which is concave down (because the coefficient -1 of x^2 is negative) and has vertex $x = \frac{-2}{2(-1)} = 1$.
Because of the large inequality $x - 1 \leq y$, the line $y = x - 1$ is included in the region, and because of the strict inequality $y < -x^2 + 2x + 1$, the parabola $y = -x^2 + 2x + 1$ is not included in the region. Then we get the following region:

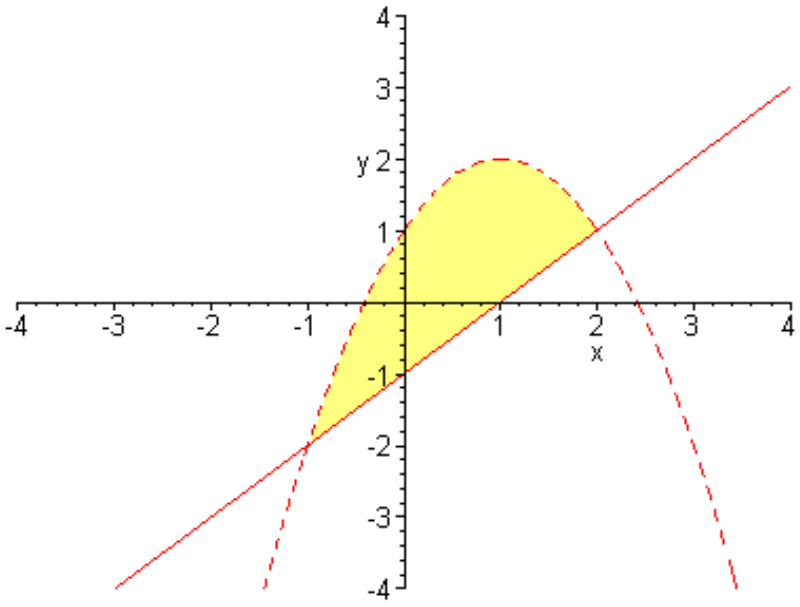


Figure 2: The enclosed region.

Problem 4. (20 points) Find all values of x in the interval $[0, 2\pi]$ that satisfy:

1. **(10 points)** $4\cos^2 x = 3$
2. **(10 points)** $2\sin x < \sqrt{3}$

Solution:

1) $4\cos^2 x = 3$ is equivalent to $\cos^2 x = 3/4$, that is, $\cos x = \pm\sqrt{3}/2$.

$$\cos x = \sqrt{3}/2 \Longrightarrow x = \frac{\pi}{6}, \frac{11\pi}{6}$$
$$\cos x = -\sqrt{3}/2 \Longrightarrow x = \frac{5\pi}{6}, \frac{7\pi}{6}$$

Then the values of x satisfying $4\cos^2 x = 3$ are $\frac{\pi}{6}$, $\frac{5\pi}{6}$, $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$.

2) $2\sin x < \sqrt{3}$ is equivalent to $\sin x < \sqrt{3}/2$. On the interval $[0, 2\pi]$, we have that x must be such that $0 \leq x < \frac{\pi}{3}$ or $\frac{2\pi}{3} < x \leq 2\pi$. Writing the solutions in terms of intervals, we obtain that x must be in $[0, \frac{\pi}{3}) \cup (\frac{2\pi}{3}, 2\pi]$.

Problem 5. (20 points) Find the vertical and horizontal asymptotes of the curve

$$y = f(x) = \frac{\sqrt{4x^2+1}}{x-1}$$

Justify your answer.

Solution:

First we find the domain of f. For that, we need $x-1 \neq 0$, that is, $x \neq 1$. Hence the domain of f is $(-\infty, 1) \cup (1, \infty)$.
The horizontal asymptotes are obtained by computing the limits of $f(x)$ at ∞ and $-\infty$.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\sqrt{4x^2+1}}{x-1} = \lim_{x\to\infty} \frac{\sqrt{4x^2}}{x} = \lim_{x\to\infty} \frac{2|x|}{x} = \lim_{x\to\infty} \frac{2x}{x} = 2$$

Hence $y = 2$ is a horizontal asymptote of the curve $y = f(x)$.

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{\sqrt{4x^2+1}}{x-1} = \lim_{x\to-\infty} \frac{\sqrt{4x^2}}{x} = \lim_{x\to-\infty} \frac{2|x|}{x} = \lim_{x\to-\infty} \frac{-2x}{x} = -2$$

Hence $y = -2$ is a horizontal asymptote of the curve $y = f(x)$.
For the vertical asymptotes, we compute the limits at the points x which are not included in the domain of $f(x)$. Here, there is only one point $x = 1$. Hence, we need to compute the right-hand limit of $f(x)$ at $x = 1$, or the left-hand limit of $f(x)$ at $x = 1$.

$$\lim_{x\to1^+} f(x) = \lim_{x\to1^+} \frac{\sqrt{4x^2+1}}{x-1} = \frac{\sqrt{5}}{0^+} = \infty$$

Hence $x = 1$ is a vertical asymptote of the curve $y = f(x)$.
We get the same vertical asymptote if we compute the left-hand limit of $f(x)$ at $x = 1$, that is,

$$\lim_{x\to1^-} f(x) = \lim_{x\to1^-} \frac{\sqrt{4x^2+1}}{x-1} = \frac{\sqrt{5}}{0^-} = -\infty$$

Problem 6. (15 points) Consider the function

$$f(x) = \begin{cases} \frac{\sqrt{x^2+9}-3}{x} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

1. (**10 points**) Use a limit of an appropriate function to find the slope of the tangent line to the curve $y = f(x)$ at $x = 0$.

2. (**5 points**) Write the equation of this tangent line to the curve $y = f(x)$ at $x = 0$.

Solution:

1) By definition, the slope of the tangent line to the curve $y = f(x)$ at $x = 0$ is

$$m = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$$

Here $f(0) = 0$ and $f(x) = \frac{\sqrt{x^2+9}-3}{x}$ for all $x \neq 0$, and in particular for $x \to 0$. Hence the slope is:

$$m = \lim_{x \to 0} \frac{\frac{\sqrt{x^2+9}-3}{x} - 0}{x - 0} = \lim_{x \to 0} \frac{\sqrt{x^2+9} - 3}{x^2}$$

Substituting $x = 0$ inside this expression, we get $\frac{0}{0}$ which is an indeterminate form. We use the conjugate of the numerator to rewrite this expression as:

$$\frac{\sqrt{x^2+9} - 3}{x^2} = \frac{\left(\sqrt{x^2+9} - 3\right)\left(\sqrt{x^2+9} + 3\right)}{x^2\left(\sqrt{x^2+9} + 3\right)} = \frac{x^2 + 9 - 9}{x^2\left(\sqrt{x^2+9} + 3\right)} = \frac{1}{\sqrt{x^2+9} + 3}$$

Therefore, the slope is

$$m = \lim_{x \to 0} \frac{1}{\sqrt{x^2+9} + 3} = \frac{1}{\sqrt{9} + 3} = \frac{1}{6}$$

2) By definition, an equation of the tangent line to the curve $y = f(x)$ at $x = 0$ is

$$y - f(0) = m(x - 0)$$

where m is the slope of the tangent line. Here $m = 1/6$ (from question (1)) and $f(0) = 0$. Then the equation of the tangent line is

$$y = \frac{1}{6}x$$