21-114: Calculus for Architecture
Homework #4 Solutions

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§3.4 #10. Differentiate

$$y = \frac{1+\sin x}{x+\cos x}.$$

Solution:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{1+\sin x}{x+\cos x}\right) \\
&= \frac{(x+\cos x)\frac{d}{dx}(1+\sin x) - (1+\sin x)\frac{d}{dx}(x+\cos x)}{(x+\cos x)^2} \text{ (Quotient Rule)} \\
&= \frac{(x+\cos x)(\cos x) - (1+\sin x)(1-\sin x)}{(x+\cos x)^2} \\
&= \frac{x\cos x + \cos^2 x - 1 + \sin^2 x}{(x+\cos x)^2} \\
&= \frac{x\cos x}{(x+\cos x)^2}.
\end{aligned}$$

§3.4 #16. Differentiate $y = x\sin x\cos x$.

Solution:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}(x\sin x\cos x) \\
&= (x\sin x)\frac{d}{dx}(\cos x) + (\cos x)\frac{d}{dx}(x\sin x) \text{ (Product Rule)} \\
&= (x\sin x)(-\sin x) + (\cos x)\left(x\frac{d}{dx}(\sin x) + (\sin x)\frac{d}{dx}(x)\right) \text{ (Product Rule)} \\
&= (x\sin x)(-\sin x) + (\cos x)\left(x(\cos x) + (\sin x)(1)\right) \\
&= -x\sin^2 x + x\cos^2 x + \sin x\cos x \\
&= x(\cos^2 x - \sin^2 x) + \sin x\cos x \\
&= x\cos 2x + \frac{1}{2}\sin 2x.
\end{aligned}$$

§3.4 #22. Find an equation of the tangent line to the curve $y = e^x \cos x$ at the point $(0, 1)$.

Solution: Since we have the point, we need the slope of the tangent line at $x = 0$:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left(e^x \cos x\right) \\
&= (e^x)\frac{d}{dx}(\cos x) + (\cos x)\frac{d}{dx}(e^x) \text{ (Product Rule)} \\
&= (e^x)(-\sin x) + (\cos x)(e^x) \\
&= e^x(\cos x - \sin x).
\end{aligned}$$

Therefore the slope of the line is

$$\left.\frac{dy}{dx}\right|_{x=0} = e^0(\cos(0) - \sin(0)) = 1,$$

and hence the tangent line is $f(x) = 1(x - 0) + 1 = x + 1$.

§3.4 #30. Find the points on the curve

$$y = \frac{\cos x}{2 + \sin x}$$

at which the tangent is horizontal.

Solution: The tangent is horizontal exactly when the derivative is 0, so we first must differentiate:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{\cos x}{2 + \sin x}\right) \\
&= \frac{(2 + \sin x)\frac{d}{dx}(\cos x) - (\cos x)\frac{d}{dx}(2 + \sin x)}{(2 + \sin x)^2} \text{ (Quotient Rule)} \\
&= \frac{(2 + \sin x)(-\sin x) - (\cos x)(\cos x)}{(2 + \sin x)^2} \\
&= \frac{-2\sin x - \sin^2 x - \cos^2 x}{(2 + \sin x)^2} \\
&= \frac{-2\sin x - 1}{(2 + \sin x)^2}.
\end{aligned}$$

Setting $\frac{dy}{dx} = 0$, we note that $(2+\sin x)^2 \neq 0$, so $-2\sin x - 1 = 0$, and hence $\sin x = -\frac{1}{2}$. In the interval $[0, 2\pi)$, the only solutions are $x = \frac{7\pi}{6}, \frac{11\pi}{6}$. Additionally, at $x = \frac{7\pi}{6}$,

$$\frac{\cos\frac{7\pi}{6}}{2 + \sin\frac{7\pi}{6}} = \frac{-\sqrt{3}/2}{3/2} = -\frac{1}{\sqrt{3}},$$

and at $x = \frac{11\pi}{6}$,

$$\frac{\cos\frac{11\pi}{6}}{2+\sin\frac{11\pi}{6}} = \frac{\sqrt{3}/2}{3/2} = \frac{1}{\sqrt{3}}.$$

Since both $\sin(x)$ and y are periodic with periods 2π, the set of all points on the curve where the tangent is horizontal is

$$\left\{\left(\frac{7\pi}{6}+k(2\pi),-\frac{1}{\sqrt{3}}\right) : k \text{ is an integer}\right\} \bigcup \left\{\left(\frac{11\pi}{6}+k(2\pi),\frac{1}{\sqrt{3}}\right) : k \text{ is an integer}\right\}.$$

§3.5 #26. Differentiate

$$G(y) = \frac{(y-1)^4}{(y^2+2y)^5}.$$

Solution: We can write $f(y) = (y-1)^4$ and $g(y) = (y^2+2y)^5$, so $G(y) = f(y)/g(y)$. Hence, we must apply the Quotient Rule, which first requires us to calculate $f'(y), g'(y)$:

By the Chain Rule,

$$\frac{d}{dy}(f(y)) = \frac{d}{dy}((y-1)^4) = 4(y-1)^3\frac{d}{dy}(y-1) = 4(y-1)^3,$$

and

$$\frac{d}{dy}(g(y)) = \frac{d}{dy}((y^2+2y)^5) = 5(y^2+2y)^4\frac{d}{dy}(y^2+2y) = 5(y^2+2y)^4(2y+2)$$

$$\begin{aligned}
\frac{d}{dy}(G(y)) &= \frac{d}{dy}\left(\frac{(y-1)^4}{(y^2+2y)^5}\right) \\
&= \frac{(y^2+2y)^5\frac{d}{dy}((y-1)^4) - (y-1)^4\frac{d}{dy}((y^2+2y)^5)}{((y^2+2y)^5)^2} \\
&= \frac{(y^2+2y)^5(4(y-1)^3) - (y-1)^4 5(y^2+2y)^4(2y+2)}{(y^2+2y)^{10}} \\
&= \frac{(y^2+2y)4(y-1)^3 - (y-1)^4 5(2y+2)}{(y^2+2y)^6} \\
&= \frac{(y-1)^3((y^2+2y)(4) - (y-1)5(2y+2))}{(y^2+2y)^6} \\
&= \frac{(y-1)^3(-6y^2+8y+10)}{(y^2+2y)^6}.
\end{aligned}$$

§3.5 #34. Differentiate

$$y = x \sin \frac{1}{x}.$$

Solution:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left(x \sin \frac{1}{x}\right) \\
&= x\frac{d}{dx}\left(\sin \frac{1}{x}\right) + \left(\sin \frac{1}{x}\right)\frac{d}{dx}(x) \quad \text{(Product Rule)} \\
&= x\left(\left(\cos \frac{1}{x}\right)\frac{d}{dx}\left(\frac{1}{x}\right)\right) + \sin \frac{1}{x} \quad \text{(Chain Rule)} \\
&= x\left(\left(\cos \frac{1}{x}\right)\left(\frac{-1}{x^2}\right)\right) + \sin \frac{1}{x} \\
&= \frac{-\cos(1/x)}{x} + \sin \frac{1}{x}.
\end{aligned}$$

§3.5 #52. Find the x-coordinates of all points on the curve $y = \sin(2x) - 2\sin x$ at which the tangent line is horizontal.

Solution: The tangent line is horizontal when $\frac{dy}{dx} = 0$, so we calculate the derivative:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}(\sin 2x) - \frac{d}{dx}(2\sin x) \\
&= (\cos 2x)\frac{d}{dx}(2x) - (2\cos x) \text{ (Chain Rule)} \\
&= 2\cos 2x - 2\cos x.
\end{aligned}$$

Setting this equal to 0 and dividing by 2, we get $\cos 2x = \cos x$. But $\cos a = \cos b$ can happen if and only if either $a = b + 2k\pi$ or $a = -b + 2k\pi$, where k is some integer (negative, positive or 0). From the first equation, we get that $2x = x + 2k\pi$, so $x = 2k\pi$. Otherwise, from the second equation, we get that $2x = -x + 2k\pi$, so $x = 2k\pi/3$.