21-112 Calculus II:
Homework #6 Solutions

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§8.3 #26 Differentiate $\frac{e^t}{\cos 2t}$ with respect to x.

A: Use the Quotient Rule and the Chain Rule:

$$\begin{aligned}\frac{d}{dt}\left(\frac{e^t}{\cos 2t}\right) &= \frac{e^t \cos 2t - e^t(-2\sin 2t)}{\cos^2 2t} \\ &= \frac{e^t(\cos 2t + 2\sin 2t)}{\cos^2 2t}\end{aligned}$$

§8.3 #39 Suppose that a person's blood pressure P at time t (in seconds) is given by $P = 100 + 20\cos 6t$.

(a) Find the maximum value of P (called the systolic pressure) and the minimum value of P (called the diastolic pressure) and give one or two values of t where these maximum and minimum values of P occur.

A:

Method 1: We must first take the derivative of P with respect to time:

$$\frac{dP}{dt} = -20\sin 6t(6) = -120\sin 6t$$

Then we set it equal to 0 and solve for t:

$$-120\sin 6t = 0 \Rightarrow 6t = 0 + 2\pi k, 6t = \pi + 2\pi k \Rightarrow t = \frac{k\pi}{3}, \frac{\pi}{6} + \frac{k\pi}{3}$$

Now plug these values into P and compare:

$$t = 0 \Rightarrow P = 100 + 20\cos(6(0)) = 100 + 20(1) = 120$$

$$t = \frac{\pi}{6} \Rightarrow P = 100 + 20\cos\left(6\left(\frac{\pi}{6}\right)\right) = 100 + 20(-1) = 80$$

We see here that the maximum value is 120 (occurring at $t = 0, \frac{\pi}{3}, \frac{2\pi}{3}$, etc.) and the minimum value is 80 (occurring at $t = \frac{\pi}{6}, \frac{3\pi}{6}, \frac{5\pi}{6}$, etc.).

Method 2: Notice that $-1 \leq \cos 6t \leq 1$. Using this fact, we know that the maximum value is attained when $\cos 6t = 1$, and the minimum value is attained when $\cos 6t = -1$. Based on this, the maximum value is $100 + 20(1) = 120$, and the minimum value is $100 + 20(-1) = 80$. To find the points where the maximum value occurs, we know that $\cos 6t = 1 \Rightarrow 6t = 0 + 2\pi k \Rightarrow t = k\frac{\pi}{3}$. To find the points where the minimum value occurs, we use the fact that $\cos 6t = -1 \Rightarrow 6t = \pi + 2\pi k \Rightarrow t = \frac{\pi}{6} + k\frac{\pi}{3}$.

(b) If time is measured in seconds, approximately how many heartbeats per minute are predicted by equation for P?

A: The question is asking us to find the number of maxima in the interval $0 < t \leq 60$. From part (a), we know that all maxima are attained at $t = k\frac{\pi}{3}$. So we get

$$\begin{aligned} 0 &< \frac{k\pi}{3} \leq 60 \\ 0 &< k\pi \leq 180 \\ 0 &< k \leq \frac{180}{\pi} \\ 0 &< k \leq 57.29578 \end{aligned}$$

But since k is an integer, it can only take on integer values from 1 to 57. So the equation predicts approximately 57 beats per minute.

§8.3 #44 The number of hours of daylight per day in Washington, D.C. t weeks after the beginning of the year is $f(t) = 12.18 + 2.725 \sin[\frac{2\pi}{52}(t - 12)]$. The graph of this function is sketched in Figure 13 on page 469.

(a) How many hours of daylight are there after 42 weeks?

A: Reading from the graph of $y = f(t)$, we find that $f(42) \approx 11$ hours. Or, using a calculator, we can rigorously evaluate $f(42)$:

$$\begin{aligned} f(42) &= 12.18 + 2.725 \sin\left[\frac{2\pi}{52}(42 - 12)\right] \\ &= 12.18 + 2.725 \sin\left(\frac{15\pi}{13}\right) \\ &\approx 10.91363 \; hours \end{aligned}$$

(b) After 32 weeks, how fast is the number of hours of daylight decreasing?

A: Reading the graph of $y = f'(t)$, we get $f'(32) \approx -0.25$. Or, we can evaluate $f'(32)$ rigorously using a calculator by taking the derivative and evaluating it at 32:

$$\begin{aligned}
\frac{df}{dt} = f'(t) &= 2.725 \cos\left[\frac{2\pi}{52}(t-12)\right]\left(\frac{2\pi}{52}\right) \\
&= \frac{2\pi}{52}(2.725)\cos\left[\frac{2\pi}{52}(t-12)\right] \\
f'(32) &= \frac{2\pi}{52}(2.725)\cos\left[\frac{2\pi}{52}(32-12)\right] \\
&= \frac{2\pi}{52}(2.725)\cos\left[\frac{10\pi}{13}\right] \\
&\approx -0.24646
\end{aligned}$$

The number of hours of daylight is decreasing by 0.24646 hours per week.

(c) When is there 14 hours of daylight per day?

A: Read it off from the graph. Or, evaluate rigorously by setting $f(x)$ equal to 14:

$$\begin{aligned}
12.18 + 2.725 \sin\left[\frac{2\pi}{52}(t-12)\right] &= 14 \\
\sin\left[\frac{2\pi}{52}(42-12)\right] &= \frac{14-12.18}{2.725}
\end{aligned}$$

We need a calculator to solve this. After plugging it in, we get 2 values for t: $t = 18.05286$ and $t = 31.94714$ (weeks into the year).

(d) When is the number of hours of daylight increasing at the rate of 15 minutes per week?

A: Read it off from the graph. Or evaluate rigorously. This is similar to (c), except we set the derivative equal to $\frac{1}{4}$, since we need to convert to hours:

$$\begin{aligned}
\frac{2\pi}{52}(2.725)\cos\left[\frac{2\pi}{52}(t-12)\right] &= \frac{1}{4} \\
\cos\left[\frac{2\pi}{52}(t-12)\right] &= \frac{\frac{1}{4}}{\frac{2\pi}{52}(2.725)}
\end{aligned}$$

Again, using a calculator, we find that there are 2 values of t that satisfies this equation: $t = 6.13556$ and $t = 17.86445$ (weeks into the year).

(e) When are the days longest? Shortest?

A: Here, we need to find the points where the maximum and minimum values occur. So we set the derivative equal to zero and either read it off from the graph of $f'(t)$ or solve for t using a calculator:

$$\begin{aligned}\frac{2\pi}{52}(2.725)\cos\left[\frac{2\pi}{52}(t-12)\right] &= 0\\ \cos\left[\frac{2\pi}{52}(t-12)\right] &= 0\\ t &= 25, 51.\end{aligned}$$

Plug these values into $f(t)$ to figure out where the maximum and minimum occurs:

$$\begin{aligned}t = 25 \Rightarrow f(25) &= 12.18 + 2.725\sin\left[\frac{2\pi}{52}(25-12)\right]\\ &= 14.905\ hours\\ t = 51 \Rightarrow f(51) &= 12.18 + 2.725\sin\left[\frac{2\pi}{52}(51-12)\right]\\ &= 9.455\ hours\end{aligned}$$

Therefore the days are longest 25 weeks into the year and shortest 51 weeks into the year.

(f) When is the number of hours of daylight increasing fastest? Decreasing fastest?

A: Now we are looking for the points where the maximum and minimum values occur in the derivative (also called the *points of inflection*). This requires us to find $f''(t)$ and set it equal to 0. We can either read this off from the graph of $f''(t)$ or calculate it rigorously:

$$\begin{aligned}f'(t) &= \frac{2\pi}{52}(2.725)\cos\left[\frac{2\pi}{52}(t-12)\right]\\ f''(t) &= \frac{2\pi}{52}(2.725)\left(-\sin\left[\frac{2\pi}{52}(t-12)\right]\right)\left(\frac{2\pi}{52}\right)\\ &= -\left(\frac{2\pi}{52}\right)^2(2.725)\sin\left[\frac{2\pi}{52}(t-12)\right]\\ 0 &= -\left(\frac{2\pi}{52}\right)^2(2.725)\sin\left[\frac{2\pi}{52}(t-12)\right]\\ 0 &= \sin\left[\frac{2\pi}{52}(t-12)\right]\\ t &= 12, 38\end{aligned}$$

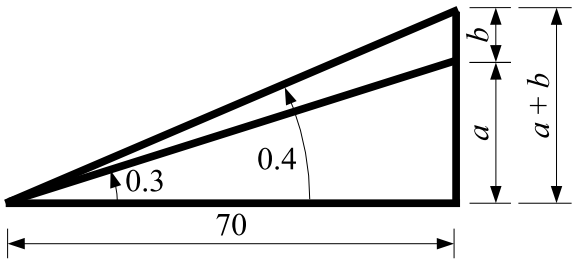
Now plug these back into $f'(t)$:

$$t = 12 \Rightarrow f'(12) = \frac{2\pi}{52}(2.725)\cos\left[\frac{2\pi}{52}(12-12)\right]$$

$$
\begin{aligned}
&= \frac{2\pi}{52}(2.725) \\
&\approx 0.32926 \\
t = 38 \Rightarrow f'(38) &= \frac{2\pi}{52}(2.725)\cos\left[\frac{2\pi}{52}(38-12)\right] \\
&= -\frac{2\pi}{52}(2.725) \\
&\approx -0.32926
\end{aligned}
$$

The number of hours of daylight is increasing fastest 12 weeks into the year, and decreasing the fastest 38 weeks into the year.

§8.4 #12 The angle of elevation from an observer to the top of a church is .3 radian, while the angle of elevation from the observer to the top of the church spire is .4 radian. If the observer is 70 meters from the church, how tall is the spire on top of the church?



A: Let a be the height of the church, and let b be the height of the spire on top of the church. So the height of the church and spire is $a+b$. In this problem, we need to solve for b. From the problem, we know that the observer is 70 meters away, which is exactly the length of the adjacent side. We also know that

$$
\begin{aligned}
\tan(.3) &= \frac{opp}{adj} = \frac{a}{70} \\
\tan(.4) &= \frac{opp}{adj} = \frac{a+b}{70}.
\end{aligned}
$$

We can solve these equations for a and $a+b$:

$$
\begin{aligned}
70\tan(.3) &= a \qquad (1) \\
70\tan(.4) &= a+b \qquad (2)
\end{aligned}
$$

Plugging the value for a in Equation (1) into (2), and solving for b, we get

$$
\begin{aligned}
70\tan(.4) &= 70\tan(.3) + b \\
b &= 70\tan(.4) - 70\tan(.3)\ meters.
\end{aligned}
$$

§8.4 #24 Differentiate $y = 2\tan\sqrt{x^2-4}$ with respect to x.

A: Use the Chain Rule:

$$\begin{aligned}\frac{dy}{dx} &= 2\sec^2\sqrt{x^2-4}\left(\frac{1}{2}(x^2-4)^{-\frac{1}{2}}(2x)\right)\\ &= \frac{2x}{\sqrt{x^2-4}}\sec^2\sqrt{x^2-4}\end{aligned}$$

§8.4 #26 Differentiate $y = e^{3x}\tan 2x$ with respect to x.

A: Use the Product Rule:

$$\begin{aligned}\frac{dy}{dx} &= e^{3x}(\sec^2 2x(2)) + (\tan 2x)(e^{3x}(3))\\ &= 4xe^{3x}\sec^2 2x + 3e^{3x}\tan 2x\end{aligned}$$

§8.4 #32 Differentiate $y = \ln(\tan t)$ with respect to t.

A: Use the Chain Rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\tan t}(\sec^2 t)\\ &= \frac{\sec^2 t}{\tan t}\end{aligned}$$

§8.4 #36 Evaluate $\int \sec^2(2x+1)\,dx$.

A: Use u-substiution, letting $u = 2x+1$. Then $du = 2\,dx$, and so

$$\begin{aligned}\int \sec^2(2x+1)\,dx &= \frac{1}{2}\int \sec^2 u\,du\\ &= \frac{1}{2}\tan u + C\\ &= \frac{1}{2}\tan(2x+1) + C.\end{aligned}$$

§8.4 #40 Evaluate $\int \frac{3}{\cos^2 2x}\,dx$.

A: The first thing we need to do is to use the trigonometric identity $\frac{1}{\cos\theta} = \sec\theta$. Therefore

$$\int \frac{3}{\cos^2 2x}\,dx = \int 3\sec^2 2x\,dx.$$

Now make a u-substitution, letting $u = 2x$, and $du = 2\,dx$. So

$$\begin{aligned}
\int 3\sec^2 2x\,dx &= \frac{1}{2}\int 3\sec^2 u\,du \\
&= \frac{3}{2}\int \sec^2 u\,du \\
&= \frac{3}{2}\tan u + C \\
&= \frac{3}{2}\tan 2x + C.
\end{aligned}$$