21-114: Calculus for Architecture
Homework #5 Solutions

December 9, 2004
Mike Picollelli

§4.3 #14. Let

$$f(x) = \frac{x^2}{x^2+3}.$$

(a) Find the intervals on which f is increasing and decreasing.

Solution: To do this, we note that f is increasing exactly where $f'(x) > 0$, and f is decreasing exactly where $f'(x) < 0$, so we first calculate $f'(x)$:

$$f'(x) = \frac{(x^2+3)(2x) - (x^2)(2x)}{(x^2+3)^2} = \frac{6x}{(x^2+3)^2}$$

by the Quotient Rule. Since its denominator is positive for all x, $f'(x)$ has the same sign as its numerator, $6x$. Therefore

$$f'(x) > 0 \iff 6x > 0 \iff x > \frac{0}{6} = 0,$$

and hence f is increasing on the interval $(0, \infty)$. Similarly,

$$f'(x) < 0 \iff 6x < 0 \iff x < \frac{0}{6} = 0,$$

so f is decreasing on the interval $(-\infty, 0)$.

(b) Find the local maximum and minimum values of f.

Solution: Local extrema occur at points where $f'(x) = 0$:

$$f'(x) = \frac{6x}{(x^2+3)^2} = 0 \iff 6x = 0 \iff x = 0,$$

so the only possible local maximum/minimum occurs at $x = 0$. However, from (a) we know that $f'(x) < 0$ for $x < 0$ and $f'(x) > 0$ for $x > 0$, so by the First Derivative Test, f has a local minimum at $x = 0$, of value $f(0) = 0$, and f doesn't have a local maximum.

(c) Find the intervals of concavity and the inflection points.

Solution: To find the intervals of concavity and the inflection points, we need to consider $f''(x)$:

$$\begin{aligned} f''(x) &= \frac{d}{dx}\left(\frac{6x}{(x^2+3)^2}\right) \\ &= \frac{(x^2+3)^2\frac{d}{dx}(6x) - (6x)\frac{d}{dx}((x^2+3)^2)}{((x^2+3)^2)^2} \\ &= \frac{(x^2+3)^2(6) - (6x)(2(x^2+3)(2x))}{(x^2+3)^4} \\ &= \frac{(x^2+3)(6) - (6x)(2)(2x)}{(x^2+3)^3} \\ &= \frac{18-18x^2}{(x^2+3)^3}. \end{aligned}$$

First, we find the intervals of concavity: by the Concavity Test, f is concave upward on the intervals where $f''(x) > 0$: since the denominator of $f''(x)$ is strictly positive,

$$f''(x) > 0 \iff 18 - 18x^2 > 0 \iff x^2 < 1 \iff |x| < 1,$$

so f is concave upwards on the interval $(-1, 1)$. The Concavity Test also shows that f is concave downward on the intervals where $f''(x)$ is strictly negative, so

$$f''(x) < 0 \iff 18 - 18x^2 < 0 \iff x^2 > 1 \iff |x| > 1,$$

so f is concave downward on the intervals $(-\infty, -1)$ and $(1, \infty)$.

Finally, f has an inflection point wherever the concavity changes direction: since f is concave downward on $(-\infty, -1)$, concave upward on $(-1, 1)$, and concave downward again on $(1, \infty)$, f has two inflection points:

$$(-1, f(-1)) = \left(-1, \frac{1}{4}\right)$$

and

$$(1, f(1)) = \left(1, \frac{1}{4}\right).$$

§4.3 #46. Let

$$f(x) = \frac{x^2}{(x-2)^2}$$

(a) Find the vertical and horizontal asymptotes.

Solution: The vertical asymptotes correspond to values of a where $\lim_{x\to a^+} f(x)$ or $\lim_{x\to a^-} f(x)$ is $\pm\infty$. Since f is rational, they can only exist where the denominator, $(x-2)^2$ is 0, which occurs at $x = 2$. As $x \to 2$, $(x-2) \to 0$, so $(x-2)^2 \to 0^+$, and hence

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2}{(x-2)^2} = \frac{2^2}{0^+} = +\infty,$$

so f has a vertical asymptote at $x = 2$.

Horizontal asymptotes are determined by taking limits at $\pm\infty$:

$$\lim_{x\to-\infty} \frac{x^2}{(x-2)^2} = \lim_{x\to-\infty} \frac{x^2}{x^2-4x+4} = \lim_{x\to-\infty} \frac{x^2}{x^2} = 1,$$

and

$$\lim_{x\to\infty} \frac{x^2}{(x-2)^2} = \lim_{x\to\infty} \frac{x^2}{x^2} = 1,$$

so f has a horizontal asymptote at $y = 1$.

(b) Find the intervals of increase or decrease.

Solution: The intervals of increase or decrease correspond to when $f'(x) > 0$ and $f'(x) < 0$ respectively:

$$f'(x) = \frac{(x-2)^2(2x) - (x^2)(2(x-2))}{((x-2)^2)^2} = \frac{(x-2)(2x) - (x^2)(2)}{(x-2)^3} = \frac{-4x}{(x-2)^3}.$$

We have $f'(x) > 0$ when either $-4x > 0$ and $(x-2)^3 > 0$ or $-4x < 0$ and $(x-2)^3 < 0$. The first condition corresponds to $x < 0$ and $x > 2$, an impossibility. The second condition corresponds to $x > 0$ and $x < 2$, so $f'(x) > 0$ only occurs on the interval $(0, 2)$.

We have $f'(x) < 0$ when either $-4x > 0$ and $(x-2)^3 < 0$ or $-4x < 0$ and $(x-2)^3 > 0$. The first condition corresponds to $x < 0$ and $x < 2$, so $f'(x) < 0$ on $(-\infty, 2)$. The second condition corresponds to $x > 0$ and $x > 2$, so $f'(x) < 0$ on $(2, \infty)$.

Therefore the interval of increase is $(0, 2)$, and the intervals of decrease are $(-\infty, 0)$ and $(2, \infty)$.

(c) Find the local maximum or minimum values.

Solution: The local maximum and minimum values can only occur where $f'(x) = 0$:

$$0 = \frac{-4x}{(x-2)^3} \iff 0 = -4x \iff x = 0,$$

and we note that by part (b) $f(x)$ is decreasing on $(-\infty, 0)$ and increasing on $(0, 2)$, so $x = 0$ corresponds to a local minimum with value $f(0) = 0$ by the First Derivative Test. There is no local maximum.

(d) Find the intervals of concavity and the inflection points.

Solution: To find intervals of concavity, we need the second derivative:

$$f''(x) = \frac{(x-2)^3(-4) - (-4x)(3(x-2)^2)}{((x-2)^3)^2} = \frac{(x-2)(-4) - (-4x)(3)}{(x-2)^4} = \frac{8x+8}{(x-2)^4}.$$

By the Concavity Test, $f(x)$ is concave upward on intervals where $f''(x) > 0$, and since $(x-2)^4 > 0$ for $x \neq 2$, that means that $f'(x) > 0 \iff (8x+8) > 0 \iff x > -1$, so f is concave upward on the intervals $(-1, 2)$ and $(2, \infty)$.

Similarly, $f(x)$ is concave downward on intervals where $f''(x) < 0$, which occurs only when $8x + 8 < 0$, or $x < -1$, so $f(x)$ is concave downward on $(-\infty, -1)$.

Finally, from the above we see that the only value of x which corresponds to an inflection point is $x = -1$, since $f(x)$ is concave downward on $(-\infty, -1)$ and concave upward on $(-1, 2)$, so the only inflection point is $(-1, f(1)) = (-1, 1/9)$.

(e) Use the information from (a)-(d) to sketch the graph of f.

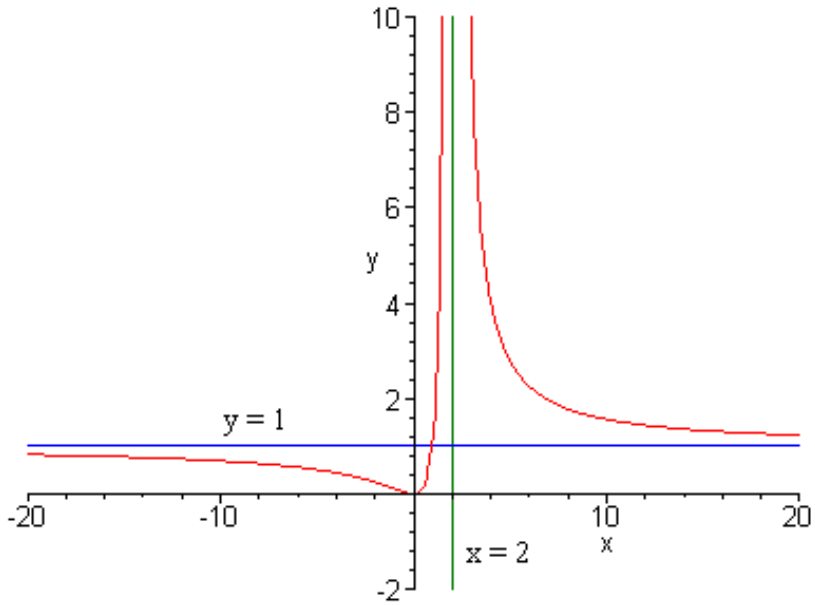


Figure 1: Graph of $f(x)$.

§4.5 #2. Sketch the curve $y = x^3 + 6x^2 + 9x$.

Solution: A. The domain: since $y(x)$ is a polynomial, the domain is $\mathbb{R}$.

B. The intercepts: the y-intercept is $y(0) = 0$, and the x-intercepts occur when $y = 0$:

$$0 = x^3 + 6x^2 + 9x = x(x+3)^2,$$

so $x = 0$ and $x = -3$.

C. Symmetry: since y is a polynomial with both even and odd powers of x, it is neither even nor odd, and is not periodic, so there is no useful symmetry.

D. Asymptotes: Since the domain is $\mathbb{R}$ and y is continuous, there are no vertical asymptotes (since $\lim_{x\to a} y(x) = y(a)$, which is never $\pm\infty$, for every a). We then check for horizontal asymptotes:

$$\lim_{x\to-\infty} y(x) = \lim_{x\to-\infty} (x^3 + 6x^2 + 9x) = \lim_{x\to-\infty} x^3 = -\infty,$$

and

$$\lim_{x\to\infty} y(x) = \lim_{x\to\infty} x^3 = \infty,$$

so y has no horizontal asymptotes.

E. Intervals of increase/decrease: $y'(x) = 3x^2 + 12x + 9 = 3(x+1)(x+3)$. Since $y'(x)$ is a quadratic with positive leading coefficient and roots $x = -3, -1$, so $y'(x) > 0$ on the intervals $(-\infty, -3)$ and $(-1, \infty)$, so y is increasing on $(-\infty, -3)$ and $(-1, \infty)$. Similarly, $y'(x) < 0$ only on $(-3, -1)$, so y is decreasing on $(-3, -1)$.

F. Local maximum and minimum values: $y'(x) = 0$ only when $x = -3, -1$: y is increasing on $(-\infty, -3)$ and decreasing on $(-3, -1)$ and increasing on $(-1, \infty)$, so by the First Derivative Test, $x = -3$ corresponds to a local maximum with value $y(-3) = 0$ and $x = -1$ corresponds to a local minimum with value $y(-1) = -4$.

G. Concavity and inflection points: $y''(x) = 6x + 12$, so therefore $y''(x) < 0$ when $x < -2$ and $y''(x) > 0$ when $x > -2$, and hence y is concave downward on $(-\infty, -2)$ and concave upward on $(-2, \infty)$. Since $y''(x)$ goes from negative to positive, the point $(-2, y(-2)) = (-2, -2)$ is the only inflection point.

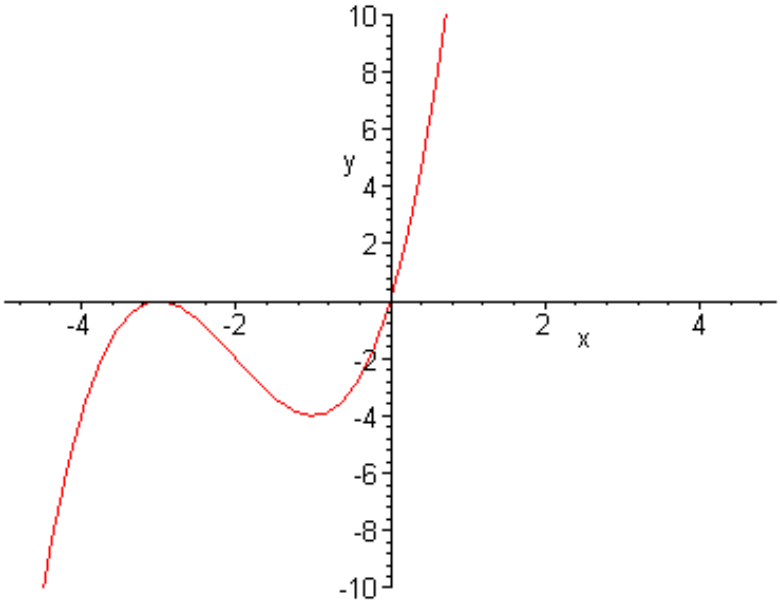


Figure 2: Graph of $y(x)$.

§4.5 #10. Sketch the curve

$$y = \frac{x}{(x-1)^2}.$$

Solution: A. The domain: Since y is rational, the domain is everywhere except where the denominator is 0: $(x-1)^2 = 0 \iff x = 1$, so the domain is $(-\infty, 1) \cup (1, \infty)$.

B. The intercepts: The y-intercept is $y = 0$, and the x-intercept is clearly $x = 0$.

C. Symmetry: Since the function is rational, it's not periodic, and since it involves both even and odd coefficients of x, it's nether even nor odd, so $y(x)$ is not symmetric.

D. Asymptotes: Since the domain is $(-\infty, 1) \cup (1, \infty)$, we take the one-sided limits of each of the endpoints:

$$\lim_{x\to-\infty} \frac{x}{(x-1)^2} = \lim_{x\to-\infty} \frac{x}{x^2} = \lim_{x\to-\infty} \frac{1}{x} = 0,$$

$$\lim_{x\to1^-} \frac{x}{(x-1)^2} = \frac{1}{0^+} = +\infty,$$

$$\lim_{x\to1^+} \frac{x}{(x-1)^2} = \frac{1}{0^+} = +\infty,$$

$$\lim_{x\to\infty} \frac{x}{(x-1)^2} = \lim_{x\to\infty} \frac{x}{x^2} = 0,$$

and we see that $y(x)$ has a horizontal asymptote at $y = 0$ and vertical asymptote $x = 1$.

E. Intervals of increase/decrease:

$$y'(x) = \frac{(x-1)^2(1) - x(2(x-1))}{((x-1)^2)^2} = \frac{-(x+1)}{(x-1)^3}$$

First, we find the intervals of increase: they correspond to values in the domain where $y'(x) > 0$, which can only happen if either $-(x+1) > 0$ and $(x-1)^3 > 0$ or $-(x+1) < 0$ and $(x-1)^3 < 0$. In the former case, we get $x < -1$ and $x > 1$, which is impossible. In the latter, we get $x > -1$ and $x < 1$, so $y(x)$ is increasing on $(-1, 1)$.

The intervals of decrease correspond to $y'(x) < 0$: either $-(x+1) > 0$ and $(x-1)^3 < 0$, in which case $x < -1$ and $x < 1$, or $-(x+1) < 0$ and $(x-1)^3 > 0$, in which case $x > -1$ and $x > 1$. Thus, we see the intervals of decrease are $(-\infty, -1)$ and $(1, \infty)$.

F. Local maximum and minimum values: the only solution to $y'(x) = 0$ is $x = -1$, and since y goes from decreasing to increasing, it is a local minimum with value $y(-1) = -1/4$. There is no local maximum.

G. Concavity and inflection points:

$$y''(x) = \frac{(x-1)^3(-1) - (-(x+1))(3(x-1)^2)}{((x-1)^3)^2} = \frac{2x+4}{(x-1)^4},$$

and since its denominator is positive in the domain, that means the sign of $y''(x)$ is the sign of its numerator $2x + 4$. Hence $y''(x) < 0$ when $x < -2$, $y''(x) > 0$ when $x > -2$, and we get that y is concave downward on $(-\infty, -2)$, and y is concave upward on the intervals $(-2, 1)$ and $(1, \infty)$. Since the concavity changes at $x = -2$, y has an inflection point at $(-2, y(-2)) = (-2, -2/9)$.

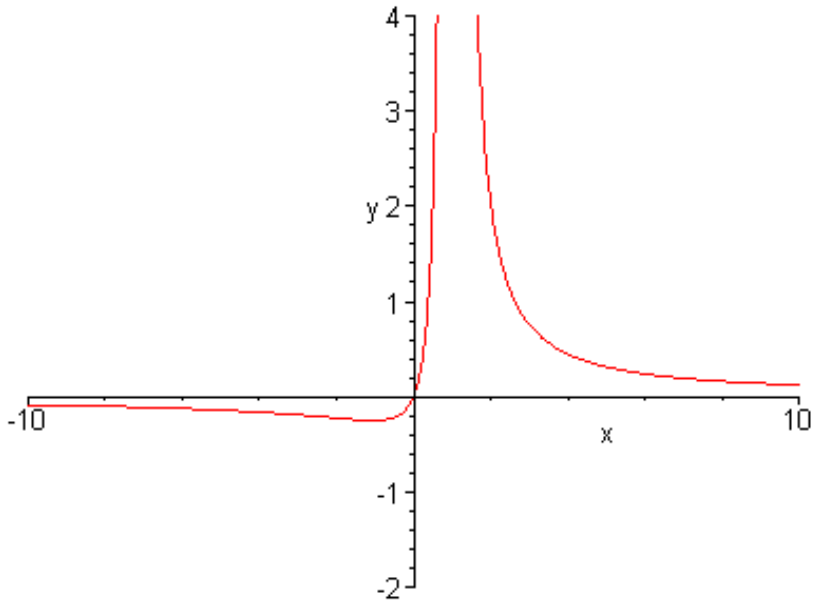


Figure 3: Graph of $y(x)$.