Rates of Decay to Equilibria for p-Laplacian Type Equations

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Abstract

The long-time asymptotics for p-Laplacian type equations $\rho_t = \Delta_p \rho^m = \mathrm{div}(|\nabla\rho^m|^{p-2}\nabla\rho^m)$ in $\mathbb{R}^n$, is studied for $p > 1$ and $m \geq \frac{n-p+1}{n(p-1)}$. The non-negative solutions of the equations are shown to behave asymptotically, as $t \to \infty$, like Barenblatt-type solutions, and the explicit rates of decay are established for the convergence of the relative energy, the convergence with respect to the Wasserstein distances and the convergence with respect to the L^1-norm. The rates are proved to be optimal for $p = 2$. The method used is based on mass transportation inequalities.

Keywords: Asymptotic behavior, rate of convergence, displacement convexity, energy inequality, generalized logarithmic Sobolev inequalities, generalized Talagrand's inequalities, Csiszàr-Kullback type inequalities.

AMS Subject Classifications: 35K55, 35K65, 35B40.

1 Introduction

The present paper deals with the asymptotic behavior, as $t \to \infty$, of the doubly nonlinear parabolic equations

$$\frac{\partial \rho}{\partial t} = \Delta_p \rho^m = \mathrm{div}(|\nabla\rho^m|^{p-2}\nabla\rho^m) \quad \text{in} \quad \mathbb{R}^n \times (0,\infty) \tag{1}$$

$$\rho(t=0) = \rho_0 \quad \text{in} \quad \mathbb{R}^n \tag{2}$$

where $0 \leq \rho_0 \in L^1(\mathbb{R}^n)$ and $n > 1$. Our purpose in this work is to estimate the rates at which solutions to equations (1)-(2) converge to equilibrium.

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The existence of weak solutions to the Cauchy problem (1)-(2) is well-studied in the literature under some mild conditions on the initial datum ρ_0 and the parameters m and p. We refer to Agueh [3] for existence results in a bounded domain of $I\!R^n$, and to Ishige [18], Del Pino-Dolbeault-Gentil [15] and references therein, for the existence of weak solutions in the entire $I\!R^n$. Furthermore, for a non-negative initial datum $\rho_0 \in L^1(I\!R^n)$, and under suitable conditions on the parameters m and p, it is well-known that the solutions to equations (1)-(2) remain non-negative in time, and moreover, they share the same mass as the initial datum ρ_0. This is in particular the case when $m > \frac{n-p}{n(p-1)}$, $p > 1$ and $0 \leq \rho_0 \in L^1(I\!R^n)$ with $H_c^F(\rho_0) < \infty$, where H_c^F is defined by (19) and (6)-(7).

In the sequel, we assume existence of non-negative mass-preserving solutions to the Cauchy problem (1)-(2) (see [16, 18, 15, 31] for more details and further references), and we propose to estimate the rates at which these solutions converge to some equilibrium solution, when the parameters m and p satisfy the conditions

$$m \geq \frac{n-p+1}{n(p-1)} \quad \text{and} \quad p > 1. \tag{3}$$

For $p = 2$, where equation (1) reads as the heat equation (i.e. $m = 1$), the porous-medium equation (i.e. $m > 1$) and the fast diffusion equation (i.e. $0 < m < 1$), the asymptotic behavior for these equations has been studied by several authors, and the rates of convergence have been entirely described in the regime (3) under consideration. Indeed, Friedmann-Kamin [17] proved that, if $m > 1 - \frac{2}{n}$, then the solutions to the equations converge to some fundamental solution whose initial value is the Dirac mass at the origin (up to the multiplicative constant $\int_{I\!R^n} \rho_0 \, dx$). This solution is known as the Barenblatt-Prattle solution [4, 26] if $m \neq 1$, and the Gaussian if $m = 1$. Rates of convergence to the fundamental solutions were computed by Carrillo-Toscani [5] for $m > 1$, and independently by Del Pino - Dolbeault [12] and Otto [24] for $m \geq 1 - \frac{1}{n}$. For more details on these developments, we refer to [8, 6, 7] and the survey paper of Vázquez [30] and the references therein.

But for $p \neq 2$, where equation (1) reads as the p-Laplacian equation (i.e. $m = 1$), the rates of convergence to equilibrium still remain unknown for certain values of p. Indeed, Kamin-Vázquez [20] proved existence and uniqueness of a fundamental solution to the p-Laplacian equation when $p > \frac{2n}{n+1}$. Furthermore, they derived an L^1 and L^∞ convergence of the non-negative solutions to the fundamental solution, with no rates. Rates of convergence were obtained by Del Pino - Dolbeault [13, 14] for $\frac{2n+1}{n+1} \leq p < n$, and more recently by Agueh [1] for all $p \geq \frac{2n+1}{n+1}$. For the range $\frac{2n}{n+1} < p < \frac{2n+1}{n+1}$, the rates of convergence are still unknown.

The main purpose of this paper is to provide detailed proofs of these convergence results presented in our previous paper [1], then extend them to more general equations of the form (1), and to other types of convergence.

From now on, we assume that the initial datum ρ_0 is a probability density function on $I\!R^n$. Therefore, for m and p satisfying condition (3), the solutions to equations (1)-(2) are probability densities as well. For convenience and to achieve maximum generality in our results, we rewrite equations (1)-(2) in the form

$$\frac{\partial \rho}{\partial t} = \mathrm{div}\Big\{\rho \nabla c^* \left[\nabla \left(F'(\rho)\right)\right]\Big\} \quad \text{in} \quad I\!\!R^n \times (0,\infty) \tag{4}$$

$$\rho(t=0) = \rho_0 \quad \text{in} \quad I\!\!R^n \tag{5}$$

where $c^*(x) = \frac{|x|^p}{p}$ is the Legendre transform of the convex function

$$c(x) = \frac{|x|^q}{q}, \quad \frac{1}{p} + \frac{1}{q} = 1, \tag{6}$$

and the energy density function $F : [0,\infty) \to I\!\!R$ is defined by

$$F(x) = \begin{cases} \frac{1}{p-1} x \ln x & \text{if} \quad m = \frac{1}{p-1} \\ \\ \frac{m x^\gamma}{\gamma(\gamma-1)}, \ \gamma = m + \frac{p-2}{p-1} & \text{if} \quad m \neq \frac{1}{p-1}. \end{cases} \tag{7}$$

Note that $\gamma \geq 1 - \frac{1}{n} > 0$ because of condition (3). To equation (4) is associated the free energy functional

$$H^F(\rho) = \int_{R^n} F\left(\rho(x)\right) \, \mathrm{d}x \tag{8}$$

which reads as

- the entropy functional $H^F(\rho) = \int_{R^n} \rho \ln \rho \, \mathrm{d}x$, if equation (4) is the heat equation,
- $H^F(\rho) = \int_{R^n} \frac{\rho^m}{m-1} \, \mathrm{d}x$, if (4) is the porous medium or the fast diffusion equations, and
- $H^F(\rho) = \int_{R^n} \frac{\rho^\gamma}{\gamma(\gamma-1)} \, \mathrm{d}x$, $\gamma = \frac{2p-3}{p-1}$, if (4) is the parabolic p-Laplacian equation.

The fundamental solution $\rho_\infty(t,x)$ - that is, the solution whose initial value is the Dirac mass at the origin - which determines the long-time asymptotics of equation (4), exists whenever $m > \frac{n-p}{n(p-1)}$, and it is unique [20]. Furthermore, there exists a time dependent scaling $R(t)$ satisfying $R(0) = 1$, such that

$$\rho_\infty(t,x) = \frac{1}{R(t)^n} \hat{\rho}_\infty \left(\frac{x}{R(t)}\right), \tag{9}$$

where $\hat{\rho}_\infty$ is the unique stationary solution of the convection-diffusion equation

$$\frac{\partial \hat{\rho}}{\partial \tau} = \mathrm{div}\Big\{\hat{\rho} \nabla c^* \left[\nabla \left(F'(\hat{\rho})\right)\right] + \hat{\rho} y\Big\} \quad \text{in} \quad I\!\!R^n \times (0,\infty) \tag{10}$$

$$\hat{\rho}(\tau = 0) = \rho_0 \quad \text{in} \quad I\!\!R^n \tag{11}$$

which is simply equations (4)-(5) rescaled in time and space as follows:

$$\hat{\rho}(\tau, y) = R(t)^n \rho(t,x), \quad \tau = \ln R(t), \quad y = \frac{x}{R(t)} \tag{12}$$

and

$$R(t) = (1+\delta_p t)^{1/\delta_p}, \quad \delta_p = (p-1)(nm+1)+1-n. \tag{13}$$

In fact $\hat{\rho}_\infty$ is the unique probability density function that satisfies on its support

$$\nabla \left(F'(\hat{\rho}_\infty) + c\right) = 0. \tag{14}$$

It is explicitly given by

$$\hat{\rho}_\infty(y) = \begin{cases} \frac{1}{\sigma} \exp\left(-\frac{p-1}{q}|y|^q\right) & \text{if} \quad m = \frac{1}{p-1} \\ \\ \left(K - \frac{\gamma-1}{mq}|y|^q\right)_+^{1/\gamma-1} & \text{if} \quad m \neq \frac{1}{p-1} \end{cases} \tag{15}$$

where $\gamma = m + \frac{p-2}{p-1}$, $\frac{1}{p}+\frac{1}{q}=1$, and σ, K are the unique constants such that $\int_{\mathbb{R}^n} \hat{\rho}_\infty(y)\,\mathrm{d}y = 1$. It is easily checked that $\rho(t,x)$ solves (4)-(5) if and only if $\hat{\rho}(\tau,y)$ solves (10)-(11). Therefore, the asymptotic behavior of solutions $\hat{\rho}(\tau,y)$ of the rescaled equations (10)-(11) to the stationary solution $\hat{\rho}_\infty(y)$, gives the long-time asymptotics of solutions $\rho(t,x)$ of the original equations (1)-(2) to the fundamental solution $\rho_\infty(t,x)$. It is then sufficient to study the long-time behavior for the rescaled equations (10)-(11). For this purpose, we establish some mass transportation inequalities, from which different types of convergence (convergence of the relative energy, convergence w.r.t. the Wasserstein distances, and convergence w.r.t. the L^1-norm) are derived. Two ingredients are essentially used in our analysis. The first one is the superb notion of *displacement convexity* introduced by McCann [22], which leads to an *energy inequality* relating the energy $H^F(\hat{\rho}(\tau))$ at every time τ, to the energy $H^F(\hat{\rho}_\infty)$ at equilibrium [3]. This notion expresses the convexity of the energy functional $H^F(\hat{\rho})$ along geodesics in the space of probability densities equipped with the Wasserstein metric. The second ingredient is the *Young's inequality*

$$a \cdot b \leq \frac{|a|^p}{p} + \frac{|b|^q}{q}, \quad \forall a, b \in \mathbb{R}^n, \quad \text{where} \quad \frac{1}{p}+\frac{1}{q} = 1. \tag{16}$$

Combining these two ingredients, we are able to establish rates of decay to equilibrium for the solutions of equations (10)-(11), from which we derive the decay rates to the fundamental solution $\rho_\infty(t,x)$ of equations (1)-(2).

The outline of the paper is as follows. In section 2, we prove the convergence of the free energy $H^F(\hat{\rho}(\tau))$ to $H^F(\hat{\rho}_\infty)$, as $\tau \to \infty$. In section 3, we study the convergence of the q-Wasserstein distance $W_q(\hat{\rho}(\tau), \hat{\rho}_\infty)$ (defined by (18)), and section 4 concludes with the convergence of the L^1-norms $\|\hat{\rho}(\tau) - \hat{\rho}_\infty\|_{L^1}$ and $\|\rho(t) - \rho_\infty\|_{L^1}$, as $\tau \to \infty$ and $t \to \infty$ respectively. Finally, in section 5, we apply the results obtained in the previous sections 2-4 to some concrete examples of equations of the form (1)-(2), namely, the heat equation, the porous medium and fast diffusion equations, the parabolic p-Laplacian equation, and some doubly degenerate parabolic equations.

Throughout this paper, $\mathcal{P}_a(I\!R^n)$ denotes the set of probability densities over $I\!R^n$, i.e., $\mathcal{P}_a(I\!R^n) = \{\rho : I\!R^n \to [0,\infty),\ \rho \geq 0 \text{ and } \int_{R^n} \rho(x)\,\mathrm{d}x = 1\}$, and $\mathrm{supp}\,(\rho)$ stands for the support of $\rho \in \mathcal{P}_a(I\!R^n)$, that is, the closure of $\{x \in I\!R^n :\ \rho(x) \neq 0\}$. For $\rho_0, \rho_1 \in \mathcal{P}_a(I\!R^n)$, $H_c^F(\rho_0|\rho_1) = H_c^F(\rho_0) - H_c^F(\rho_1)$ denotes the relative free energy of ρ_0 with respect to ρ_1, where $H_c(\rho) = \int_{R^n} c(x)\rho(x)\,\mathrm{d}x$ is the potential energy, $H^F(\rho) = \int_{R^n} F(\rho(x))\,\mathrm{d}x$ is the internal energy, and $H_c^F(\rho) = H^F(\rho) + H_c(\rho) = \int_{R^n} (F(\rho) + c\rho)\,\mathrm{d}x$ is the total free energy associated with $\rho \in \mathcal{P}_a(I\!R^n)$. The c-Wasserstein work from $\rho_0 \in \mathcal{P}_a(I\!R^n)$ to $\rho_1 \in \mathcal{P}_a(I\!R^n)$ is defined as

$$\mathcal{W}_c(\rho_0, \rho_1) = \inf\Big\{ \int_{R^n} c\,(x - T(x))\,\mathrm{d}x,\ T_{\#}\rho_0 = \rho_1 \Big\} \tag{17}$$

where $T_{\#}\rho_0 = \rho_1$ means that $\rho_1(B) = \rho_0\left(T^{-1}(B)\right)$ for all Borel sets $B \subset I\!R^n$. If $c(x) = \frac{|x|^q}{q}$, then $\mathcal{W}_c = \frac{1}{q} W_q^q$, where

$$W_q(\rho_0, \rho_1) = \Big[\inf\Big\{ \int_{R^n} |x - T(x)|^q\,\mathrm{d}x,\ T_{\#}\rho_0 = \rho_1 \Big\}\Big]^{1/q} \tag{18}$$

is called the q-Wasserstein distance (see Villani [32] for more details on this topics). In order to state general results, we will often use in place of F, the function $G : [0,\infty) \to I\!R$ which satisfies the hypotheses

(H): $G \in C[0,\infty) \cap C^2(0,\infty)$ is convex, $G(0) = 0$ and $(0,\infty) \ni x \mapsto x^n G(x^{-n})$ is convex and non-increasing.

The last two conditions in (H) implies the displacement convexity of the energy functional H^G defined by (8) [22]. We point out here that, under condition (3), the function F defined by (7) satisfies (H).

2 Convergence of the relative energy

In this section, we estimate the rate at which the relative energy $H_c^F(\hat\rho(\tau)|\hat\rho_\infty)$ between a solution $\hat\rho(\tau)$ of the rescaled equations (10)-(11) and the equilibrium solution $\hat\rho_\infty$ (defined by (15)) converges to 0. To equation (10) is associated the free energy

$$H_c^F(u) = \int_{R^n} \left(F(u(y)) + c(y)u(y)\right)\,\mathrm{d}y, \quad u \in \mathcal{P}_a(I\!R^n) \tag{19}$$

which is the sum of the internal energy $H^F(u) = \int_{R^n} F(u(y))\,\mathrm{d}y$ and the potential energy $H_c(u) = \int_{R^n} c(y)u(y)\,\mathrm{d}y$. Here and after, we shall assume that the initial probability density $\rho_0 \in \mathcal{P}_a(I\!R^n)$ satisfies $H_c^F(\rho_0) < \infty$. Below, we will use the *entropy method* to prove the convergence in relative entropy - or relative energy - of the solutions to equations (10)-(11). This method consists of establishing a logarithmic Sobolev type inequality, which essentially compares the relative energy between solutions of equations (10)-(11) at times τ and $\tau \to \infty$, with the dissipation of the relative energy. Displacement convexity [22] will play an important role in this inequality. For the sake of illustration, we consider the following ODE, which can be viewed as a toy model for equations (10)-(11) when $c(x) = \frac{|x|^2}{2}$,

$$\begin{cases} x_t = -\nabla f\left(x(t)\right) & \text{in} \quad I\!R^n \times (0, \infty) \\ \\ x(0) = x_0 & \text{in} \quad I\!R^n \end{cases} \tag{20}$$

where $f : I\!R^n \to I\!R$ is a uniformly convex function such that $D^2 f \geq \lambda I,\ \lambda > 0$. Equation (20) is a gradient flow whose unique equilibrium solution is the critical point $x_\infty \in I\!R^n$ of f, that is, $\nabla f(x_\infty) = 0$. We will show that the rates of convergence of the solution $x(t)$ of equation (20), to the equilibrium solution x_∞ can be derived from the following *basic uniform convexity inequality* for f, which is obtained from the second order Taylor expansion combined with the uniform convexity property of f, that is, $D^2 f \geq \lambda I$. For $z_0, z_1 \in I\!R^n$,

$$f(z_1) - f(z_0) \geq \nabla f(z_0) \cdot (z_1 - z_0) + \frac{\lambda}{2}|z_1 - z_0|^2. \tag{21}$$

Here are the main steps leading to the rates of convergence to equilibrium for equation (20).

Step 1: *The Lyapunov function.*
From equation (20), we have that

$$\frac{d}{dt} f\left(x(t)\right) = -\left|\nabla f\left(x(t)\right)\right|^2 \tag{22}$$

which shows that f is decreasing along the solution $x(t)$ of (20), and exactly at the equilibrium solution x_∞, $\frac{d}{dt} f\left(x(t)\right) = 0$. Therefore, f is a Lyapunov function for (20). Equation (22) is known as the dissipation of the Lyapunov function f.

Step 2: *A toy model of the logarithmic Sobolev inequality.*
The basic uniform convexity inequality (21), combined with the Young's inequality

$$\nabla f(z_0) \cdot (z_0 - z_1) \leq \frac{\lambda}{2}|z_1 - z_0|^2 + \frac{1}{2\lambda}|\nabla f(z_0)|^2$$

gives the toy model logarithmic Sobolev inequality

$$f(z_0) - f(z_1) \leq \frac{1}{2\lambda}|\nabla f(z_0)|^2. \tag{23}$$

Step 3: *The convergence to equilibrium for the Lyapunov function.*
Using $z_0 = x(t)$ and $z_1 = x_\infty$ in (23), and combining this inequality with (22), we obtain

$$\frac{d}{dt}\left[f\left(x(t)\right) - f(x_\infty)\right] \leq -2\lambda \left[f\left(x(t)\right) - f(x_\infty)\right]$$

i.e.

$$f\left(x(t)\right) - f(x_\infty) \leq e^{-2\lambda t}\left[f(x_0) - f(x_\infty)\right], \tag{24}$$

which shows the exponential decay to equilibrium of the Lyapunov function $f(x(t))$, at the the rate 2λ.

Step 4: *A toy model of the Talagrand's inequality.*
The basic uniform convexity inequality (21) applied with $z_0 = x_\infty$ and $z_1 = x(t)$ gives the toy model Talagrand's inequality

$$|x(t) - x_\infty|^2 \leq \frac{2}{\lambda} \left[f(x(t)) - f(x_\infty) \right]. \tag{25}$$

Step 5: *The convergence to equilibrium for the Euclidean distance.*
Combining (24) and (25), we obtain

$$|x(t) - x_\infty| \leq e^{-\lambda t} \left[\frac{2}{\lambda} \left(f(x_0) - f(x_\infty) \right) \right]^{1/2}$$

that is, the exponential decay to equilibrium of the Euclidean distance at the rate λ.

For the quadratic cost $c(x) = \frac{|x|^2}{2}$, equation (10) is a gradient flow of the entropy functional H_c^F with respect to the Wasserstein metric W_2 [19]. Therefore, one obtains an exponential decay to equilibrium in relative entropy and in the Wasserstein distance by following the Steps 1-5 above, while identifying the set $I\!R^n$ to the set of probability densities $\mathcal{P}_a(I\!R^n)$, the Euclidean distance $|z_0 - z_1|$ to the Wasserstein distance $W_2(\rho_0, \rho_1)$, the Lyapunov function f to the free energy functional H_c^F [23, 27], and the uniform convexity of f to the uniform displacement convexity of H_c^F (see [25, 24, 6, 7] and the references therein for further details). But for a general cost $c(x) = \frac{|x|^q}{q}$, $q > 1$, this method does not apply directly because equation (10) is not known to be a gradient flow when $q \neq 2$. Here, we use instead some argument from [10, 2] to establish logarithmic Sobolev type inequalities appropriate to equation (10). These inequalities will lead to an exponential decay in relative entropy of the solutions to equation (10) for all $q > 1$. In particular when $q = 2$, we recover the optimal rates obtained by the previous authors. On the other hand, when $q < 2$, that is $p > 2$, $\frac{1}{p} + \frac{1}{q} = 1$, the rate obtained here for the convergence of the relative energy for the p-Laplacian equation is sharper than the rate of Del Pino-Dolbeault [14], but this is not true when $p < 2$. Therefore, the optimal rate for the convergence to equilibrium for the p-Laplacian equation is still unknown.

The next lemma states the *energy inequality* needed for the proof of the main theorem of this section. This inequality is the analogue of the basic convexity inequality (21) in the space of probability densities when $\lambda = 0$, in the sense that it follows from the *displacement convexity* of the internal energy H^F, that is, the convexity of H^F along the geodesic connecting two probability densities u_0 and u_1 in $\mathcal{P}_a(I\!R^n)$ equipped with the Wasserstein metric W_2 [22]. For a complete proof of this inequality, we refer to [3]. Here, we just sketch the proof.

Lemma 2.1 *Let $\Omega \subset I\!R^n$ be open, convex and bounded, $G : [0, \infty) \to I\!R$ satisfy $G \in C[0, \infty) \cap C^2(0, \infty)$, and $u_0, u_1 \in \mathcal{P}_a(\Omega)$ be such that $0 < u_0 \in W^{1,\infty}(\Omega)$. Let T be the* *optimal map that pushes u_0 forward to u_1 in (17), with the quadratic cost $c(x) = \frac{|x|^2}{2}$. If $G(0) = 0$ and $x \mapsto x^n G(x^{-n})$ is convex and non-increasing, then the following energy inequality holds:*

$$H^G(u_1) - H^G(u_0) \geq \int_\Omega u_0 \nabla \left(G'(u_0) \right) \cdot (T - I) \, dy. \tag{26}$$

Remark. The energy inequality (26) holds for more general costs, e.g. $c(x) = \frac{|x|^q}{q}$, $q > 1$ (see [3], Theorem 2.8). It also holds when $\Omega = \mathbb{R}^d$, provided that the assumptions on u_0 and u_1 are replaced by $u_0, u_1 \in \mathcal{P}_a(\mathbb{R}^d)$, $u_0 \in W^{1,\infty}(\mathbb{R}^d)$ and u_0, u_1 have compact supports (see [9], Proposition 4.1).

Sketch of Proof of Lemma 2.1. For $t \in [0,1]$, define $T_t = (1-t)I + tT$. Then the path $[0,1] \ni t \mapsto u_t = (T_t)_{\#} u_0$ is a minimal geodesic connecting u_0 and u_1 in $(\mathcal{P}_a(\Omega), W_2)$, that is, $W_2(u_0, u_1) = W_2(u_0, u_t) + W_2(u_t, u_1)$, and for any other path $\tilde{u}_t$ connecting u_0 and u_1, we have $W_2(u_0, u_1) < W_2(u_0, \tilde{u}_t) + W_2(\tilde{u}_t, u_1)$ (see [32] for the details). Moreover, if $G(0) = 0$ and $x \mapsto x^n G(x^{-n})$ is convex and non-increasing, then $[0,1] \ni t \mapsto H^G(u_t)$ is convex [22] (we say that H^G is displacement convex). Therefore,

$$\begin{aligned} H^G(u_1) - H^G(u_0) &\geq \left[\frac{d}{dt} H^G(u_t) \right]_{t=0} \\ &= \int_\Omega G'(u_0) \frac{\partial u_t}{\partial t}\Big|_{t=0} \, \mathrm{d}y \\ &= -\int_\Omega G'(u_0) \, \mathrm{div} \left(u_0 (T - I) \right) \, \mathrm{d}y \\ &= \int_\Omega u_0 \nabla \left(G'(u_0) \right) \cdot (T - I) \, \mathrm{d}y. \end{aligned}$$

Here is the main theorem of this section.

Theorem 2.2 *Let $c(x) = \frac{|x|^q}{q}$, $q > 1$, and assume that $G : [0, \infty) \to \mathbb{R}$ satisfies (H). Then, for all $u_0, u_1 \in \mathcal{P}_a(\mathbb{R}^n)$ such that $u_0 \in C(\mathbb{R}^n)$, we have*

$$H_c^G(u_0|u_1) \leq H_c(u_0) + \int_{R^n} u_0 \nabla \left(G'(u_0) \right) \cdot y \, dy + \int_{R^n} u_0 c^* \left(-\nabla \left(G'(u_0) \right) \right) \, dy \tag{27}$$

with equality if $u_0 = u_1 = u_\infty$ satisfies on its support

$$\nabla \left(G'(u_\infty) + c \right) = 0. \tag{28}$$

Therefore, the following generalizations of the logarithmic Sobolev inequality hold: If $q = 2$, then

$$H_c^G(u_0|u_1) \leq \frac{1}{2} I_G(u_0), \tag{29}$$

and it is sharp, in the sense that equality occurs if $u_0 = u_1 = u_\infty$. If $q \neq 2$, then

$$H_c^G(u_0|u_1) \le I_G(u_0), \tag{30}$$

where

$$I_G(u_0) = \int_{R^n} u_0 \nabla \left(G'(u_0) + c\right) \cdot \left[\nabla c^* \left(\nabla (G'(u_0))\right) + y\right] \, dy. \tag{31}$$

Furthermore, if $G = F$ is defined by (7) with $m > \frac{n-p+1}{n(p-1)}$ and $\frac{1}{p} + \frac{1}{q} = 1$, and if the initial density $\rho_0 \in \mathcal{P}_a(I\!R^n)$ is such that $H_c^F(\rho_0) < \infty$, then any solution $\hat\rho(\tau)$ of equations (10)-(11) with finite energy $H_c^F(\hat\rho(\tau)) < \infty$, satisfies the decay rates: If $q = 2$, then

$$H_c^F\left(\hat\rho(\tau)|\hat\rho_\infty\right) \le e^{-2\tau} H_c^F\left(\rho_0|\hat\rho_\infty\right), \tag{32}$$

and the rate is optimal. If $q \neq 2$, then

$$H_c^F\left(\hat\rho(\tau)|\hat\rho_\infty\right) \le e^{-\tau} H_c^F\left(\rho_0|\hat\rho_\infty\right), \tag{33}$$

where $\hat\rho_\infty$ is the unique equilibrium solution of equations (10)-(11), defined by (15).

Proof. First, we prove (27)-(30). The proof will be done in two steps. In Step 1, we assume that u_0 and u_1 have compact supports, and in Step 2 we approximate u_0 and u_1 by compactly supported functions $\{u_{0,j}\}_j$ and $\{u_{1,j}\}_j$ with equal mass, and we obtained (27)-(30) in the limit as $j \to \infty$.

Step 1. We assume that $\operatorname{supp} u_0$, $\operatorname{supp} u_1 \subset \Omega$, $u_0 \in C(\Omega)$, where $\Omega \subset I\!R^n$ is open, convex and bounded, and $\int_\Omega u_0(y)\,\mathrm{d}y = \int_\Omega u_1(y)\,\mathrm{d}y$. Then, the energy inequality (26) holds, that is,

$$H^G(u_0|u_1) \le \int_\Omega u_0 \nabla\left(G'(u_0)\right) \cdot y \,\mathrm{d}y - \int_\Omega u_0 \nabla\left(G'(u_0)\right) \cdot T(y) \,\mathrm{d}y. \tag{34}$$

Applying Young's inequality with the convex function c, that is,

$$-\nabla\left(G'(u_0)\right) \cdot T(y) \le c\left(T(y)\right) + c^*\left(-\nabla\left(G'(u_0)\right)\right), \tag{35}$$

we have that

$$H^G(u_0|u_1) \le \int_\Omega u_0 \nabla\left(G'(u_0)\right) \cdot y \,\mathrm{d}y + \int_\Omega u_0 c^*\left(-\nabla\left(G'(u_0)\right)\right) \,\mathrm{d}y + \int_\Omega u_0 c\left(T(y)\right) \,\mathrm{d}y.$$

Then we use that $T_\# u_0 = u_1$, to conclude (27). Next, set $u_0 = u_1$ in (34). We have that $T = I$ and equality holds in (34), where all the integrals can be taken over the support of $u_0 = u_1$. Therefore, equality holds in (27) if equality holds in the Young's inequality (35). This occurs if $u_0 = u_1 = u_\infty$ satisfies (28) on its support.

Now, we use that $c(x) = \frac{|x|^q}{q}$, to rewrite (27) and (31) as

$$H_c^G(u_0|u_1) \le \frac{1}{p} I_1(u_0) + \frac{1}{q} I_2(u_0) + I_3(u_0) \tag{36}$$

and

$$I_G(u_0) = \sum_{i=1}^{4} I_i(u_0), \tag{37}$$

where

$$I_1(u_0) = \int u_0 |\nabla (G'(u_0))|^p \, dy \quad , \quad I_2(u_0) = \int u_0 |y|^q \, dy$$
$$I_3(u_0) = \int u_0 \nabla (G'(u_0)) \cdot y \, dy \quad \text{and} \quad I_4(u_0) = \int u_0 \nabla c^* [\nabla (G'(u_0))] \cdot \nabla c(y) \, dy,$$

and $\frac{1}{p} + \frac{1}{q} = 1$. If $q = 2$, then $I_3(u_0) = I_4(u_0)$. Therefore, (36)-(37) give (29), with equality if $u_0 = u_1 = u_\infty$ satisfies (28) on $\text{supp}(u_\infty)$. If $q \neq 2$, we use again Young's inequality with the convex function c, that is,

$$-\nabla c^* [\nabla (G'(u_0))] \cdot \nabla c(y) \leq c^* (\nabla c(y)) + c (\nabla c^* \nabla (G'(u_0))) = \frac{|x|^q}{p} + \frac{|\nabla (G'(u_0))|^p}{q},$$

to have that

$$-I_4(u_0) \leq \frac{1}{q} I_1(u_0) + \frac{1}{p} I_2(u_0). \tag{38}$$

Then, we combine (36)-(38) to conclude that

$$H_c^G(u_0|u_1) \leq I_1(u_0) + I_2(u_0) + I_3(u_0) - \left(\frac{1}{q} I_1(u_0) + \frac{1}{p} I_2(u_0) \right) \leq I_G(u_0).$$

Step 2. Assume that u_0 and u_1 satisfy the assumptions in Theorem 2.2. It suffices to prove (27)-(28), then (29) and (30) will follow as in Step 1. For simplicity, we will assume that either $G \geq 0$ or $G \leq 0$. This assumption is not restrictive because it is satisfied by the function F defined by (7). In fact, if $m = \frac{1}{p-1}$, then $F(x) = \frac{1}{p-1} x \ln x \geq \frac{-e^{-1}}{p-1}$ can be replaced by $F(x) + \frac{e^{-1}}{p-1}$ and (27) still holds. Without loss of generality, we assume that $\int_{I\!R^n} u_0 c^* (-\nabla G'(u_0)) \, dy < \infty$ and $H_c(u_0) < \infty$, otherwise (27) is trivial. Let $\Omega_j \subset I\!R^n$ be open, convex and bounded, such that $\overline{\Omega}_j \subset \Omega_{j+1}$ and $\cup_{j=1}^{\infty} \Omega_j = I\!R^n$. Set $u_{1,j} = u_1 \chi_{\Omega_j}$, where χ_{Ω_j} denotes the characteristic function of Ω_j. By Lemma 6.1 [9], we can approximate u_0 by a sequence $\{u_{0,j}\}_j$ of positive functions in $L^1(\Omega_j) \cap C(\Omega_j)$ with support in Ω_j, such that

$$\int_{\Omega_j} u_{0,j} \, dy = \int_{\Omega_j} u_{1,j} \, dy \quad , \quad u_{0,j} = u_0 \text{ on } \Omega_j \setminus \Omega_1, \quad u_{0,j} \to u_0 \text{ in } L^1(I\!R^n),$$
$$u_{0,j} \to u_0 \text{ in } W^{1,\infty}(\Omega_1) \quad \text{and} \quad H_c^G(u_{0,j}) \to H_c^G(u_0). \tag{39}$$

From Step 1, we have that

$$H_c^G(u_{0,j}|u_{1,j}) \leq H_c(u_{0,j}) + \int_{\Omega_j} u_{0,j} \nabla (G'(u_{0,j})) \cdot y \, dy + \int_{\Omega_j} u_{0,j} c^* (-\nabla (G'(u_{0,j}))) \, dy \tag{40}$$

with equality if $u_{0,j} = u_{1,j}$ satisfies

$$\nabla \left(G'(u_{1,j}) + c \right) = 0 \ \text{ on } \ \Omega_j \cap \operatorname{supp}(u_{1,j}). \tag{41}$$

As $j \to \infty$, the equality $u_{0,j} = u_{1,j}$ obviously gives that $u_0 = u_1$. And since $u_{1,j} = u_1$ on Ω_j and $\cup_{j=1}^{\infty} \Omega_j = I\!R^n$, then (41) reads as (28). Next, we study the limit, as $j \to \infty$, of the terms in (40) to prove (27). Since $u_{0,j} = u_0$ in $\Omega_j \setminus \Omega_1$, we have that

$$H_c(u_{0,j}) - H_c(u_0) = \int_{\Omega_1} (u_{0,j} - u_0) c \, \mathrm{d}y - \int_{R^n \setminus \Omega_j} u_0 c \, \mathrm{d}y.$$

We let $j \to \infty$ in the subsequent inequality, and we use that $H_c(u_0) < \infty$, $u_{0,j} \to u_0$ in $L^1(I\!R^n)$, and the dominated convergence theorem, to obtain that

$$\lim_{j \to \infty} H_c(u_{0,j}) = H_c(u_0). \tag{42}$$

Similarly, using that $u_{1,j} = u_1 \chi_{\Omega_j}$ and $G(0) = 0$, we have that

$$H^G(u_{1,j}) = \int_{R^n} G(u_1) \chi_{\Omega_j} \, \mathrm{d}y \quad \text{and} \quad H_c(u_{1,j}) = \int_{R^n} c u_1 \chi_{\Omega_j} \, \mathrm{d}y.$$

We use that $c \geq 0$, $G \geq 0$ or $G \leq 0$, and the monotone convergence theorem, to have that $H^G(u_{i,j}) \to H^G(u_1)$ and $H_c(u_{1,j}) \to H_c(u_1)$. Then we deduce, using (39) that

$$\lim_{j \to \infty} H_c^G(u_{0,j} | u_{1,j}) = H_c^G(u_0 | u_1). \tag{43}$$

Furthermore, using again that $u_{0,j} = u_0$ on $\Omega_j \setminus \Omega_1$, we have that

$$\begin{aligned} &\int_{\Omega_j} u_{0,j} c^* \left(-\nabla \left(G'(u_{0,j}) \right) \right) \mathrm{d}y - \int_{R^n} u_0 c^* \left(-\nabla \left(G'(u_0) \right) \right) \mathrm{d}y \\ &\quad = \int_{\Omega_1} \left[u_{0,j} c^* \left(-\nabla \left(G'(u_{0,j}) \right) \right) - u_0 c^* \left(-\nabla \left(G'(u_0) \right) \right) \right] \mathrm{d}y \\ &\qquad - \int_{R^n \setminus \Omega_j} u_0 c^* \left(-\nabla \left(G'(u_0) \right) \right) \mathrm{d}y. \end{aligned}$$

We let $j \to \infty$ in the above equality, and we use (39) and the fact that $\int_\Omega u_0 c^* \left(-\nabla \left(G'(u_0) \right) \right) \mathrm{d}y < \infty$, to obtain that

$$\lim_{j \to \infty} \int_{\Omega_j} u_{0,j} c^* \left(-\nabla \left(G'(u_{0,j}) \right) \right) \mathrm{d}y = \int_{R^n} u_0 c^* \left(-\nabla \left(G'(u_0) \right) \right) \mathrm{d}y. \tag{44}$$

Finally, we note that $\int_{R^n} |u_0 \nabla \left(G'(u_0) \right) \cdot y| \, \mathrm{d}y < \infty$, because of the Young's inequality $\pm \nabla \left(G'(u_0) \right) \cdot y \leq c^* \left(-\nabla G'(u_0) \right) + c(x)$ and the fact that $\int_\Omega u_0 c^* \left(-\nabla \left(G'(u_0) \right) \right) \mathrm{d}y < \infty$ and $H_c(u_0) < \infty$. Then we proceed as in (44) to have that

$$\lim_{j \to \infty} \int_{\Omega_j} u_{0,j} \nabla \left(G'(u_{0,j}) \right) \cdot y \, \mathrm{d}y = \int_{R^n} u_0 \nabla \left(G'(u_0) \right) \cdot y \, \mathrm{d}y. \tag{45}$$

We combine (40) and (42)-(45) to conclude (27).

Now we prove (32) and (33). Here also, we split the proof into two parts. In Part 1, we assume that ρ_0 is a positive, smooth function with finite mass, defined on a bounded domain of $\mathbb{R}^n$. In Part 2, we approximate $\rho_0 \in \mathcal{P}_a(\mathbb{R}^n)$ by a sequence $\{\rho_0^{k,j}\}_{k,j}$ of functions satisfying the assumptions in Part 1, and we obtained (32) and (33) in the limit as $(k,j) \to (\infty,\infty)$.

Part 1. Let $\Omega \subset \mathbb{R}^n$ be open, convex and bounded with a smooth boundary $\partial\Omega$. Assume that $\rho_0 \in L^1(\Omega) \cap C^\infty(\overline{\Omega})$ and $N \le \rho_0 \le M$, for some positive constants N and M. Then the equation

$$\begin{cases} \frac{\partial \hat\rho}{\partial \tau} = \operatorname{div}\Big\{\hat\rho \nabla c^* \left[\nabla \left(F'(\hat\rho)\right)\right] + \hat\rho y\Big\} & \text{in} \quad \Omega \times (0,\infty) \\ \hat\rho \left(\nabla c^* \left[\nabla \left(F'(\hat\rho)\right)\right] + y\right) \cdot \nu = 0 & \text{on} \quad \partial\Omega \times (0,\infty) \\ \hat\rho(\tau = 0) = \rho_0 & \text{in} \quad \Omega \end{cases} \tag{46}$$

has a unique weak solution $\hat\rho \in L^\infty_{loc}(\Omega \times (0,\infty))$ that satisfies $\int_\Omega \hat\rho(\tau)\,\mathrm{d}y = \int_\Omega \rho_0\,\mathrm{d}y$ for all $\tau > 0$, and $0 < N(T) \le \hat\rho(\tau) \le M(T)$ for all $\tau \in (0,T)$, where $0 < T < \infty$ [3, 16]. Moreover, since $\rho_0 \in C^\infty(\overline{\Omega})$ and $0 < N(T) \le \hat\rho(\tau) \le M(T)$ for all $\tau \in (0,T)$, then the regularity theory for quasilinear equations gives that $\hat\rho \in C^{1,\alpha}_{loc}(\Omega \times (0,T))$ for some $\alpha \in (0,1)$ and for all $0 < T < \infty$ [16]. Therefore, we can compute the energy dissipation equation by a direct differentiation:

$$\begin{aligned} \frac{dH_c^F(\hat\rho(\tau))}{d\tau} &= \int_\Omega \left(F'(\hat\rho(\tau)) + c\right) \operatorname{div}\Big\{\hat\rho(\tau)\left[\nabla c^*\left(\nabla F'(\hat\rho(\tau))\right) + y\right]\Big\}\,\mathrm{d}y \\ &= -\int_\Omega \hat\rho(\tau) \nabla \left(F'(\hat\rho(\tau)) + c\right) \cdot \Big\{\nabla c^*\left[\nabla F'(\hat\rho(\tau)) + y\right]\Big\}\,\mathrm{d}y \\ &= -I_F(\hat\rho(\tau)). \end{aligned} \tag{47}$$

If $q = 2$, then (29) and (47) give that

$$\frac{dH_c^F(\hat\rho(\tau)|\hat\rho_\infty)}{d\tau} \le -2H_c^F(\hat\rho(\tau)|\hat\rho_\infty) \quad \text{i.e.} \quad \frac{d}{d\tau}\left[e^{2\tau} H_c^F(\hat\rho(\tau)|\hat\rho_\infty)\right] \le 0. \tag{48}$$

We integrate (48) over $(0,\tau)$ to conclude (32). Note that the rate in (32) is optimal because inequality (29) is sharp. if $q \ne 2$, then we combine (30) and (47) to conclude (33).

Part 2. Assume that $\rho_0 \in \mathcal{P}_a(\mathbb{R}^n)$ satisfies $H_c^F(\rho_0) < \infty$. For simplicity, we assume that $F \ge 0$. This assumption includes the functions F in (7), when $\gamma > 1$ and $m = \frac{1}{p-1}$, because for $m = \frac{1}{p-1}$, $F(x) = \frac{1}{p-1} x \ln x \ge \frac{-e^{-1}}{p-1}$ can be replaced by $F(x) + \frac{e^{-1}}{p-1} \ge 0$ without altering inequalities (32) and (33). Remark that under the assumption $F \ge 0$, F has a superlinear growth. For the case $\gamma < 1$, i.e. $F \le 0$, we refer to [8] where (32) and (33) were established when $q = 2$. Now, for $k \in \mathbb{N}$, set $\Omega_k = \{y \in \mathbb{R}^n,\ |y| < k\}$. We can approximate $\rho_0 \in \mathcal{P}_a(\mathbb{R}^n)$ by a sequence $\{\rho_0^{k,j}\}_{k,j}$ of positive functions in $L^1(\Omega_k) \cap C^\infty(\overline{\Omega}_k)$ such that

$$\rho_0^{k,j} \to \rho_0 \quad \text{in} \quad L^1(\mathbb{R}^n), \qquad \int_{\Omega_k} \rho_0^{k,j}\,\mathrm{d}y \to \int_{\mathbb{R}^n} \rho_0\,\mathrm{d}y = 1, \quad H_c^F(\rho_0^{k,j}) \to H_c^F(\rho_0). \tag{49}$$

Indeed, one can obtain $\rho_0^{k,j}$ by simply mollifying the function

$$\begin{cases} \frac{1}{j} & \text{if } \ \rho_0\chi_{\Omega_k} \le \frac{1}{j} \\ \rho_0\chi_{\Omega_k} & \text{if } \ \frac{1}{j} \le \rho_0\chi_{\Omega_k} \le j \\ j & \text{if } \ \rho_0\chi_{\Omega_k} \ge j. \end{cases}$$

Consider the solution $\hat{\rho}^{k,j}$ of the equation

$$\begin{cases} \frac{\partial \hat{\rho}^{k,j}}{\partial \tau} = \operatorname{div}\Big\{\hat{\rho}^{k,j}\nabla c^*\Big[\nabla\Big(F'(\hat{\rho}^{k,j})\Big)\Big] + \hat{\rho}^{k,j} y\Big\} & \text{in } \ \Omega_k \times (0,\infty) \\ \hat{\rho}^{k,j}\Big(\nabla c^*\Big[\nabla\Big(F'(\hat{\rho}^{k,j})\Big)\Big] + y\Big)\cdot\nu = 0 & \text{on } \ \partial\Omega_k \times (0,\infty) \\ \hat{\rho}^{k,j}(\tau=0) = \rho_0^{k,j} & \text{in } \ \Omega_k \end{cases} \tag{50}$$

From Part 1, we have for $q=2$,

$$H_c^F\left(\hat{\rho}^{k,j}(\tau)|\hat{\rho}_\infty^{k,j}\right) \le e^{-2\tau} H_c^F\left(\rho_0^{k,j}|\hat{\rho}_\infty^{k,j}\right), \tag{51}$$

where $\hat{\rho}_\infty^{k,j}$ is the unique equilibrium solution of (50), defined by

$$\nabla\left(F'(\hat{\rho}_\infty^{k,j}) + c\right) = 0 \ \text{ in } \ \Omega_k \cap \operatorname{supp}(\hat{\rho}_\infty^{k,j}), \quad \text{and} \quad \int_{\Omega_k} \hat{\rho}_\infty^{k,j}\,\mathrm{d}y = \int_{\Omega_k} \rho_0^{k,j}\,\mathrm{d}y. \tag{52}$$

Next, we let $(k,j)\to(\infty,\infty)$ in (51) to show (32). To proceed, we will assume that for all $\tau>0$, the sequence $\{\hat{\rho}^{k,j}(\tau)\}_{k,j}$ of solutions of (50) converges a.e. to the solution $\hat{\rho}(\tau)$ of equations (10)-(11). Actually, this assumption is true for $q=2$ (see [8, 24]). For $q\neq 2$, we intend to give the details in a forthcoming paper where we investigate the existence of solutions to equations (10)-(11) in the entire domain $I\!R^n\times(0,\infty)$. Since F and c are non-negative, then Fatou's lemma gives that

$$H_c^F\left(\hat{\rho}(\tau)\right) \le \liminf_{k,j\to\infty} H_c^F\left(\hat{\rho}^{k,j}(\tau)\right). \tag{53}$$

It is now sufficient to show that

$$\lim_{k,j\to\infty} H_c^F(\hat{\rho}_\infty^{k,j}) = H_c^F(\hat{\rho}_\infty) \tag{54}$$

to conclude (32), via (49), (51), (53) and (54). First, we show that $\hat{\rho}_\infty^{k,j}$ and $\hat{\rho}_\infty$ are the unique minimizers of $H_c^F(\hat{\rho})$ on Ω_k and $I\!R^n$ respectively, that is,

$$H_c^F(\hat{\rho}_\infty^{k,j}) = \inf\{H_c^F(\hat{\rho}):\ \hat{\rho}:\Omega_k\to[0,\infty),\ \int_{\Omega_k}\hat{\rho}\,\mathrm{d}y = \int_{\Omega_k}\rho_0^{k,j}\} \tag{55}$$

and

$$H_c^F(\hat{\rho}_\infty) = \inf\{H_c^F(\hat{\rho}),\ \hat{\rho}\in\mathcal{P}_a(I\!R^n)\}. \tag{56}$$

It is easy to check that (55) and (56) follow from the (total) energy inequality (71) established in the next section. Indeed, by (71) and (52), we have that

$$H_c^F(\hat{\rho}) - H_c^F(\hat{\rho}_\infty^{k,j}) \ge \frac{\lambda_q}{q} W_q\left(\hat{\rho}, \hat{\rho}_\infty^{k,j}\right) \ge 0,$$

which shows (55). Furthermore, if $\bar{\rho}_\infty^{k,j}$ is another minimizer of H_c^F in (55), then

$$0 = H_c^F(\bar{\rho}_\infty^{k,j}) - H_c^F(\hat{\rho}_\infty^{k,j}) \geq \frac{\lambda_q}{q} W_q\left(\hat{\rho}, \hat{\rho}_\infty^{k,j}\right) \geq 0,$$

that is, $W_q\left(\bar{\rho}_\infty^{k,j}, \hat{\rho}_\infty^{k,j}\right) = 0$ or $\bar{\rho}_\infty^{k,j} = \hat{\rho}_\infty^{k,j}$. For the proof of (56), we use (28) in place of (52).

Since F and c are non-negative, and F has a superlinear growth, the arguments used in [24] yield that $H_c^F(\hat{\rho}_\infty^{k,j}) \leq H_c^F(\hat{\rho}_\infty)$ and $\hat{\rho}_\infty^{k,j} \rightharpoonup \hat{\rho}_\infty$ in $L^1(\mathbb{R}^n)$. Then we use that $c \geq 0$ and the weak lower semicontinuity of H^F (which holds because $F \geq 0$ is convex) to conclude that

$$H_c^F(\hat{\rho}_\infty) \leq \liminf_{k,j\to\infty} H_c^F(\hat{\rho}_\infty^{k,j}) \leq \limsup_{k,j\to\infty} H_c^F(\hat{\rho}_\infty^{k,j}) \leq H_c^F(\hat{\rho}_\infty).$$

This proves (54) and then concludes the proof of (32). The proof of (33) is similar.

3 Convergence w.r.t. the Wasserstein distances

Here we estimate the rates at which the q-Wasserstein distance $W_q(\hat{\rho}(\tau), \hat{\rho}_\infty)$ between a solution $\hat{\rho}(\tau)$ of the rescaled equations (10)-(11) and the equilibrium solution $\hat{\rho}_\infty$ converges to 0. We start by recalling the notion of *uniform c-convexity* introduced in [9]. This notion will be used to generalize the Talagrand's inequality [28] (which is an estimate of the 2-Wasserstein distance in terms of the relative entropy), from which the convergence to equilibrium in the q-Wasserstein distance for equations (10)-(11) will be derived. A function $V : \mathbb{R}^n \to \mathbb{R}$ is uniformly c-displacement convex with $\mathrm{Hess}_c V \geq \lambda I$ for some $\lambda \in \mathbb{R}$, if for all $a, b \in \mathbb{R}^n$, we have

$$V(b) - V(a) \geq \nabla V(a) \cdot (b-a) + \lambda c(b-a). \tag{57}$$

The following lemma expresses the uniform c-convexity of $c(x) = \frac{|x|^q}{q}$, $q > 1$.

Lemma 3.1 *Let $c(x) = \frac{|x|^q}{q}$, $q > 1$. Then, there exists $\lambda \geq 0$ such that, for all $a, b \in \mathbb{R}^n$, we have*

$$c(b) - c(a) \geq \nabla c(a) \cdot (b-a) + \lambda c(b-a). \tag{58}$$

Furthermore, the optimal constant $\lambda = \lambda_q$ in (58) satisfies:

(i). If $1 < q < 2$, then $\lambda_q = 0$.

(ii). If $q = 2$, then $\lambda_q = 1$.

(iii). If $q > 2$, then $\lambda_q = \frac{q-1}{(r+1)^{q-2}}$, where $r > 0$ uniquely solves the equation

$$r^{q-1} - (q-1)r - (q-2) = 0.$$

Example. If $q = 4$, then $r = 2$ is the unique positive solution of $r^3 - 3r - 2 = 0$, and therefore $\lambda_4 = \frac{1}{3}$.

Proof of Lemma 3.1. Since $c(x) = \frac{|x|^q}{q}$, $q > 1$, is convex, then inequality (58) holds for $\lambda = 0$. Therefore, the optimal constant λ_q in (58) satisfies

$$\lambda_q \geq 0. \tag{59}$$

If $a = 0$, inequality (58) gives that $c(b) \geq \lambda_q c(b)$, and then

$$\lambda_q \leq 1. \tag{60}$$

If $a \neq 0$, dividing (58) by $c(a)$, we have that

$$\Big|\frac{b}{|a|}\Big|^q - 1 \geq q\frac{a}{|a|} \cdot \left(\frac{b}{|a|} - \frac{a}{|a|}\right) + \lambda_q \Big|\frac{b}{|a|} - \frac{a}{|a|}\Big|^q.$$

So, it is sufficient to prove that, $\forall a, b \in I\!R$ such that $|a| = 1$,

$$|b|^q - 1 \geq qa \cdot (b - a) + \lambda_q |b - a|^q,$$

that is,

$$r^q - 1 \geq q(rx - 1) + \lambda_q (1 + r^2 - 2rx)^{q/2} \tag{61}$$

where $r = |b| \geq 0$ and $x = \cos\theta \in [-1, 1]$, with θ being the angle between the vectors a and b.

(ii). If $q = 2$, then c is twice continuously differentiable, and Hess c $= I$. Hence $\lambda_2 = 1$.

(i). If $1 < q < 2$, setting $r = 1$ in (61), we have that

$$\lambda_q \leq \frac{q(1-x)^{1-q/2}}{2^{q/2}}. \tag{62}$$

We let $x \to 1$ in (62), and we use (59) to conclude that $\lambda_q = 0$.

(iii). If $q > 2$, inequality (61) gives that

$$\lambda_q = \min_{(r,x)\in D} \left[f(r,x) := \frac{r^q - 1 - q(rx - 1)}{(1 + r^2 - 2rx)^{q/2}} \right] \tag{63}$$

where $D = \{(r, x) : \ r \geq 0, -1 \leq x \leq 1\}$.

In the interior $(r, x) \in (0, \infty) \times (-1, 1)$ of D, there is no critical points, because $(f_r(r, x), f_x(r, x)) = (0, 0)$ gives that

$$\begin{cases} (r^{q-1} - x)(1 + r^2 - 2rx) - (r - x)\left(r^q - 1 - q(rx - 1)\right) & = \ 0 \\ -(1 + r^2 - 2rx) + \left(r^q - 1 - q(rx - 1)\right) & = \ 0 \end{cases}$$

i.e. $r = 1$ and $2(1 - x) = q(1 - x)$, which leads to the contradiction $q = 2$.

On the boundary $(r,x) \in [0,\infty) \times \{1\}$, we have that $f(r,1) = \frac{r^q-1-q(r-1)}{|r-1|^q}$. By a direct computation, we have that $f_r(r,1) = \frac{q}{|r-1|^q}\left(q-1-\frac{r^{q-1}-1}{r-1}\right)$, and then $\min\{f(r,1),\ 1 \neq r \geq 0\} = \min\{1, q-1\} = 1$, i.e. $\lambda_q \leq 1$.

On the boundary $(r,x) \in [0,\infty) \times \{-1\}$, we have that $f(r,-1) = \frac{r^q-1+q(r+1)}{|r+1|^q}$. It is easy to check that $f_r(r,-1) = \frac{q}{|r+1|^{q+1}}\left[r^{q-1}-(q-1)r-(q-2)\right]$. Then $\min\{f(r,-1),\ r \geq 0\}$ is attained at the point $\bar{r} > 0$, solution of the equation

$$\bar{r}^{q-1} - (q-1)\bar{r} - (q-2) = 0. \tag{64}$$

Noting that $f(\bar{r},-1) \leq f(1,-1) = \frac{q}{2^{q-1}} < 1$, and using (60) and (63), we deduce that

$$\lambda_q = f(\bar{r},-1) = \frac{\bar{r}^q - 1 + q(\bar{r}+1)}{(\bar{r}+1)^q}. \tag{65}$$

Next, we multiply (64) by $\bar{r}$ to have $\bar{r}^q = (q-1)\bar{r}^2 + (q-2)\bar{r}$, and we substitute this expression into (65) to conclude that $\lambda_q = \frac{q-1}{(\bar{r}+1)^{q-2}}$, where $\bar{r} > 0$ solves equation (64).

The next theorem gives the convergence to equilibrium in the q-Wasserstein distance for equations (10)-(11).

Theorem 3.2 *Let $c(x) = \frac{|x|^q}{q}$, $q \geq 2$, and assume that $G : [0,\infty) \to \mathbb{R}$ satisfies (H). Then for all probability densities $u, u_\infty \in \mathcal{P}_a(\mathbb{R}^n)$, such that $\nabla\left(G'(u_\infty) + c\right) = 0$, the following generalization of the Talagrand's inequality holds:*

$$W_q(u,u_\infty) \leq \left[\frac{q}{\lambda_q} H_c^G(u|u_\infty)\right]^{1/q}, \tag{66}$$

where λ_q is defined as in Lemma 3.1.

Furthermore, if $G = F$ is defined by (7) with $m \geq \frac{n-p+1}{n(p-1)}$ and $\frac{1}{p} + \frac{1}{q} = 1$, and if the initial probability density $\rho_0 \in \mathcal{P}_a(\mathbb{R}^n)$ is such that $H_c^F(\rho_0) < \infty$, then any solution $\hat{\rho}(\tau)$ of equations (10)-(11) with finite energy $H_c^F(\hat{\rho}(\tau)) < \infty$, satisfies the decay rates: If $q = 2$, then

$$W_2(\hat{\rho}(\tau), \hat{\rho}_\infty) \leq e^{-\tau}\sqrt{2H_c^F(\rho_0|\hat{\rho}_\infty)} \tag{67}$$

and if $q > 2$, then

$$W_q(\hat{\rho}(\tau), \hat{\rho}_\infty) \leq e^{-\tau/q}\left[\frac{qH_c^F(\rho_0|\hat{\rho}_\infty)}{\lambda_q}\right]^{1/q}, \tag{68}$$

where $\hat{\rho}_\infty$ is the unique equilibrium solution of equations (10)-(11), defined by (15).

Proof. For $q \neq 2$, (66) was first established by Cordero-Gangbo-Houdré [9]. The proof that we present below is taken from this paper. Here, we will assume that $u \in C(\mathbb{R}^n)$ and u_∞ have compact supports contained in some bounded domain $\Omega \subset \mathbb{R}^n$. The complete proof is obtained by an approximation argument, as done in Theorem 2.2.

The idea of the proof is based on the toy model Talagrand's inequality (25) established in Section 2. Indeed, let T be the optimal map that pushes u_∞ forward to u in (17) with the cost $c(x) = \frac{|x|^q}{q}$, $q \geq 2$. Then, the energy inequality (26) holds, that is,

$$H^G(u) - H^G(u_\infty) \geq \int u_\infty \nabla\left(G'(u_\infty)\right) \cdot (T - I)\, \mathrm{d}y. \tag{69}$$

Since $q \geq 2$, then $c(x) = \frac{|x|^q}{q}$ satisfies (58) with $\lambda_q > 0$. Then,

$$\begin{aligned} H_c(u) - H_c(u_\infty) &= \int \left[c\left(T(y)\right) - c(y)\right] u_\infty(y)\, \mathrm{d}y \\ &\geq \int u_\infty \nabla c \cdot (T - I)\, \mathrm{d}y + \lambda_q \int c(T - I) u_\infty\, \mathrm{d}y \\ &= \int u_\infty \nabla c \cdot (T - I)\, \mathrm{d}y + \lambda_q W_c(u, u_\infty). \end{aligned} \tag{70}$$

Adding (69) and (70), and using that $W_c = \frac{1}{q} W_q^q$, we obtain the total free energy inequality

$$H_c^G(u) - H_c^G(u_\infty) \geq \int u_\infty \nabla\left(F'(u_\infty) + c\right) \cdot (T - I)\, \mathrm{d}y + \frac{\lambda_q}{q} W_q(u, u_\infty)^q, \tag{71}$$

which is analogous to the basic uniform convexity inequality (21), in the sense that it expresses the c-uniform displacement convexity of H_c^G. Next, we use that $\nabla\left(G'(u_\infty) + c\right) = 0$ to conclude (66).

If $q = 2$, Lemma 3.1 gives that $\lambda_2 = 1$. Then we combine (32) and (66) - where we use $G = F$ - to conclude (67). If $q \neq 2$, we combine (33) and (66) to conclude (68).

4 Convergence w.r.t the L^1-norm

In this section, we establish the rates at which the L^1-distance $\|\hat\rho(\tau) - \hat\rho_\infty\|_{L^1(R^n)}$ between a solution $\hat\rho(\tau)$ of the rescaled equations (10)-(11), and the equilibrium solution $\hat\rho_\infty$ converges to 0. Then, we derive the rate of convergence to 0 of the L^1-distance $\|\rho(t) - \rho_\infty\|_{L^1(R^n)}$ between a solution $\rho(t)$ of the initial equations (1)-(2), and the fundamental solution ρ_∞, defined by (9), (13) and (15). The L^1-decay for equations (10)-(11) will be obtained from the convergence of the relative energy (Section 2), via some Csiszár-Kullback [11, 21] type inequalities, which are estimates of the L^1-distance $\|\hat\rho(\tau) - \hat\rho_\infty\|_{L^1(R^n)}$ in terms of the relative energy $H_c^F\left(\hat\rho(\tau)|\hat\rho_\infty\right)$.

Theorem 4.1 *Let $c(x) = \frac{|x|^q}{q}$, $q > 1$, and $F : [0, \infty) \to I\!R$ be defined by (7) with $m \geq \frac{n-p+1}{n(p-1)}$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then for any probability density $u \in \mathcal{P}_a(I\!R^n)$, the following Csiszár-Kullback type inequalities hold: If $m = \frac{1}{p-1}$, then*

$$\|u - \hat\rho_\infty\|^2_{L^1(R^n)} \leq 8(p-1) H_c^F(u|\hat\rho_\infty) \tag{72}$$

and if $m \neq \frac{1}{p-1}$, then

$$\|u - \hat{\rho}_\infty\|^2_{L^1(R^n)} \leq \frac{8}{m} \left(\int_{R^n} \hat{\rho}_\infty^{2-\gamma} \right) H_c^F(u|\hat{\rho}_\infty), \tag{73}$$

where $\hat{\rho}_\infty \in \mathcal{P}_a(\mathbb{R}^n)$ is the unique equilibrium solution of equations (10)-(11), defined by (15).

Furthermore, if the initial probability density $\rho_0 \in \mathcal{P}_a(\mathbb{R}^n)$ is such that $H_c^F(\rho_0) < \infty$, then any solution $\hat{\rho}(\tau)$ of equations (10)-(11) with finite energy $H_c^F(\hat{\rho}(\tau)) < \infty$, satisfies the decay rates: If $q = 2$, then

$$\|\hat{\rho}(\tau) - \hat{\rho}_\infty\|_{L^1(R^n)} \leq M e^{-\tau} \sqrt{H_c^F(\rho_0|\hat{\rho}_\infty)} \tag{74}$$

and if $q \neq 2$, then

$$\|\hat{\rho}(\tau) - \hat{\rho}_\infty\|_{L^1(R^n)} \leq M e^{-\tau/2} \sqrt{H_c^F(\rho_0|\hat{\rho}_\infty)}, \tag{75}$$

where $M > 0$ is a constant.

The scalings (9), (12), (13) and the exponential decays (74) and (75) of the solutions $\hat{\rho}(\tau)$ of equations (10)-(11) to the stationary solution $\hat{\rho}_\infty$, give an algebraic decay in the L^1-norm for the solutions $\rho(t)$ of equations (1)-(2) to the fundamental solution ρ_∞, defined by (9), (13) and (15), as shown in the next theorem.

Theorem 4.2 *Let $c(x) = \frac{|x|^q}{q}$, $q > 1$, and $F : [0, \infty) \to \mathbb{R}$ be defined by (7) with $m \geq \frac{n-p+1}{n(p-1)}$ and $\frac{1}{p} + \frac{1}{q} = 1$. If the initial probability density $\rho_0 \in \mathcal{P}_a(\mathbb{R}^n)$ is such that $H_c^F(\rho_0) < \infty$, then any solution $\rho(t)$ of equations (1)-(2) satisfies the decay rates: If $q = 2$, then*

$$\|\rho(t) - \rho_\infty\|_{L^1(R^n)} \leq \frac{M\sqrt{H_c^F(\rho_0|\hat{\rho}_\infty)}}{(1 + \delta_2 t)^{1/\delta_2}} \tag{76}$$

and if $q \neq 2$, then

$$\|\rho(t) - \rho_\infty\|_{L^1(R^n)} \leq \frac{M\sqrt{H_c^F(\rho_0|\hat{\rho}_\infty)}}{(1 + \delta_p t)^{1/2\delta_p}}, \tag{77}$$

where $\delta_p = (p-1)(nm+1) + 1 - n$ for all $p > 1$, $M > 0$ is a constant, and $\hat{\rho}_\infty$ is the unique equilibrium solution of equations (10)-(11) defined by (15), and ρ_∞ is defined by (9) and (13).

Proof of Theorem 4.2. Straightforward, using (9)-(13), (74) and (75).

Proof of Theorem 4.1. The proof of the Csiszár-Kullback type inequalities (72) and (73) is standard. For $q = 2$, (72) is first established in [11, 21] for $F(x) = x \ln x$ and $\hat{\rho}_\infty = \frac{e^{-|x|^2/2}}{2\pi}$, and (73) first appeared in [5] for $F(x) = \frac{x^m}{m-1}$, $m > 1$, and $\hat{\rho}_\infty = \left(K - \frac{m-1}{2m}|x|^2\right)_+^{\frac{1}{m-1}}$. Some generalizations of this inequality can also be found in [29, 8].

Here, we prove (72) and (73) for all $q>1$. The proof follows the same ideas as in the case $q=2$. First, we observe that the relative energy $H_c^F(u|\hat\rho_\infty)$ can be written as

$$\begin{aligned} H_c^F(u|\hat\rho_\infty) &= \int_{[\hat{\rho_\infty}\neq 0]} [F(u)-F(\hat\rho_\infty)-F'(\hat\rho_\infty)(u-\hat\rho_\infty)]\,\mathrm{d}y \\ &\quad + M\int_{[\hat{\rho_\infty}\neq 0]}(u-\hat\rho_\infty)\,\mathrm{d}y + \int_{[\hat\rho_\infty=0]}[F(u)+cu]\,\mathrm{d}y, \end{aligned} \tag{78}$$

for some real M. Indeed, since $\nabla\left(F'(\hat\rho_\infty)+c\right)=0$ on $\mathrm{supp}(\hat\rho_\infty)$, we have that $c(y)=M-F'\left(\hat\rho_\infty(y)\right)$ for all $y\in[\hat\rho_\infty\neq 0]$, and for some real M. Therefore,

$$\begin{aligned} H_c^F(u|\hat\rho_\infty) &= \int_{\mathbb{R}^n}[F(u)-F(\hat\rho_\infty)+c(u-\hat\rho_\infty)]\,\mathrm{d}y \\ &= \int_{[\hat\rho_\infty\neq 0]}[F(u)-F(\hat\rho_\infty)-F'(\hat\rho_\infty)(u-\hat\rho_\infty)]\,\mathrm{d}y + M\int_{[\hat\rho_\infty\neq 0]}(u-\hat\rho_\infty)\,\mathrm{d}y \\ &\quad + \int_{[\hat\rho_\infty=0]}[F(u)+cu-(F(\hat\rho_\infty)+c\hat\rho_\infty)]\,\mathrm{d}y, \end{aligned}$$

which proves (78), because on $[\hat\rho_\infty=0]$, $F(\hat\rho_\infty)+c\hat\rho_\infty=F(0)=0$. The second order Taylor expansion of F around 1, evaluated at $t=\frac{u(y)}{\hat\rho_\infty(y)}$ for $y\in[\hat\rho_\infty\neq 0]$, gives on $[\hat\rho_\infty\neq 0]$

$$F\left(\frac{u}{\hat\rho_\infty}\right)-F(1)-\frac{F'(1)}{\hat\rho_\infty}(u-\hat\rho_\infty)=\frac{1}{2\hat\rho_\infty^2}|u-\hat\rho_\infty|^2F''\left(1+\theta(\frac{u}{\hat\rho_\infty}-1)\right), \tag{79}$$

for some $\theta\in(0,1)$.

Case 1: $m=\frac{1}{p-1}$.

Here $F(t)=\frac{1}{p-1}t\ln t$ and $\hat\rho_\infty(y)=\frac{1}{\sigma}\exp\left(-\frac{p-1}{q}|y|^q\right)$. Then $[\hat\rho_\infty\neq 0]=\mathbb{R}^n$, and (78) read as

$$H_c^F(u|\hat\rho_\infty)=\int_{\mathbb{R}^n}[F(u)-F(\hat\rho_\infty)-F'(\hat\rho_\infty)(u-\hat\rho_\infty)]\,\mathrm{d}y, \tag{80}$$

where we use that $\int_{\mathbb{R}^n}u\,\mathrm{d}y=\int_{\mathbb{R}^n}\hat\rho_\infty\,\mathrm{d}y$. Multiplying (79) by $\hat\rho_\infty$, and integrating over $\mathbb{R}^n$, we have, after using (80),

$$\begin{aligned} H_c^F(u|\hat\rho_\infty) &= \frac{1}{2(p-1)}\int_{\mathbb{R}^n}\frac{1}{\hat\rho_\infty}|u-\hat\rho_\infty|^2\frac{1}{1+\theta\left(\frac{u}{\hat\rho_\infty}-1\right)}\,\mathrm{d}y \\ &\geq \frac{1}{2(p-1)}\int_{[u<\hat\rho_\infty]}\frac{1}{\hat\rho_\infty}|u-\hat\rho_\infty|^2\,\mathrm{d}y. \end{aligned} \tag{81}$$

Since u and $\hat\rho_\infty$ have equal mass, we have that $\frac{1}{2}\int_{\mathbb{R}^n}|u-\hat\rho_\infty|\,\mathrm{d}y=\int_{[u<\hat\rho_\infty]}|u-\hat\rho_\infty|\,\mathrm{d}y$. Then, we use Hölder's inequality and (81) to obtain that

$$
\begin{aligned}
\frac{1}{2}\int_{R^n} |u-\hat\rho_\infty|\,\mathrm{d}y &= \int_{[u<\hat\rho_\infty]} \frac{1}{\sqrt{\hat\rho_\infty}}|u-\hat\rho_\infty|\sqrt{\hat\rho_\infty}\,\mathrm{d}y \\
&\le \left(\int_{[u<\hat\rho_\infty]} \frac{1}{\hat\rho_\infty}|u-\hat\rho_\infty|^2\,\mathrm{d}y\right)^{1/2}\left(\int_{R^n}\hat\rho_\infty\,\mathrm{d}y\right)^{1/2} \\
&= \left(\int_{[u<\hat\rho_\infty]} \frac{1}{\hat\rho_\infty}|u-\hat\rho_\infty|^2\,\mathrm{d}y\right)^{1/2} \\
&\le \Big[2(p-1)H_c^F(u|\hat\rho_\infty)\Big]^{1/2}.
\end{aligned}
$$

This proves (72).

Case 2: $m<\frac{1}{p-1}$.

Here $F(t)=\frac{mt^\gamma}{\gamma(\gamma-1)}$ where $\gamma=m+\frac{p-2}{p-1}<1$, and $\hat\rho_\infty(y)=\left(K-\frac{\gamma-1}{mq}|y|^q\right)^{1/\gamma-1}$ is positive. Therefore $[\hat\rho_\infty\neq 0]=I\!R^n$ and (80) holds. We multiply (79) by $\hat\rho_\infty^\gamma$, and we observe that

$$
\hat\rho_\infty^\gamma F\left(\frac{u}{\hat\rho_\infty}\right)=F(u),\quad \rho_\infty^\gamma F(1)=F(\hat\rho_\infty)\ \text{ and }\ \hat\rho_\infty^{\gamma-1}F'(1)=F'(\hat\rho_\infty)
$$

to have that

$$
F(u)-F(\hat\rho_\infty)-F'(\hat\rho_\infty)(u-\hat\rho_\infty)=\frac{1}{2}\hat\rho_\infty^{\gamma-2}|u-\hat\rho_\infty|^2F''\left(1+\theta\left(\frac{u}{\hat\rho_\infty}-1\right)\right). \tag{82}
$$

Then, we use (80) to obtain that

$$
H_c^F(u|\hat\rho_\infty)=\frac{m}{2}\int_{R^n}\hat\rho_\infty^{\gamma-2}|u-\hat\rho_\infty|^2\left(1+\theta\left(\frac{u}{\hat\rho_\infty}-1\right)\right)^{\gamma-2}. \tag{83}
$$

Since $\gamma<1$, then (83) gives that

$$
H_c^F(u|\hat\rho_\infty)\ge\frac{m}{2}\int_{[u<\hat\rho_\infty]}\hat\rho_\infty^{\gamma-2}|u-\hat\rho_\infty|^2\,\mathrm{d}y. \tag{84}
$$

We use (84), Hölder's inequality and the fact that u and $\hat\rho_\infty$ have equal mass, to obtain that

$$
\begin{aligned}
\frac{1}{2}\int_{R^n}|u-\hat\rho_\infty|\,\mathrm{d}y &= \int_{[u<\hat\rho_\infty]}|u-\hat\rho_\infty|\,\mathrm{d}y \\
&\le \left(\int_{[u<\hat\rho_\infty]}|u-\hat\rho_\infty|^2\hat\rho_\infty^{\gamma-2}\,\mathrm{d}y\right)^{1/2}\left(\int_{R^n}\hat\rho_\infty^{2-\gamma}\,\mathrm{d}y\right)^{1/2} \\
&\le \left(\frac{2}{m}H_c^F(u|\hat\rho_\infty\right)^{1/2}\left(\int_{R^n}\hat\rho_\infty^{2-\gamma}\,\mathrm{d}y\right)^{1/2}.
\end{aligned}
$$

This proves (73). It is easy to check that $\int_{R^n} \hat\rho_\infty^{2-\gamma}\,\mathrm{d}y < \infty$ since $\gamma \geq 1-\frac{1}{n}$ because of (3).

Case 3: $m > \frac{1}{p-1}$.

Here $F(t) = \frac{mt^\gamma}{\gamma(\gamma-1)}$ where $\gamma = m + \frac{p-2}{p-1} > 1$, and $\hat\rho_\infty(y) = \left(K - \frac{\gamma-1}{mq}|y|^q\right)_+^{1/\gamma-1}$ has a compact support, $\mathrm{supp}(\hat\rho_\infty) = \{y \in \mathbb{R}^n : \; M - c(y) \geq 0\} = \{y \in \mathbb{R}^n : \; |y| < (qM)^{1/q}\}$, where M and K are related by $K = \frac{\gamma-1}{m}M$. Using that $c(y) \geq M$ on $[\hat\rho_\infty = 0]$, we have from (78)

$$\begin{aligned}
H_c^F(u|\hat\rho_\infty) \;\; &\geq \;\; \int_{[\hat\rho_\infty\neq 0]} [F(u) - F(\hat\rho_\infty) - F'(\hat\rho_\infty)(u-\hat\rho_\infty)]\,\mathrm{d}y \\
&\quad + M\int_{[\hat\rho_\infty\neq 0]} (u-\hat\rho_\infty)\,\mathrm{d}y + M\int_{[\hat\rho_\infty = 0]} (u-\hat\rho_\infty)\,\mathrm{d}y + \int_{[\hat\rho_\infty = 0]} F(u)\,\mathrm{d}y \\
&= \;\; \int_{[\hat\rho_\infty\neq 0]} [F(u) - F(\hat\rho_\infty) - F'(\hat\rho_\infty)(u-\hat\rho_\infty)]\,\mathrm{d}y + M\int_{R^n}(u-\hat\rho_\infty)\,\mathrm{d}y \\
&\quad + \int_{[\hat\rho_\infty = 0]} F(u)\,\mathrm{d}y.
\end{aligned}$$

And since u and $\hat\rho_\infty$ have equal mass and $F \geq 0$, we obtain that

$$H_c^F(u|\hat\rho_\infty) \geq \int_{[\hat\rho_\infty\neq 0]} [F(u) - F(\hat\rho_\infty) - F'(\hat\rho_\infty)(u-\hat\rho_\infty)]\,\mathrm{d}y. \tag{85}$$

As is Case 2, (82) holds on $[\hat\rho_\infty \neq 0]$. Then using (85), we have that

$$\begin{aligned}
H_c^F(u|\hat\rho_\infty) \;\; &\geq \;\; \int_{[\hat{\rho_\infty}\neq 0]} [F(u) - F(\hat\rho_\infty) - F'(\hat\rho_\infty)(u-\hat\rho_\infty)]\,\mathrm{d}y \\
&= \;\; \frac{m}{2}\int_{[\hat\rho_\infty\neq 0]} \hat\rho_\infty^{\gamma-2}|u-\hat\rho_\infty|^2 \left(1+\theta\left(\frac{u}{\hat\rho_\infty}-1\right)\right)^{\gamma-2}\,\mathrm{d}y. \qquad (86)
\end{aligned}$$

If $1 < \gamma \leq 2$, then (86) gives that

$$H_c^F(u|\hat\rho_\infty) \geq \frac{m}{2}\int_{[\hat\rho_\infty\neq 0]\cap[u<\hat\rho_\infty]} \hat\rho_\infty^{\gamma-2}|u-\hat\rho_\infty|^2\,\mathrm{d}y = \frac{m}{2}\int_{[u<\hat\rho_\infty]} \hat\rho_\infty^{\gamma-2}|u-\hat\rho_\infty|^2\,\mathrm{d}y$$

because $[u < \hat\rho_\infty] \subset [\hat\rho_\infty \neq 0]$ since $u \geq 0$. Then we follow the same argument as in Case 2 to obtain (73).

If $\gamma > 2$, then (86) gives that

$$H_c^F(u|\hat\rho_\infty) \geq \frac{m}{2}\int_{[u>\hat\rho_\infty]\cap[\hat\rho_\infty\neq 0]} \hat\rho_\infty^{\gamma-2}|u-\hat\rho_\infty|^2\,\mathrm{d}y \geq \frac{m}{2}\int_{[u>\hat\rho_\infty]} \hat\rho_\infty^{\gamma-2}|u-\hat\rho_\infty|^2\,\mathrm{d}y \tag{87}$$

because $\hat\rho_\infty^{\gamma-2} = 0$ on $[\hat\rho_\infty = 0]$. Using (87), Hölder's inequality and the fact that u and $\hat\rho_\infty$ have equal mass, we obtain as in Case 2

$$\frac{1}{2}\int_{R^n} |u-\hat\rho_\infty|\,\mathrm{d}y = \int_{[u>\hat\rho_\infty]} |u-\hat\rho_\infty|\,\mathrm{d}y \leq \left(\frac{2}{m}H_c^F(u|\hat\rho_\infty)\right)^{1/2}\left(\int_{R^n}\hat\rho_\infty^{2-\gamma}\,\mathrm{d}y\right)^{1/2},$$

which proves (73). Note that $\int_{R^n} \hat{\rho}_\infty^{2-\gamma}\,\mathrm{d}y < \infty$ because

$$\int_{R^n} \hat{\rho}_\infty^{2-\gamma}\,\mathrm{d}y = \int_{|y|\le (qM)^{1/q}} \left(K - \frac{\gamma-1}{m}|y|^q\right)^{\frac{2-\gamma}{\gamma-1}} \mathrm{d}y = \mathrm{cst}\int_{|z|\le 1} (1-|z|^q)^{\frac{2-\gamma}{\gamma-1}}\,\mathrm{d}z$$

converges since $\frac{\gamma-2}{\gamma-1} < 1$.

If $q = 2$, we combine (32) and (72)-(73) to deduce (74). And if $q \neq 2$, we combine (33) and (72)-(73) to conclude (75).

5 Examples

Here, we apply Theorems 2.2, 3.2, 4.1 and 4.2 proved in Sections 2-4 to some examples of equations of the form (1)-(2).

Heat equation ($p = 2, m = 1$). If $c(x) = \frac{|x|^2}{2}$ and $F(x) = x\ln x$, then (1)-(2) is the heat equation

$$\begin{cases} \frac{\partial \rho}{\partial t} = \Delta\rho & \text{in } \mathbb{R}^n \times (0,\infty) \\ \rho(t=0) = \rho_0 & \text{in } \mathbb{R}^n. \end{cases} \tag{88}$$

Scaling in time and space as

$$\tau = \ln(\sqrt{1+2t}),\ y = \frac{x}{\sqrt{1+2t}}\ ,\hat{\rho}(\tau, y) = (1+2t)^{\frac{n}{2}}\rho(t,x),$$

(88) reads as the linear Fokker-Planck equation

$$\begin{cases} \frac{\partial \hat{\rho}}{\partial \tau} = \Delta\hat{\rho} + \mathrm{div}\,(y\hat{\rho}) & \text{in } \mathbb{R}^n \times (0,\infty) \\ \hat{\rho}(\tau=0) = \rho_0 & \text{in } \mathbb{R}^n. \end{cases}$$

Theorems 2.2, 3.2 and 4.1 assert that the self-similar solutions $\hat{\rho}$ of (88) decay exponentially fast to the normalized Gaussian $\hat{\rho}_\infty(y) = \frac{e^{-|y|^2/2}}{2\pi}$ in relative entropy, in the 2-Wasserstein distance and in the L^1-norm, at the rates 2, 1 and 1 respectively. Theorem 4.2 gives the algebraic decay of the solutions ρ of (88) to the fundamental solution $\rho_\infty(t,x) = \frac{e^{-|x|^2/2(1+2t)}}{2\pi(1+2t)^{n/2}}$ in L^1-norm, at the rate 1/2. These rates are all sharp.

Porous medium and fast diffusion equations ($p = 2$, $1 \neq m > 1 - \frac{1}{n}$). If $c(x) = \frac{|x|^2}{2}$ and $F(x) = \frac{x^m}{m-1}$, then (1)-(2) is the porous medium equation ($m > 1$), or the fast diffusion equation ($m < 1$)

$$\begin{cases} \frac{\partial \rho}{\partial t} = \Delta\rho^m & \text{in } \mathbb{R}^n \times (0,\infty) \\ \rho(t=0) = \rho_0 & \text{in } \mathbb{R}^n. \end{cases} \tag{89}$$

After scaling in time and space as in (12)-(13) with $R(t) = (1+\delta_2 t)^{1/\delta_2}$ and $\delta_2 = n(m-1)+2$, then (89) reads as

$$\begin{cases} \frac{\partial \hat{\rho}}{\partial \tau} = \Delta\hat{\rho}^m + \mathrm{div}\,(y\hat{\rho}) & \text{in } \mathbb{R}^n \times (0,\infty) \\ \hat{\rho}(\tau=0) = \rho_0 & \text{in } \mathbb{R}^n. \end{cases}$$

Theorems 2.2, 3.2 and 4.1 give the same exponential decay rates to the equilibrium solution $\hat{\rho}_\infty(y) = \left(K_\infty + \frac{1-m}{2m}|y|^2\right)_+^{\frac{1}{m-1}}$ as for the heat equation, while Theorem 4.2 gives the algebraic decay of solutions ρ of (89) to the Barenblatt-Prattle profile
$\rho_\infty(t,x) = \frac{1}{(1+\delta_2 t)^{n/\delta_2}}\left(K_\infty + \frac{1-m}{2m(1+\delta_2)^{2/\delta_2}}|x|^2\right)_+^{\frac{1}{m-1}}$ in L^1-norm, at the rate $\frac{1}{n(m-1)+2}$. These rates are all sharp.

Parabolic p-Laplacian equation ($2 \neq p \geq \frac{2n+1}{n+1}$, $m = 1$). If $c(x) = \frac{|x|^q}{q}$ and $F(x) = \frac{x^\gamma}{\gamma(\gamma-1)}$ where $\gamma = \frac{2p-3}{p-1}$ and $\frac{1}{p} + \frac{1}{q} = 1$, then (1)-(2) is the parabolic p-Laplacian equation

$$\begin{cases} \frac{\partial \rho}{\partial t} = \Delta_p \rho = \operatorname{div}\left(|\nabla \rho|^{p-2}\nabla\rho\right) & \text{in} \quad I\!\!R^n \times (0,\infty) \\ \rho(t=0) = \rho_0 & \text{in} \quad I\!\!R^n. \end{cases} \tag{90}$$

After using the scaling (12)-(13) with $R(t) = (1+\delta_p t)^{1/\delta}$ and $\delta_p = p(n+1) - 2n$, (90) reads as

$$\begin{cases} \frac{\partial \hat{\rho}}{\partial \tau} = \Delta_p \hat{\rho} + \operatorname{div}(y\hat{\rho}) & \text{in} \quad I\!\!R^n \times (0,\infty) \\ \hat{\rho}(\tau=0) = \rho_0 & \text{in} \quad I\!\!R^n. \end{cases}$$

Theorems 2.2, 3.2 and 4.1 show that the self-similar solutions $\hat{\rho}$ of (90) decay exponentially fast to the equilibrium solution $\hat{\rho}_\infty(y) = \left(K - \frac{p-2}{p}|y|^q\right)_+^{\frac{p-1}{p-2}}$ in relative entropy, in the q-Wasserstein distance (for $q > 2$ i.e. $p < 2$), and in L^1-norm, at the rates 1, $\frac{1}{q}$ and $\frac{1}{2}$, respectively. And Theorem 4.2 gives the algebraic decay of the solutions ρ of (90) to the Barenblatt type solution $\rho_\infty(t,x) = \frac{1}{(1+\delta_p t)^{n/\delta_p}}\left(K - \frac{p-2}{p(1+\delta_p t)^{q/\delta_p}}|x|^q\right)_+^{\frac{p-1}{p-2}}$ in L^1-norm, at the rate $\frac{1}{2p(n+1)-4n}$. Note that when $p > 2$, the rate, 1, obtained here for the convergence in relative entropy is sharper than the rate $q\left(1 - \frac{1}{p}(p-1)^{1/q}\right)$ of Del Pino-Dolbeault [14], but this is not true when $p > 2$. Therefore, the optimal rate of convergence to equilibrium for the p-Laplacian equation is still unknown.

Generalized heat equation ($2 \neq p > 1$, $m = \frac{1}{p-1}$). If $c(x) = \frac{|x|^q}{q}$ and $F(x) = \frac{1}{p-1}x\ln x$, $\frac{1}{p} + \frac{1}{q} = 1$, then (1)-(2) is the generalized heat equation

$$\begin{cases} \frac{\partial \rho}{\partial t} = \Delta_p \rho^{\frac{1}{p-1}} = \operatorname{div}\left(|\nabla \rho^{\frac{1}{p-1}}|^{p-2}\nabla\rho^{\frac{1}{p-1}}\right) & \text{in} \quad I\!\!R^n \times (0,\infty) \\ \rho(t=0) = \rho_0 & \text{in} \quad I\!\!R^n. \end{cases} \tag{91}$$

The scalings (12)-(13) applied with $R(t) = (1+\delta_p t)^{1/\delta_p}$, $\delta_p = (p-1)\left(\frac{n}{p-1} + 1\right) + 1 - n$, and we get the same exponential decay rates as in the p-Laplacian equation, for the convergence to $\hat{\rho}_\infty(y) = \frac{1}{\sigma}e^{-(p-1)|y|^q/q}$, and the algebraic decay rate $\frac{1}{2\delta_p}$ for the L^1-convergence to $\rho_\infty(t,x) = \frac{1}{R(t)^n}\hat{\rho}_\infty\left(\frac{x}{R(t)}\right)$.

Generalized p-Laplacian equation ($2 \neq p > 1$, $m \neq \frac{1}{p-1}$). Here $c(x) = \frac{|x|^q}{q}$, $F(x) = \frac{mx^\gamma}{\gamma(\gamma-1)}$, $\gamma = m + \frac{p-2}{p-1}$ and $\frac{1}{p-1} \neq m \geq \frac{n-(p-1)}{n(p-1)}$. Then, Theorems 2.2, 3.2, 4.1

and 4.2 give the same decay rates as for the generalized heat equation with $\hat{\rho}_\infty(y) = \left(K - \frac{\gamma-1}{mq}|y|^q\right)_+^{\frac{1}{\gamma-1}}$ and $\rho_\infty(t,x) = \frac{1}{R(t)^n}\left(K - \frac{\gamma-1}{mqR(t)^q}|x|^q\right)_+^{\frac{1}{\gamma-1}}$, where $R(t) = (1+\delta_p t)^{1/\delta_p}$ and $\delta_p = (p-1)(nm+1)+1-n$.

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