21-112 Calculus II:
Homework #4 Solutions

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§6.6 #18 Determine $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$ by making an appropriate substitution.

A: In this case the appropriate subsitution to make is $u = \sqrt{x} = x^{\frac{1}{2}}$. Then $du = \frac{1}{2}x^{-\frac{1}{2}}\,dx$. The issue here is that the denominator and du differ by a constant. But we can re-write the integral as follows:

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \int x^{-\frac{1}{2}}e^{x^{\frac{1}{2}}}\,dx = \int 2\left(\frac{1}{2}\right)x^{-\frac{1}{2}}e^{x^{\frac{1}{2}}}\,dx,$$

and then substitute:

$$\begin{aligned}\int 2\left(\frac{1}{2}\right)x^{-\frac{1}{2}}e^{x^{\frac{1}{2}}}\,dx &= \int 2e^u\,du \\ &= 2\int e^u\,du \\ &= 2e^u + C \\ &= 2e^{\sqrt{x}} + C.\end{aligned}$$

§6.6 #28 Use integration by parts to determine $\int x^3 \ln x\,dx$.

A: We first need to define $f(x), g(x), G(x)$, and $f'(x)$:

$$\begin{aligned}f(x) &= \ln x \\ f'(x) &= \frac{1}{x} \\ g(x) &= x^3 \\ G(x) &= \frac{x^4}{4}.\end{aligned}$$

Now we invoke the formula for integration by parts found on page 362, and substitute:

$$\begin{aligned}\int x^3 \ln x \, dx &= \left(\frac{x^4}{4}\right) \ln x - \int \left(\frac{1}{x}\right)\left(\frac{x^4}{4}\right) dx \\ &= \frac{x^4 \ln x}{4} - \int \frac{x^3}{4} \, dx \\ &= \frac{x^4 \ln x}{4} - \frac{x^4}{16} + C.\end{aligned}$$

§6.6 #36 Determine $\int \frac{2x-1}{3x^2-3x+1} \, dx$.

A: Use u-substitution. Let $u = 3x^2 - 3x + 1$. Then

$$\begin{aligned}du &= 6x - 3 \, dx \\ &= 3(2x - 1) \, dx \\ \frac{du}{3} &= 2x - 1 \, dx\end{aligned}$$

Now substitute:

$$\begin{aligned}\int \frac{2x-1}{3x^2-3x+1} \, dx &= \frac{1}{3} \int \frac{1}{u} \, du \\ &= \frac{1}{3} \ln |u| + C \\ &= \frac{1}{3} \ln |3x^2 - 3x + 1| + C.\end{aligned}$$

§6.6 #44 Determine $\int \frac{(\ln x)^4}{x} \, dx$.

A: Again, use u-substitution, letting $u = \ln x$. Then $du = \frac{1}{x} \, dx$. Now substitute:

$$\begin{aligned}\int \frac{(\ln x)^4}{x} \, dx &= \int u^4 \, du \\ &= \frac{u^5}{5} + C \\ &= \frac{(\ln x)^5}{5} + C.\end{aligned}$$

§6.6 #46 Determine $\int \sqrt{x} \ln \sqrt{x} \, dx$.

A: Use integration by parts, since u-substitution will not work here.

$$\begin{aligned} f(x) &= \ln \sqrt{x} \\ f'(x) &= \frac{1}{2x} \\ g(x) &= \sqrt{x} \\ G(x) &= \frac{2x^{\frac{3}{2}}}{3}. \end{aligned}$$

Invoking the integration by parts formula, we have

$$\begin{aligned} \int \sqrt{x} \ln \sqrt{x}\, dx &= \frac{2x^{\frac{3}{2}} \ln \sqrt{x}}{3} - \int \frac{2x^{\frac{3}{2}}}{6x}\, dx \\ &= \frac{2x^{\frac{3}{2}} \ln \sqrt{x}}{3} - \int \frac{\sqrt{x}}{3}\, dx \\ &= \frac{2x^{\frac{3}{2}} \ln \sqrt{x}}{3} - \frac{2x^{\frac{3}{2}}}{9} + C \end{aligned}$$

§6.6 #48 Figure 2 (on page 366) shows several functions $f(x)$ whose slope at each x is $(2\sqrt{x}+1)/\sqrt{x}$. Find the expression for the function $f(x)$ whose graph passes through $(4, 15)$.

A: From the given information in the problem, we know that

$$f'(x) = \frac{2\sqrt{x}+1}{\sqrt{x}}.$$

So we'll have to integrate $f'(x)$. We can use u-substitution for this, letting $u = 2\sqrt{x}+1$. Then $du = \frac{1}{\sqrt{x}}\, dx$:

$$\begin{aligned} \int \frac{2\sqrt{x}+1}{\sqrt{x}}\, dx &= \int u\, du \\ &= \frac{u^2}{2} + C \\ &= \frac{(2\sqrt{x}+1)^2}{2} + C. \end{aligned}$$

We also know that $f(4) = 15$, and so

$$15 = \frac{(2\sqrt{4}+1)^2}{2} + C$$

$$15 = \frac{25}{2} + C$$
$$\frac{30}{2} = \frac{25}{2} + C$$
$$C = \frac{5}{2}.$$

So our desired function is $f(x) = \frac{(2\sqrt{x}+1)^2}{2} + \frac{5}{2}$.

§6.7 #34 Evaluate $\int_2^\infty \frac{1}{x \ln x}\, dx$ if it is convergent.

A: By definition,

$$\int_2^\infty \frac{1}{x \ln x}\, dx = \lim_{a\to\infty} \int_2^a \frac{1}{x \ln x}\, dx.$$

Now, we compute $\int \frac{1}{x \ln x}\, dx$ using u-substitution, with $u = \ln x$ and $du = \frac{1}{x}\, dx$:

$$\begin{aligned}\int \frac{1}{x \ln x}\, dx &= \int \frac{1}{u}\, du \\ &= \ln|u| + C = \ln|\ln|x|| + C.\end{aligned}$$

Now we work with the definite integral:

$$\begin{aligned}\int_2^\infty \frac{1}{x \ln x}\, dx &= \lim_{a\to\infty} \ln|\ln x| \Big]_2^a \\ &= \lim_{a\to\infty} (\ln|\ln a| - \ln|\ln 2|) \\ &= \infty - \ln|\ln 2| = \infty.\end{aligned}$$

So the integral is divergent.

§6.7 #44 Evaluate $\int_{-\infty}^\infty \frac{e^{-x}}{(e^{-x}+2)^2}\, dx$ if it is convergent.

A: The first thing we need to do is compute $\int \frac{e^{-x}}{(e^{-x}+2)^2}\, dx$. We can use u-substitution here, letting $u = e^{-x} + 2$, and $du = -e^{-x}\, dx \Rightarrow -du = e^{-x}\, dx$:

$$\int \frac{e^{-x}}{(e^{-x}+2)^2}\, dx = \int \frac{-1}{u^2}\, du = \frac{1}{u} + C = \frac{1}{e^{-x}+2} + C.$$

Since we have 2 infinite limits, we will have to split the integral at $x = 0$. Then, one piece at a time, let the lower limit $a \to -\infty$, and the upper limit $b \to \infty$, and add the results together:

$$\int_{-\infty}^\infty \frac{e^{-x}}{(e^{-x}+2)^2}\, dx = \lim_{a\to-\infty} \int_a^0 \frac{e^{-x}}{(e^{-x}+2)^2}\, dx + \lim_{b\to\infty} \int_0^b \frac{e^{-x}}{(e^{-x}+2)^2}\, dx$$

Using the information we found already, we know that

$$\begin{aligned}\lim_{a\to-\infty}\int_a^0 \frac{e^{-x}}{(e^{-x}+2)^2}\,dx &= \lim_{a\to-\infty}\frac{1}{e^{-x}+2}\Bigg]_a^0\\ &= \lim_{a\to-\infty}\left(\frac{1}{3}-\frac{1}{e^{-a}+2}\right)\\ &= \frac{1}{3}.\end{aligned}$$

And now for the second limit:

$$\begin{aligned}\lim_{b\to\infty}\int_0^b \frac{e^{-x}}{(e^{-x}+2)^2}\,dx &= \lim_{b\to\infty}\frac{1}{e^{-x}+2}\Bigg]_0^b\\ &= \lim_{b\to\infty}\left(\frac{1}{e^{-b}+2}-\frac{1}{3}\right)\\ &= \frac{1}{2}-\frac{1}{3}.\end{aligned}$$

Now add the results together and we get $\frac{1}{3}+\frac{1}{2}-\frac{1}{3}=\frac{1}{2}$. Since $\int_{-\infty}^{\infty}\frac{e^{-x}}{(e^{-x}+2)^2}\,dx=\frac{1}{2}$, the integral is convergent.

§6.7 #46 If $k>0$, show that $\int_1^\infty \frac{k}{x^{k+1}}\,dx$.

A: We will integrate as before, applying the usual rules for integration, remembering that k is a constant:

$$\begin{aligned}\int_1^\infty \frac{k}{x^{k+1}}\,dx &= \lim_{b\to\infty} k\int_1^b \frac{1}{x^{k+1}}\,dx\\ &= \lim_{b\to\infty} k\int_1^b x^{-(k+1)}\,dx\\ &= \lim_{b\to\infty} k\frac{x^{-(k+1)+1}}{-(k+1)+1}\Bigg]_1^b\\ &= \lim_{b\to\infty}\frac{-1}{x^k}\Bigg]_1^b\\ &= \lim_{b\to\infty}\left(\frac{-1}{b^k}+1\right)\\ &= 0+1=1.\end{aligned}$$

§6.7 #48 Find the capital value of an asset that generates income at the rate of \$5,000 per year, assuming an interest rate of 10%.

A: Here we want to find the present value of an income stream $K(t) = 5,000$ over the period $t \geq 0$ at an annual rate of $r = 10\% = .1$. The formula is *[capital value]* $= \int_0^\infty K(t)e^{-rt}\,dt$. So:

$$\begin{aligned}
\int_0^\infty K(t)e^{-rt}\,dt &= \lim_{b\to\infty}\int_0^b 5,000e^{-.1t}\,dt \\
&= \lim_{b\to\infty} 5,000\frac{e^{-.1t}}{-.1}\Big]_0^b \\
&= \lim_{b\to\infty}(-50,000e^{-.1t})\Big]_0^b \\
&= \lim_{b\to\infty}(-50,000e^{-.1b} + 50,000) \\
&= \$50,000.00
\end{aligned}$$

§6.7 #50 Suppose that a large farm with a known reservoir of gas beneath the ground sells the gas rights to a company for a guaranteed payment at the rate of $10,000e^{.04t}$ dollars per year. Find the present value of this perpetual stream of income, assuming an interest rate of 12%, compounded continuously.

A: We use the same formula as we did in #48, with $K(t) = 10,000e^{.04t}$ and $r = 12\% = .12$:

$$\begin{aligned}
\int_0^\infty K(t)e^{-rt}\,dt &= \lim_{b\to\infty}\int_0^b 10,000e^{.04t}e^{-.12t}\,dt \\
&= \lim_{b\to\infty}\int_0^b 10,000e^{-.08t}\,dt \\
&= \lim_{b\to\infty} 10,000\left(\frac{e^{-.08t}}{-.08}\right)\Big]_0^b \\
&= \lim_{b\to\infty}\frac{10,000}{-.08}(e^{-0.8b} - 1) \\
&= \frac{10,000}{.08} = \$125,000.00
\end{aligned}$$