Perfect matchings in a random k-out subgraph of the hypercube

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Abstract

We study the threshold for perfect matchings in $Q_{n,k\text{-out}}$, the random subgraph of the dimension n hypercube where each vertex chooses k neighbors uniformly and independently at random. We show that for any $\varepsilon > 0$, with high probability, when $k \leq (1-\varepsilon)\log_2 n$, $Q_{n,k\text{-out}}$ does not have a perfect matching, while when $k \geq 4.16 \log_2 n$, $Q_{n,k\text{-out}}$ does have a perfect matching. A key part of our argument is a counting technique due to Kostochka.

1 Introduction

As a mature subject, the study of random graphs encompasses a large number of models. The first models to be studied were $G_{n,p}$ by Gilbert [13] and $G_{n,m}$ by Erdős and Rényi [8, 9].[1] It became apparent fairly early on that for several important properties (for example, connectivity and Hamiltonicity), the dominant obstruction was a vertex of small degree; once the parameters (p or m) of the random graph were large enough to ensure a certain minimum degree, the property would hold with high probability. This leads to various models where a lower bound on the minimum degree is built into the model, that is, the minimum degree is ensured deterministically. One such model is $G_{n,k\text{-out}}$.

The random graph $G_{n,k\text{-out}}$ has vertex set $[n]$ and each vertex independently chooses k uniformly random neighbors.[2] Thus every vertex of $G_{n,k\text{-out}}$ has degree at least k, although

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[1]The random graph $G_{n,p}$ has vertex set $[n]$ and each pair of vertices is selected to be an edge independently with probability p. The random graph $G_{n,m}$ is a uniformly random graph on the labeled vertex set $[n]$ with exactly m edges. See [15] for a reference on random graphs.

[2]We ignore multiple edges and disallow loops. There is no orientation involved. One advantage that $G_{n,k\text{-out}}$ has over, say, a random regular graph, which also inherently satisfies a minimum degree condition, is that $G_{n,k\text{-out}}$ has each vertex choosing (at least some of) its neighbors independently of other vertices.

it is possible for a vertex to have much larger degree, and the total number of edges of $G_{n,k\text{-out}}$ is between $nk/2$ and nk. The general question is: given a graph property, what is the minimum k such that $G_{n,k\text{-out}}$ has this property w.h.p.?[3] For example, Fenner and Frieze [10] showed that $k = 2$ is necessary and sufficient for connectivity, Frieze [11] showed that $k = 2$ is necessary and sufficient for the existence of a perfect matching, and Bohman and Frieze [2] showed that $k = 3$ is necessary and sufficient for the existence of a Hamilton cycle.

In $G_{n,p}$, $G_{n,m}$, and $G_{n,k\text{-out}}$, the "host graph" is the complete n-vertex graph K_n, meaning that these random graphs are naturally viewed as random subgraphs of K_n. There has been considerable interest in the study of random subgraphs of other graphs. For example, in percolation theory, one typically takes a lattice like $\mathbb{Z}^2$ and retains each edge independently with probability p. On the other hand, random subgraphs of arbitrary graphs are also studied: Krivelevich, Lee, and Sudakov [21] studied the existence of long paths and cycles, and Christoph, Morawski, and Wigderson [7] very recently studied the size of the largest components. A popular host graph is the hypercube Q_n, which has vertex set $\{0,1\}^n$ and an edge between $x, y \in V(Q_n)$ if and only if they differ in exactly one coordinate. We let $Q_{n,p}$ be the random subgraph of Q_n where we retain each edge independently with probability p. The earliest result on $Q_{n,p}$ was by Burtin [5], who proved that $p = 1/2$ is the threshold for connectivity. Kostochka, Sapozhenko, and Weber [19] have an old survey about $Q_{n,p}$ and the related model where we retain vertices at random instead of edges; an updated survey is needed, especially given impressive recent results such as the threshold for Hamiltonicity in $Q_{n,p}$ by Condon, Espuny Díaz, Girão, Kühn, and Osthus [6].

Analogous to $G_{n,k\text{-out}}$, for a host graph H, let $H_{k\text{-out}}$ be the random subgraph of H where each vertex of H chooses k of its neighbors in H independently to be included in $H_{k\text{-out}}$. Frieze and Johansson [12] showed that if H has n vertices and minimum degree at least $(\frac{1}{2} + \varepsilon)n$, then there exists $k = O(1)$ such that $H_{k\text{-out}}$ is Hamiltonian w.h.p. The minimum value of k for which this is true is not known at present. In this paper we study $Q_{n,k\text{-out}}$, where the host graph is the hypercube Q_n. The first author determined the threshold for connectivity in $Q_{n,k\text{-out}}$.

Theorem 1.1 (Anastos [1])**.** *Let $k_0 = \log_2 n - 2\log_2\log_2 n$. Then w.h.p. $Q_{n,k\text{-out}}$ is not connected for $k \le \lfloor k_0 \rfloor$, and w.h.p. $Q_{n,k\text{-out}}$ is k-connected for all $k \ge \lceil k_0 \rceil + 1$.*

In this paper, we study the existence of a perfect matching in $Q_{n,k\text{-out}}$. Bollobás [4] and Kostochka [17] (extended in [16]) independently proved that $p = 1/2$ is the threshold for the existence of a perfect matching in $Q_{n,p}$; we note that for $p < 1/2$, $Q_{n,p}$ has an isolated vertex. Our main results show that a model which enforces a minimum degree condition means that a far sparser random subgraph will contain a perfect matching.

Theorem 1.2. *For every constant $\varepsilon > 0$, w.h.p. $Q_{n,k\text{-out}}$ does have a perfect matching for $k \ge (6 + \varepsilon)\ln n$.*

[3] "W.h.p." stands for "with high probability," and means that the probability tends to 1 as n tends to infinity (with related parameters like k possibly depending on n).

Theorem 1.3. *Let $k_0 = \log_2 n - 2\log_2\log_2 n$. For every constant $\varepsilon > 0$, w.h.p. $Q_{n,k\text{-out}}$ does not have a perfect matching for $k \le k_0 - 1 - \varepsilon$.*

The proof of Theorem 1.3, in Section 2, is a variation on the lower bound on the connectivity threshold in Theorem 1.1: we show that there exists an empty cut of $Q_{n,k\text{-out}}$ where each side has an odd number of vertices. The proof of Theorem 1.2, in Section 3, comes from an old argument of Kostochka [18] about the number of subsets of the hypercube with a given size of edge-boundary. While over 30 years old, we believe Kostochka's argument deserves to be more well-known, and so we reproduce it here with a simplified presentation and in modern language. The method also bears some similarity to the so-called Sapozhenko's graph container method, which is still being developed today (see the very recent survey of Park [22]). The main difference between these two methods is that Kostochka's counts sets with a given size edge-boundary, while Sapozhenko's counts sets with a given size vertex-boundary; also Sapozhenko's method produces an intermediate object which usefully functions as a collection of 'containers.' Still, there may be some useful synergy between the two methods.

We note that the logarithms in Theorem 1.3 and Theorem 1.2 have different bases; put in the same base, these functions are a multiplicative factor of about 4.16 off from each other. We believe that Theorem 1.3 should be close to tight; that is, we conjecture that the $(6+\varepsilon)\ln n$ in Theorem 1.2 can be replaced by $(1+\varepsilon)\log_2 n$.

Notation. Observe that Q_n is bipartite, with parts $\mathcal{E} = \{x : |x| \text{ is even}\}$ and $\mathcal{O} = \{x : |x| \text{ is odd}\}$, where $|x| = |\{j : x_j = 1\}|$. For disjoint $A, B \subseteq V(Q_n)$, we denote by $E(A,B)$ the set of edges of Q_n joining A and B, and we let $e(A,B) = |E(A,B)|$. For $x \in V(Q_n) \setminus B$, we let $d(x,B) = e(\{x\},B)$ be the degree of x to B in Q_n. For $X \subseteq V(Q_n)$, $N(X)$ denotes the *external* neighborhood of X in Q_n, which is the set of vertices in $V(Q_n) \setminus X$ that are adjacent to a vertex of X in Q_n, while $N[X] := X \cup N(X)$ is the *closed* neighborhood.

We consider the k-out model of a random subgraph of Q_n, denoted $Q_{n,k\text{-out}}$. We let $E_k(A,B)$ be the set of edges of $Q_{n,k\text{-out}}$ joining A and B, $e_k(A,B) = |E_k(A,B)|$, and $d_k(x,B) = e_k(\{x\},B)$. For $X \subseteq V(Q_n)$, $N_k(X)$ denotes the external neighborhood of X in $Q_{n,k\text{-out}}$, which is the set of vertices in $V(Q_n) \setminus X$ that are adjacent to a vertex of X in $Q_{n,k\text{-out}}$.

AI Declaration. No AI was used in the creation of this paper.

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2 Lower bound: proof of Theorem 1.3

We say $K \subseteq V(Q_n)$ is a *k-cube* if the subgraph of Q_n induced by K is a dimension k hypercube. We call a pair K_1, K_2 of k-cubes *almost disjoint* if they have exactly one vertex in common. Let I_{K_1,K_2} be the event that

$$e_k(K_1 \cup K_2, \overline{K_1 \cup K_2}) = 0,$$

and let Z be the number of almost disjoint pairs K_1, K_2 of k-cubes such that I_{K_1,K_2} holds. We will use a second moment calculation to show that $Z > 0$ w.h.p. Since $|K_1 \cup K_2| = 2^{k+1} - 1$ is odd for almost disjoint k-cubes, if some such I_{K_1,K_2} holds, then there can be no perfect matching in $Q_{n,k\text{-out}}$.[4]

Note that the number of almost disjoint pairs of k-cubes is $2^n \binom{n}{k}\binom{n-k}{k} = 2^{(1+o(1))n}$. The probability that every $x \in K_1 \cup K_2$ chooses its neighbors in $K_1 \cup K_2$ is

$$\binom{2k}{k}\binom{n}{k}^{-2^{k+1}+1} \geq \exp\left(-(1+o(1))k2^{k+1}\ln n\right).$$

Note that

$$e(K_1 \cup K_2, \overline{K_1 \cup K_2}) = \left(2^{k+1}-1\right)n - k2^k \leq 2^{k+1}n,$$

and that $d(x, K_1 \cup K_2) \leq 2$ for every $x \in \overline{K_1 \cup K_2}$. Thus the probability that every $x \in \overline{K_1 \cup K_2}$ chooses its neighbors in $\overline{K_1 \cup K_2}$ is

$$\prod_{x \in \overline{K_1 \cup K_2}} \frac{\binom{n-d(x,K_1\cup K_2)}{k}}{\binom{n}{k}} \geq \prod_{x \in \overline{K_1 \cup K_2}} \exp\left(-(1+o(1))\frac{k}{n}d(x, K_1 \cup K_2)\right)$$
$$= \exp\left(-(1+o(1))\frac{k}{n}e(K_1 \cup K_2, \overline{K_1 \cup K_2})\right) \geq \exp\left(-(1+o(1))k2^{k+1}\right).$$

Putting these together, since $k \leq \log_2(n) - 2\log_2\log_2(n) - 1 - \varepsilon$, we have

$$\mathbb{P}(I_{K_1,K_2}) \geq \exp\left(-(1+o(1))k2^{k+1}\ln n\right) \geq 2^{-(1-\varepsilon/2)n},$$

and hence $\mathbb{E}Z \geq 2^{\varepsilon n/2}$.

To bound the second moment of Z, we note that I_{K_1,K_2} and $I_{K_1',K_2'}$ are independent if $N(K_1 \cup K_2)$ and $N(K_1' \cup K_2')$ are disjoint. Thus any one such event depends only on at most

$$|K_1 \cup K_2| n^3 \binom{n}{k} 2^k \binom{n}{k} \leq n^{2k+3} = o(\mathbb{E}Z)$$

other events. Thus $\mathbf{Var}(Z) = o(\mathbb{E}(Z)^2)$, and hence by Chebyshev's inequality, $Z > 0$ w.h.p., establishing Theorem 1.3.

[4] Since $Q_{n,k\text{-out}}$ has minimum degree at least k, there are at least 2^k vertices on each side of an empty cut by Harper's edge-isoperimetric theorem (Theorem 3.4). Thus a pair of almost disjoint k-cubes is nearly the smallest vertex subset of Q_n we could consider here, off by at most a factor of 2.

3 Upper bound: proof of Theorem 1.2

Since Q_n is bipartite, it is natural to use Hall's Theorem. We call a pair (A, B) a *witness* if

(i) $A \subseteq \mathcal{E}$ and $B \subseteq \mathcal{O}$,

(ii) $|A| = |B| \leq 2^{n-2}$,

(iii) $N_k(A) \subseteq B$, and

(iv) $A \cup B$ induces a connected subgraph of Q_n.

We call (A, B) a *potential witness* if (i), (ii), and (iv) hold. Having no witnesses (and the similar pairs with A and B reversed) is sufficient to show that Hall's condition holds; see Section 3.3. We will show that, w.h.p., no witness exists in $Q_{n,k\text{-out}}$ via a union bound, both estimating the number of potential witnesses and the probability that a potential witness is a (true) witness. We split into two cases based on $|A|$.

Lemma 3.1. *Let $k_0 = \log_2 n - 2\log_2\log_2 n$. For every constant $\varepsilon > 0$, w.h.p. $Q_{n,k\text{-out}}$ does not have a witness (A, B) with $|A| \leq 2^{n/6}$ for $k \geq k_0 + 1 + \varepsilon$.*

Lemma 3.2. *For every constant $\varepsilon > 0$, w.h.p. $Q_{n,k\text{-out}}$ does not have a witness (A, B) with $n^{10} \leq |A| \leq 2^{n-2}$ for $k \geq (6 + \varepsilon)\ln n$.*

We prove Lemma 3.1 in Section 3.1 and Lemma 3.2 in Section 3.2. Central to both small and large witnesses is the following counting lemma (several simple proofs exist; see, for example, [20, p. 396, Ex.11]).

Lemma 3.3. *Let G be a connected graph with maximum degree at most Δ. There are at most $|V(G)|(e\Delta)^{b-1}$ sets of b vertices which induce connected subgraphs of G.*

Naturally, we also need Harper's edge-isoperimetric inequality (see Chapter 16 of [3]).

Theorem 3.4. *For $S \subseteq V(Q_n)$, we have $e(S, V(Q_n) \setminus S) \geq |S| \log_2(2^n/|S|)$.*

3.1 Small witnesses: proof of Lemma 3.1

For a potential witness (A, B) with $|A| = |B| = a$, Theorem 3.4 implies that

$$2na = n|A| + n|B| = 2e(A, B) + e(A \cup B, V(Q_n) \setminus (A \cup B)) \geq 2e(A, B) + 2a\log_2(2^{n-1}/a),$$

and hence

$$e(A, B) \leq na - a\log_2(2^{n-1}/a) = a\log_2(2a).$$

Thus if $a < 2^{k-1}$, then there exists a vertex $x \in A$ with $d(x, B) < k$, and hence $N_k(A)$ is not a subset of B deterministically, so no such (A, B) can be a witness.

For $a \geq 2^{k-1}$, the probability that (A, B) is a witness is

$$\mathbb{P}(N_k(A) \subseteq B) \leq \prod_{x \in A} \frac{\binom{d(x,B)}{k}}{\binom{n}{k}} \leq \prod_{x \in A} \left(\frac{d(x,B)}{n}\right)^k \leq \left(\frac{e(A,B)}{an}\right)^{ka} \leq \left(\frac{\log_2(2a)}{n}\right)^{ka}.$$

Using Lemma 3.3, we can take a union bound over all possible witnesses: the probability that there is a witness (A, B) in $Q_{n,k\text{-out}}$ with $a = |A| \leq 2^{n/6}$ is at most

$$\sum_{a=2^{k-1}}^{2^{n/6}} 2^n (en)^{2a-1} \left(\frac{\log_2(2a)}{n}\right)^{ka} \leq \sum_{a=2^{k-1}}^{2^{n/6}} \left(2^{n/a}(en)^2 \left(\frac{\log_2(2a)}{n}\right)^k\right)^a. \tag{1}$$

Using $k \geq \log_2 n - 2\log_2\log_2 n + 1 + \varepsilon$, we have $a \geq 2^{k-1} \geq 2^\varepsilon n/\log_2^2 n$. Thus for $a \leq 2^{n^{o(1)}}$, the summand of the right side of (1) is at most

$$\left(2^{2^{-\varepsilon}\log_2^2 n}(en)^2 2^{-k\log_2(n/\log_2(2a))}\right)^a \leq \left(2^{2^{-\varepsilon}\log_2^2 n}(en)^2 2^{-(1-o(1))\log_2^2 n}\right)^a \leq 2^{-\Omega(a\log_2^2 n)}.$$

For $n \leq a \leq 2^{n/6}$, the summand of the right side of (1) is at most

$$\left(2(en)^2\left(\frac{1+n/6}{n}\right)^k\right)^a \leq \left(2(en)^2 6^{-(1-o(1))\log_2 n}\right)^a \leq n^{-\Omega(a)}.$$

Hence the union bound goes through, completing the proof of Lemma 3.1.

3.2 Large witnesses: proof of Lemma 3.2

For large potential witnesses (A, B), the relevant parameter, which we denote by t, is

$$t = 2e(A, \mathcal{O} \setminus B) = e(A \cup B, V(Q_n) \setminus (A \cup B)),$$

where the equality follows from the fact that Q_n is n-regular and that $|A| = |B|$. We more precisely estimate the probability that (A, B) is a witness by

$$\mathbb{P}(N_k(A) \subseteq B) = \mathbb{P}(e_k(A, \mathcal{O} \setminus B) = 0) \leq \prod_{x \in A} \frac{\binom{d(x,B)}{k}}{\binom{n}{k}} \prod_{x \in \mathcal{O} \setminus B} \frac{\binom{d(x,\mathcal{E}\setminus A)}{k}}{\binom{n}{k}}$$

$$\leq \prod_{x \in A} \left(\frac{d(x,B)}{n}\right)^k \prod_{x \in \mathcal{O}\setminus B} \left(\frac{d(x,\mathcal{E}\setminus A)}{n}\right)^k = \prod_{x \in A} \left(1 - \frac{d(x,\mathcal{O}\setminus B)}{n}\right)^k \prod_{x \in \mathcal{O}\setminus B} \left(1 - \frac{d(x,A)}{n}\right)^k$$

$$\leq \exp\left(-\frac{k}{n}\left(e(A, \mathcal{O}\setminus B) + e(\mathcal{O}\setminus B, A)\right)\right) = \exp\left(-\frac{kt}{n}\right). \tag{2}$$

Since this probability is stronger the larger t is, the key idea is to provide a more efficient counting of the number of potential witnesses than supplied by Lemma 3.3 when t is small. This is reflected in the following counting lemma, which is implicit in a proof due to Kostochka [18]. We recall that for $X \subseteq V(G)$, $N[X] = N_G[X]$ denotes the *closed* neighborhood of X in G, which is X together with the set of vertices adjacent to a vertex of X.

Lemma 3.5. *Let G be a graph with maximum degree at most Δ, and let $b \in \mathbb{Z}^+$. Suppose $\mathcal{S}$ is a collection of subsets of $V(G)$ such that for every $S \in \mathcal{S}$, the subgraph of G induced by S is connected, and there exists $X \subseteq S$ such that*

$$3|X| + 2|S \setminus N_G[X]| + |N_G[X] \setminus S| \leq b. \tag{3}$$

Then $|\mathcal{S}| \leq |V(G)|\,(8e\Delta)^b$.

Proof. We show that for every $S \in \mathcal{S}$, and $X \subseteq S$ such that (3) holds, there exists a set Y such that

(i) $X, S \setminus N[X], N[X] \setminus S \subseteq Y$,

(ii) $|Y| \leq b$, and

(iii) Y induces a connected subgraph of G.

Given this, by Lemma 3.3, there are at most $|V(G)|(e\Delta)^b$ choices for Y. From each Y, we identify X, $S \setminus N[X]$, and $N[X] \setminus S$, of which there are at most $(2^{|Y|})^3 \leq 8^b$ choices. From this information, we can reconstruct S as $(N[X] \setminus (N[X] \setminus S)) \cup (S \setminus N[X])$, yielding the conclusion of the lemma.

Fix $S \in \mathcal{S}$, and $X \subseteq S$ such that (3) holds. We first build a set Z iteratively, starting with an arbitrary element of X, maintaining throughout the process that Z induces a connected subgraph of G, with the ultimate goal that $X \cup (S \setminus N[X]) \subseteq Z \subseteq S$. Given the current Z, choose a $v \in (X \cup (S \setminus N[X])) \setminus Z$ minimizing the distance d to Z in the graph induced by S; we choose $v \in X$ if one exists, otherwise we choose $v \in S \setminus N[X]$. Since the graph induced by S is connected, we have that d is finite. In fact, we claim that $d \leq 3$, and furthermore, if $v \in S \setminus N[X]$, then $d \leq 2$.

Suppose $d > 3$. The shortest path (in S) $vv_1v_2 \dots v_d$ from v to Z has internal vertices in $(N[X] \setminus X) \cap S$, meaning that each has a neighbor in X. If v_2's neighbor w in X is in Z, then vv_1v_2w is a shorter path from v to Z; otherwise, w is a vertex of $X \setminus Z$ closer to Z than v, contradicting the choice of v. Thus $d \leq 3$. Furthermore, suppose $v \in S \setminus N[X]$ and $d > 2$. If v_1's neighbor u in X is in Z, then vv_1u is a shorter path from v to Z; otherwise, u is a vertex of $X \setminus Z$ at distance at most d from Z, so we would have chosen u over v.

Include v together with the shortest path (in S) from v to Z. Thus, at the end of the process, $|Z| \leq 3|X| + 2|S \setminus N[X]|$, Z induces a connected subgraph of G, and $X \cup (S \setminus N[X]) \subseteq Z \subseteq S$. Finally, let $Y = Z \cup (N[X] \setminus S)$, so $|Y| \leq b$ and satisfies all the desired properties. □

Given Lemma 3.5, it remains to show that there exists such an X for $S = A \cup B$, where (A, B) is a potential witness, optimizing the b in (3). We do this for every $S \subseteq V(Q_n)$ with $e(S, V(Q_n) \setminus S) \leq t$, not just those coming from potential witnesses. The following lemma is also implicit in [18], although with worse constant factors.

Lemma 3.6. *For every $S \subseteq V(Q_n)$ with $e(S, V(Q_n) \setminus S) \leq t$ and $|S| \leq 2^{n-1}$, there exists $X \subseteq S$ such that*

$$3|X| + 2|S \setminus N[X]| + |N[X] \setminus S| \leq (1 + o(1))\frac{6t}{n}. \tag{4}$$

Proof. We want to simultaneously minimize $|X|$, while capturing most of S with $N[X]$ and not letting too much of $N[X]$ escape S. We will choose X randomly. In a general unstructured n-regular graph, a decent bound is obtained by choosing each vertex of S to be in X independently with probability $c\log(n)/n$ for some constant c. In the hypercube, there is more structure to exploit, namely, asymptotically optimal dominating sets, which are known to exist by a result of Kabatiansky and Panchenko [14].

Take an asymptotically optimal dominating set C of Q_n, so that $|C| = (1+o(1))2^n/n$. Let R be a randomly shifted copy of C, where we add (modulo 2) an independent uniform random vector in $\{0,1\}^n$ to all the vertices. By this averaging, the probability a vertex is in R is independent of the choice of vertex; let

$$q := \mathbb{P}(v \in R) = (1+o(1))/n \tag{5}$$

be this probability.

Let $X = S \cap R$. Then

$$\mathbb{E}|X| = \sum_{v\in S} \mathbb{P}(v \in R) = q|S|.$$

Likewise, since $X \subseteq S$, we bound

$$\begin{aligned}\mathbb{E}|N[X] \setminus S| \le \mathbb{E}e(X, V(Q_n)\setminus S) &= \sum_{v\in S} \mathbb{P}(v\in R) d(v, V(Q_n)\setminus S)\\ &= q\sum_{v\in S} d(v, V(Q_n)\setminus S) = qe(S, V(Q_n)\setminus S) \le qt.\end{aligned}$$

Finally, for $S \setminus N[X]$, we bound the probability $v \in S$ is not in $N[X]$, or equivalently, the probability $N[v]$ is disjoint from X. For $v \in S$, since R is a dominating set, we have $|R \cap N[v]| \ge 1$. Thus $N[v]$ being disjoint from $X = R \cap S$ implies that $|R \cap (N[v] \setminus S)| \ge 1$, that is, some $w \in N[v] \setminus S$ is also in R. By the union bound, the probability that this happens is at most $q|N[v] \setminus S|$. Summarizing, we have

$$\begin{aligned}\mathbb{E}|S \setminus N[X]| = \sum_{v\in S} \mathbb{P}(X \cap N[v] = \emptyset) &\le \sum_{v\in S}\sum_{w\in N[v]\setminus S} \mathbb{P}(w \in R)\\ &= \sum_{v\in S} qd(v, V(Q_n)\setminus S) = qe(S, V(Q_n)\setminus S) \le qt.\end{aligned}$$

Thus

$$\mathbb{E}\left[3|X| + 2|S\setminus N[X]| + |N[X]\setminus S|\right] \le q(3|S|+3t) \le (1+o(1))\frac{6t}{n},$$

where the last inequality holds by (5), Theorem 3.4, and $|S| \le 2^{n-1}$. Thus there exists X such that (4) holds. □

Finally, we prove Lemma 3.2. The number of potential witnesses (A, B) with $t = 2e(A, \mathcal{O}\setminus B)$ is at most the number of $S \subseteq V(Q_n)$ with $e(S, V(Q_n)\setminus S) = t$ and S inducing a connected subgraph of Q_n, which is by Lemma 3.5 and Lemma 3.6 at most

$$2^n(8en)^{(1+o(1))6t/n}.$$

The probability that (A, B) is a witness is at most $\exp(-kt/n)$ by (2), so by the union bound, for $k \geq (6+\varepsilon)\ln n$, the probability that $Q_{n,k\text{-out}}$ has a witness (A, B) with $t = 2e(A, \mathcal{O} \setminus B)$ is at most

$$2^n (8en)^{(1+o(1))6t/n} \exp\left(-\frac{tk}{n}\right) \leq 2^n \exp\left(-\frac{\varepsilon t \ln n}{2n}\right).$$

For $|A| \geq n^{10}$, we have $t \geq n^{10}$ by Theorem 3.4, and hence the union bound goes through after summing over all possible t, proving Lemma 3.2.

3.3 Putting everything together: proof of Theorem 1.2

First we show that $Q_{n,k\text{-out}}$ having no witnesses is sufficient to verify Hall's condition for a perfect matching. Indeed, suppose that Hall's condition is violated in $Q_{n,k\text{-out}}$, and let A be the smallest set with $|N_k(A)| < |A|$. Note that, by minimality, $|N_k(A)| = |A| - 1$. By the symmetry of the hypercube, we may assume without loss of generality that $A \subseteq \mathcal{E}$. If $|A|$ were greater than 2^{n-2}, then $B := \mathcal{O} \setminus N_k(A)$ has size at most 2^{n-2} and also violates Hall's condition, since $N_k(B) \subseteq \mathcal{E} \setminus A$, so

$$|B| = 2^{n-1} - |N_k(A)| > 2^{n-1} - |A| \geq |N_k(B)|,$$

contradicting the minimality of A. Consider the connected components of Q_n induced on $A \cup N_k(A)$, which take the form $A_i \cup N_k(A_i)$, where the A_i partition A and the $N_k(A_i)$ partition $N_k(A)$. Since

$$\sum_i |N_k(A_i)| = |N_k(A)| < |A| = \sum_i |A_i|,$$

there exists some A_i such that $|N_k(A_i)| < |A_i|$. Choose B such that $N_k(A_i) \subseteq B \subseteq \mathcal{O}$ with $|B| = |A_i|$ such that $A_i \cup B$ induces a connected subgraph of Q_n. Then (A_i, B) is a witness.

Since Lemma 3.1 and Lemma 3.2 show that $Q_{n,k\text{-out}}$ does not have a witness w.h.p. for $k \geq (6+\varepsilon)\ln n$, this finishes the proof.

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