21-112 Calculus II:
Homework #8 Solutions

Kevin C. Lee

December 2, 2004

Section 7.3

Problem 2:
$f(x, y) = 1/2x^2 + y^2 - 3x + 2y - 5$. Take the partial derivatives with respect to x and y and equate them to 0. We have:

$$\begin{aligned} \frac{\partial f}{\partial x} &= x - 3 = 0 \\ x &= 3 \\ \frac{\partial f}{\partial y} &= 2y + 2 = 0 \\ y &= -1 \end{aligned}$$

Hence, $(3, -1)$ is a possible relative minimum or maximum.

Problem 12:
$f(x, y) = 6xy^2 - 2x^3 - 3y^4$. We need to solve for $D(x, y)$ by finding the second derivatives of the function:

$$\frac{\partial f}{\partial x} = 6y^2 - 6x^2$$
$$\frac{\partial^2 f}{\partial x^2} = -12x$$
$$\frac{\partial f}{\partial y} = 12xy - 12y^3$$
$$\frac{\partial^2 f}{\partial y^2} = 12x - 36y^2$$
$$\frac{\partial^2 f}{\partial x \partial y} = 12y$$

Since $D(x,y) = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - (\frac{\partial^2 f}{\partial x \partial y})^2$, $D(x,y) = -12x(12x - 36y^2) - (12y)^2 = -144x^2 + 432xy^2 - 144y^2$.

At $(0,0)$, $D(0,0) = 0$. Therefore, the test is inconclusive.

At $(1,1)$, $D(1,1) = -144 + 432 - 144 > 0$. $\frac{\partial^2 f}{\partial x^2}(1,1) = -12 < 0$. Therefore, (1,1) is a relative maximum.

At $(1,-1)$, $D(1,1) = -144 + 432 - 144 > 0$. $\frac{\partial^2 f}{\partial x^2}(1,-1) = -12 < 0$. Therefore, (1,1) is a relative maximum.

Problem 26:

$f(x,y) = x^2 + 4xy + 2y^4$. We need to find the possible values by equating first derivatives to 0:

$$\frac{\partial f}{\partial x} = 2x + 4y = 0 \tag{1}$$
$$x = -2y \tag{2}$$
$$\frac{\partial f}{\partial y} = 4x + 8y^3 = 0 \tag{3}$$

Now substitute $x = -2y$ in (2) into (3), we get:

$$-8y + 8y^3 = 0$$
$$y^3 - y = 0$$
$$y(y^2 - 1) = 0$$
$$y(y-1)(y+1) = 0$$
$$y = 0, 1, -1$$

By (2), x = 0, -2, and 2, respectively for y = 0, 1, and -1. Therefore, the possible values are (0,0), (-2,1), and (2, -1).

Now we need to solve for $D(x, y)$ to use the second derivative test by finding the second derivatives of the function:

$$\frac{\partial^2 f}{\partial x^2} = 2$$
$$\frac{\partial^2 f}{\partial y^2} = 24y^2$$
$$\frac{\partial^2 f}{\partial x \partial y} = 4$$

Since $D(x, y) = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - (\frac{\partial^2 f}{\partial x \partial y})^2$, $D(x, y) = 48y^2 - 16$.

At $(0, 0)$, $D(0, 0) = 0$. Therefore, the test is inconclusive.

At $(-2, 1)$, $D(-2, 1) = 32 > 0$. $\frac{\partial^2 f}{\partial x^2}(-2, 1) = 2 > 0$. Therefore, (-2,1) is a relative minimum.

At $(2, -1)$, $D(2, -1) = 32 > 0$. $\frac{\partial^2 f}{\partial x^2}(2, -1) = 2 > 0$. Therefore, (2,-1) is a relative minimum.

Problem 28:

$f(x, y, z) = 5 + 8x - 4y + x^2 + y^2 + z^2$. Take the partial derivatives with respect to x and y and equate them to 0. We have:

$$\frac{\partial f}{\partial x} = 8 + 2x = 0$$
$$x = -4$$
$$\frac{\partial f}{\partial y} = -4 + 2y = 0$$
$$y = 2$$
$$\frac{\partial f}{\partial z} = 2z = 0$$
$$z = 0$$

Hence, $(-4, 2, 0)$ is a possible relative minimum or maximum.

Problem 30:

Assume the box has width x, length y, and height z. The problem asks us to minimize surface area:

$$SA = 2xy + 2yz + 2zx \tag{4}$$

while satisfying the constraint that is:

$$xyz = 1000 \tag{5}$$

We can write z in terms of x and y in (5) as $z = 1000/xy$ and substitute it into (4). We get:

$$SA = 2xy + 2000z/x + 2000z/y \tag{6}$$

We need to find the possible values by equating first derivatives to 0:

$$\frac{\partial SA}{\partial x} = 2y - 2000/x^2 = 0 \tag{7}$$
$$x^2y = 1000 \tag{8}$$
$$y = 1000/x^2 \tag{9}$$
$$\frac{\partial SA}{\partial y} = 2x - 2000/y^2 - 0 \tag{10}$$
$$xy^2 = 1000 \tag{11}$$

Substitute y in terms of x using (9) in (11), we get:

$$x \cdot \frac{1000^2}{x^4} = 1000$$
$$x^3 = 1000$$
$$x = 10$$

By (9), $y = 10$. By the constraint function of $xyz = 1000$, $z = 10$. Since there is only one candidate value for the rectangular box, (10, 10, 10) must be the lengths of the box that minimize the surface area subject to the constraint.

Problem 32: Since $Profit = Revenue - Cost$, Profit is equal to:

$$P = 98x + 112y - 0.04xy - 0.1x^2 - 0.2y^2 - (30x + 20y)$$

We need to find the possible values by equating first derivatives to 0:

$$\frac{\partial P}{\partial x} = 98 - 0.04y - 0.2x - 30 = 68 - 0.04y - 0.2x = 0 \tag{12}$$

$$\frac{\partial f}{\partial y} = 112 - 0.04x - 0.4y - 20 = 92 - 0.04x - 0.4y = 0 \tag{13}$$

$$EQ(12)/-0.04 \Rightarrow 5x + y - 1700 = 0 \tag{14}$$

$$EQ(13)/-0.4 \Rightarrow 0.1x + y - 230 - 0 \tag{15}$$

Subtract (15) from (14), we get $4.9x = 1470$. Hence, $x = 300$. By (14), $y = 1700 - 1500 = 200$. Since there is only one candidate value, (300, 200) is the unit pair that the monopolist needs to produce to maximize profits.