21-112 Calculus II:
Homework #9 Solutions

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§7.4 #4 Minimize $\frac{1}{2}x^2 - 3xy + y^2 + \frac{1}{2}$ subject to the constraint $3x - y - 1 = 0$.

A: Use the method of Lagrange Multipliers. In this case we have

$$\begin{aligned} f(x,y) &= \frac{1}{2}x^2 - 3xy + y^2 + \frac{1}{2} \\ g(x,y) &= 3x - y - 1. \end{aligned}$$

Now we need $\frac{\partial f}{\partial x} = \lambda\frac{\partial g}{\partial x}$, $\frac{\partial f}{\partial y} = \lambda\frac{\partial g}{\partial y}$, and $g(x,y) = 0$:

$$\begin{aligned} x - 3y &= 3\lambda \\ -3x + 2y &= -\lambda \\ 3x - y - 1 &= 0. \end{aligned}$$

Solving the first two equations for λ gives

$$\begin{aligned} \lambda &= \frac{1}{3}x - y \\ \lambda &= 3x - 2y, \end{aligned}$$

and equating these gives

$$\begin{aligned} \frac{1}{3}x - y &= 3x - 2y \\ \frac{8}{3}x &= y. \end{aligned}$$

Now plug this value into the third equation, which we will solve for x:

$$\begin{aligned} 3x - \frac{8}{3}x - 1 &= 0 \\ \frac{1}{3}x &= 1 \\ x &= 3 \end{aligned}$$

which means that

$$y = \frac{8}{3}(3) = 8.$$

Therefore we have a minimum at $(3, 8)$, and the minimum value is $f(3, 8) = \frac{1}{2}(3)^2 - 3(3)(8) + (8)^2 + \frac{1}{2} = -3$.

§7.4 #6 Find the values of x, y that minimize the function

$$x^2 + xy + y^2 - 2x - 5y,$$

subject to the constraint $1 - x + y = 0$.

A: Let $f(x, y) = x^2 + xy + y^2 - 2x - 5y$ and let $g(x, y) = 1 - x + y$. Then

$$\begin{aligned} 2x + y - 2 &= -\lambda \\ x + 2y - 5 &= \lambda \\ 1 - x + y &= 0. \end{aligned}$$

Solving the first two equations for λ gives

$$\begin{aligned} \lambda &= -2x - y + 2 \\ \lambda &= x + 2y - 5. \end{aligned}$$

Now we can equate these and solve for x in terms of y:

$$\begin{aligned} -2x - y + 2 &= x + 2y - 5 \\ 3x &= y + 7 \\ x &= \frac{y + 7}{3} \end{aligned}$$

and plug this result into the constraint (i.e. the third equation):

$$\begin{aligned} 1 - \left(\frac{y + 7}{3}\right) + y &= 0 \\ -\frac{4}{3} + \frac{2}{3}y &= 0 \\ y &= 2. \end{aligned}$$

Since $y = 2, x = \frac{2+7}{3} = 3$. Therefore the minimum occurs at $(3, 2)$.

§7.4 #10 The amount of space required by a particular firm is $f(x, y) = 1000\sqrt{6x^2 + y^2}$, where x and y are, respectively, the number of units of labor and capital utilized. Suppose that labor costs \$480 per unit and capital costs \$40 per unit and that the firm has \$5000 to spend. Determine the amounts of labor and capital that should be utilized in order to minimize the amount of space required.

A: We are given that $f(x,y) = 1000\sqrt{6x^2+y^2}$. We need to find $g(x,y)$. Recall that

$$Cost = (price_x)(quantity_x) + (price_y)(quantity_y).$$

So this means that, based on the information we are given, $C(x,y) = 480x + 40y$, and so the constraint is $C(x,y) = 5000$. But notice that the constraint is not set equal to 0 like we are used to. We can easily fix this by rewriting $C(x,y)$ as $C(x,y) = 480x + 40y - 5000$. So our three equations are

$$\begin{aligned} \frac{6000x}{\sqrt{6x^2+y^2}} &= 480\lambda \\ \frac{1000y}{\sqrt{6x^2+y^2}} &= 40\lambda \\ 480x + 40y - 5000 &= 0. \end{aligned}$$

We can solve the first two equations for λ and we get

$$\begin{aligned} \lambda &= \frac{25x}{2\sqrt{6x^2+y^2}} \\ \lambda &= \frac{25y}{\sqrt{6x^2+y^2}} \end{aligned}$$

which we equate and then solve for y in terms of x:

$$\begin{aligned} \frac{25x}{2\sqrt{6x^2+y^2}} &= \frac{25y}{\sqrt{6x^2+y^2}} \\ \frac{25}{2}x &= 25y \\ y &= \frac{1}{2}x, \end{aligned}$$

and plug this result into the constraint:

$$\begin{aligned} 480x + 40\left(\frac{1}{2}x\right) - 5000 &= 0 \\ 500x - 5000 &= 0 \\ 500x &= 5000 \\ x &= 10. \end{aligned}$$

Since $x = 10, y = \frac{1}{2}(10) = 5$. Therefore 10 units of labor and 5 units of capital should be utilized in order to minimize the amount of space required.

§7.4 #20 Find the values of x, y, z that minimize the function

$$x^2 + y^2 + z^2 - 3x - 5y - z,$$

subject to the constraint $20 - 2x - y - z = 0$.

A: Here, $f(x,y,z) = x^2 + y^2 + z^2 - 3x - 5y - z$ and $g(x,y,z) = 20 - 2x - y - z$. Then

$$\begin{aligned} 2x - 3 &= -2\lambda \\ 2y - 5 &= -\lambda \\ 2z - 1 &= -\lambda \\ 20 - 2x - y - z &= 0. \end{aligned}$$

Now we can take the first three equations, and solve each of them for x, y, and z, respectively:

$$\begin{aligned} 2x = 3 - 2\lambda \Rightarrow x &= \frac{3}{2} - \lambda \\ 2y = 5 - \lambda \Rightarrow y &= \frac{5}{2} - \frac{1}{2}\lambda \\ 2z = 1 - \lambda \Rightarrow z &= \frac{1}{2} - \frac{1}{2}\lambda. \end{aligned}$$

Notice that all three of these equations are in terms of λ only. So we can just plug all three results into the constraint and solve the constraint for λ :

$$\begin{aligned} 20 - 2\left(\frac{3}{2} - \lambda\right) - \left(\frac{5}{2} - \frac{\lambda}{2}\right) - \left(\frac{1}{2} - \frac{\lambda}{2}\right) &= 0 \\ 20 - 3 + 2\lambda - \frac{5}{2} + \frac{\lambda}{2} - \frac{1}{2} + \frac{\lambda}{2} &= 0 \\ 14 + 3\lambda &= 0 \\ \lambda &= -\frac{14}{3}. \end{aligned}$$

Now we can plug this value into the equations we got for x, y, and z:

$$\begin{aligned} x &= \frac{3}{2} + \frac{14}{3} = \frac{37}{6} \\ y &= \frac{5}{2} + \frac{14}{6} = \frac{29}{6} \\ z &= \frac{1}{2} + \frac{14}{6} = \frac{17}{6}. \end{aligned}$$

So we have a minimum at $(\frac{37}{6}, \frac{29}{6}, \frac{17}{6})$.

§7.4 #24 A shelter for use at the beach has a back, two sides, and a top made of canvas. See Figure 4(b) on page 419. Find the dimensions that maximize the volume and require 96 square feet of canvas.

A: We know that the volume of the shelter is $V(x,y,z) = xyz$. Now, the surface area of the canvas, based on the figure, is $A(x,y,z) = xy + 2yz + xz - 96$. So

$$yz = \lambda y + \lambda z$$

$$\begin{aligned} xz &= \lambda x + 2\lambda z \\ xy &= 2y\lambda + \lambda x \\ xy + 2yz + xz - 96 &= 0. \end{aligned}$$

Now we will solve the first two equations for λ:

$$\begin{aligned} \lambda &= \frac{yz}{y+z} \\ \lambda &= \frac{xz}{x+2z} \end{aligned}$$

and equate them, solving for x in terms of y:

$$\begin{aligned} \frac{yz}{y+z} &= \frac{xz}{x+2z} \\ yz(x+2z) &= xz(y+z) \\ yx + 2yz &= yx + xz \\ x &= 2y. \end{aligned}$$

Notice that all the z's fell out. Now we need to equate the second and third equations:

$$\begin{aligned} \lambda &= \frac{xz}{x+2z} \\ \lambda &= \frac{xy}{x+2y} \\ \frac{xz}{x+2z} &= \frac{xy}{x+2y} \\ xz(x+2y) &= xy(x+2z) \\ xz + 2yz &= xy + 2yz \\ z &= y. \end{aligned}$$

Now we can use these two results to substitute into the constraint, and solve for y:

$$\begin{aligned} (2y)y + 2y(y) + (2y)(y) - 96 &= 0 \\ 6y^2 - 96 &= 0 \\ y^2 - 16 &= 0 \\ y &= \pm 4. \end{aligned}$$

Because we cannot have negative dimensions, we throw out the solution $y = -4$. So $y = 4$. Then $x = 2(4) = 8$, and $z = 4$. So the volume is maximized when the dimensions are $(8, 4, 4)$.