MONDAY, AUGUST 25, 2025.

BASIC COUNTING

Let $\phi(m, n)$ be the number of mappings from $[n]$ to $[m]$.

Theorem

$$\phi(m, n) = m^n$$

Proof By induction on n.

$$\phi(m, 0) = 1 = m^0.$$

$$\begin{aligned} \phi(m, n+1) &= m\phi(m, n) \\ &= m \times m^n \\ &= m^{n+1}. \end{aligned}$$

□

$\phi(m, n)$ is also the number of sequences $x_1 x_2 \cdots x_n$ where $x_i \in [m]$, $i = 1, 2, \ldots, n$.

Let $\psi(n)$ be the number of subsets of $[n]$.

Theorem

$$\psi(n) = 2^n.$$

Proof (1) By induction on n.

$\psi(0) = 1 = 2^0$.

$$\begin{aligned}
&\psi(n+1) \\
&= \#\{\text{sets containing } n+1\} + \#\{\text{sets not containing } n+1\} \\
&= \psi(n) + \psi(n) \\
&= 2^n + 2^n \\
&= 2^{n+1}.
\end{aligned}$$

There is a general principle that if there is a 1-1 correspondence between two finite sets A, B then $|A| = |B|$. Here is a use of this principle.

Proof (2).
For $A \subseteq [n]$ define the map $f_A : [n] \to \{0, 1\}$ by

$$f_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}.$$

f_A is the characteristic function of A.

Distinct A's give rise to distinct f_A's and vice-versa.

Thus $\psi(n)$ is the number of choices for f_A, which is 2^n by Theorem 50. □

Let $\psi_{odd}(n)$ be the number of odd subsets of $[n]$ and let $\psi_{even}(n)$ be the number of even subsets.

Theorem

$$\psi_{odd}(n) = \psi_{even}(n) = 2^{n-1}.$$

Proof For $A \subseteq [n-1]$ define

$$A' = \begin{cases} A & |A| \text{ is odd} \\ A \cup \{n\} & |A| \text{ is even} \end{cases}$$

The map $A \to A'$ defines a bijection between $[n-1]$ and the odd subsets of $[n]$. So $2^{n-1} = \psi(n-1) = \psi_{odd}(n)$. Futhermore,

$$\psi_{even}(n) = \psi(n) - \psi_{odd}(n) = 2^n - 2^{n-1} = 2^{n-1}.$$

Let $\phi_{1-1}(m,n)$ be the number of 1-1 mappings from $[n]$ to $[m]$.

Theorem

$$\phi_{1-1}(m,n) = \prod_{i=0}^{n-1}(m-i). \tag{1}$$

Proof Denote the RHS of (1) by $\pi(m,n)$. If $m < n$ then $\phi_{1-1}(m,n) = \pi(m,n) = 0$. So assume that $m \geq n$. Now we use induction on n.

If $n = 0$ then we have $\phi_{1-1}(m,0) = \pi(m,0) = 1$.

In general, if $n < m$ then

$$\begin{aligned} \phi_{1-1}(m,n+1) &= (m-n)\phi_{1-1}(m,n) \\ &= (m-n)\pi(m,n) \\ &= \pi(m,n+1). \end{aligned}$$

$\phi_{1-1}(m, n)$ also counts the number of length n ordered sequences distinct elements taken from a set of size m.

$$\phi_{1-1}(n, n) = n(n-1)\cdots 1 = n!$$

is the number of ordered sequences of $[n]$ i.e. the number of permutations of $[n]$.

Binomial Coefficients

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1}$$

Let X be a finite set and let

$\binom{X}{k}$ denote the collection of k-subsets of X.

Theorem

$$\left|\binom{X}{k}\right| = \binom{|X|}{k}.$$

Proof Let $n = |X|$,

$$k!\left|\binom{X}{k}\right| = \phi_{1-1}(n,k) = n(n-1)\cdots(n-k+1).$$

Let m, n be non-negative integers. Let Z_+ denote the non-negative integers. Let

$$S(m,n) = \{(i_1, i_2, \ldots, i_n) \in Z_+^n : \ i_1 + i_2 + \cdots + i_n = m\}.$$

Theorem

$$|S(m,n)| = \binom{m+n-1}{n-1}.$$

Proof imagine $m+n-1$ points in a line. Choose positions $p_1 < p_2 < \cdots < p_{n-1}$ and color these points red. Let $p_0 = 0$, $p_n = m+1$. The gap sizes between the red points

$$i_t = p_t - p_{t-1} - 1, \ t = 1, 2, \ldots, n$$

form a sequence in $S(m,n)$ and vice-versa. □

$|S(m,n)|$ is also the number of ways of coloring m *indistinguishable* balls using n colors.

Suppose that we want to count the number of ways of coloring these balls so that each color appears at least once i.e. to compute $|S(m,n)^*|$ where, if $N = \{1, 2, \ldots, \}$

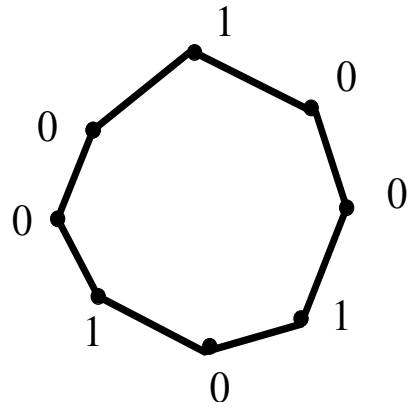
$$
\begin{aligned}
&S(m,n)^* = \\
&\{(i_1, i_2, \ldots, i_n) \in N^n : \ i_1 + i_2 + \cdots + i_n = m\} \\
&= \{(i_1 - 1, i_2 - 1, \ldots, i_n - 1) \in Z_+^n : \\
&\qquad\qquad (i_1 - 1) + (i_2 - 1) + \cdots + (i_n - 1) = m - n\}
\end{aligned}
$$

Thus,

$$|S(m,n)^*| = \binom{m-n+n-1}{n-1} = \binom{m-1}{n-1}.$$

WEDNESDAY, AUGUST 27, 2025.

Seperated 1's on a cycle



How many ways (patterns) are there of placing k 1's and $n-k$ 0's at the vertices of a polygon with n vertices so that no two 1's are adjacent?

Choose a vertex v of the polygon in n ways and then place a 1 there. For the remainder we must choose $a_1, \ldots, a_k \geq 1$ such that $a_1 + \cdots + a_k = n - k$ and then go round the cycle (clockwise) putting a_1 0's followed by a 1 and then a_2 0's followed by a 1 etc..

Each pattern π arises k times in this way. There are k choices of v that correspond to a 1 of the pattern. Having chosen v there is a unique choice of $a_1, a_2, \ldots, a_k$ that will now give π.

There are $\binom{n-k-1}{k-1}$ ways of choosing the a_i and so the answer to our question is

$$\frac{n}{k}\binom{n-k-1}{k-1.}$$

Theorem

Symmetry

$$\binom{n}{r} = \binom{n}{n-r}$$

Proof Choosing r elements to include is equivalent to choosing $n - r$ elements to exclude. □

Theorem

Pascal's Triangle

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

Proof A $k+1$-subset of $[n+1]$ either
(i) includes $n+1$ —— $\binom{n}{k}$ choices or
(ii) does not include $n+1$ —— $\binom{n}{k+1}$ choices.

Pascal's Triangle

The following array of binomial coefficents, constitutes the famous triangle:

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

. . .

Theorem

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}. \qquad (2)$$

Proof 1: Induction on n for arbitrary k.
Base case: $n = k$; $\binom{k}{k} = \binom{k+1}{k+1}$
Inductive Step: assume true for $n \geq k$.

$$\begin{aligned}
\sum_{m=k}^{n+1} \binom{m}{k} &= \sum_{m=k}^{n} \binom{m}{k} + \binom{n+1}{k} \\
&= \binom{n+1}{k+1} + \binom{n+1}{k} && \text{Induction} \\
&= \binom{n+2}{k+1}. && \text{Pascal's triangle}
\end{aligned}$$

Proof 2: Combinatorial argument.
If S denotes the set of $k+1$-subsets of $[n+1]$ and S_m is the set of $k+1$-subsets of $[n+1]$ which have largest element $m+1$ then

- $S_k, S_{k+1}, \ldots, S_n$ is a partition of S.
- $|S_k| + |S_{k+1}| + \cdots + |S_n| = |S|$.
- $|S_m| = \binom{m}{k}$.

□

Theorem

Vandermonde's Identity

$$\sum_{r=0}^{k} \binom{m}{r}\binom{n}{k-r} = \binom{m+n}{k}.$$

Proof Split $[m+n]$ into $A = [m]$ and $B = [m+n] \setminus [m]$. Let S denote the set of k-subsets of $[m+n]$ and let $S_r = \{X \in S : |X \cap A| = r\}$. Then

- $S_0, S_1, \ldots, S_k$ is a partition of S.
- $|S_0| + |S_1| + \cdots + |S_k| = |S|$.
- $|S_r| = \binom{m}{r}\binom{n}{k-r}$.
- $|S| = \binom{m+n}{k}$.

□

Theorem

Binomial Theorem

$$(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r.$$

Proof Coefficient x^r in $(1+x)(1+x)\cdots(1+x)$: choose x from r brackets and 1 from the rest. □

Applications of Binomial Theorem

- $x = 1$:

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = (1+1)^n = 2^n.$$

LHS counts the number of subsets of all sizes in $[n]$.

- $x = -1$:

$$\binom{n}{0} - \binom{n}{1} + \cdots + (-1)^n \binom{n}{n} = (1-1)^n = 0,$$

i.e.

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$$

and number of subsets of even cardinality = number of subsets of odd cardinality.

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}.$$

Differentiate both sides of the Binomial Theorem w.r.t. x.

$$n(1+x)^{n-1} = \sum_{k=0}^{n} k\binom{n}{k} x^{k-1}.$$

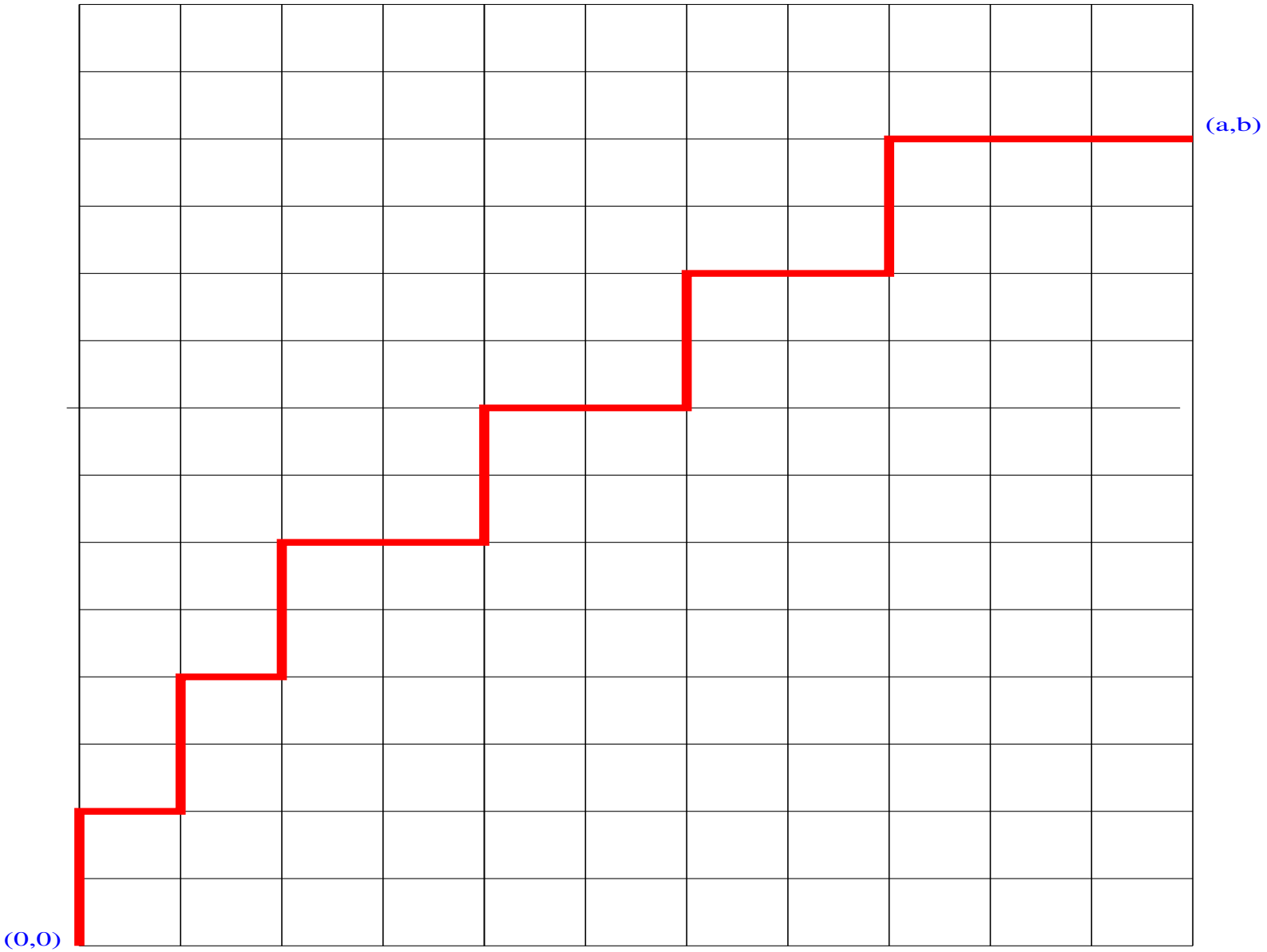
Now put $x = 1$.

Grid path problems

A *monotone path* is made up of segments $(x, y) \to (x+1, y)$ or $(x, y) \to (x, y+1)$.

$(a, b) \to (c, d))$= {monotone paths from (a, b) to (c, d)}.

We drop the $(a, b) \to$ for paths starting at $(0, 0)$.


(a,b)
(0,0)

We consider 3 questions: Assume $a, b \geq 0$.

1. How large is $PATHS(a, b)$?

2. Assume $a < b$. Let $PATHS_{>}(a, b)$ be the set of paths in $PATHS(a, b)$ which do not touch the line $x = y$ except at $(0, 0)$. How large is $PATHS_{>}(a, b)$?

3. Assume $a \leq b$. Let $PATHS_{\geq}(a, b)$ be the set of paths in $PATHS(a, b)$ which do not pass through points with $x > y$. How large is $PATHS_{\geq}(a, b)$?

1. $STRINGS(a,b) = \{x \in \{\mathrm{R},\mathrm{U}\}^* : \ x$ has a R's and b U's$\}$. [1]

There is a natural bijection between $PATHS(a,b)$ and $STRINGS(a,b)$:

Path moves to Right, add R to sequence.
Path goes up, add U to sequence.

So

$$|\mathrm{PATHS}(a,b)| = |\mathrm{STRINGS}(a,b)| = \binom{a+b}{a}$$

since to define a string we have state which of the $a+b$ places contains an R.

[1] $\{\mathrm{R},\mathrm{U}\}^*$ = set of strings of R's and U's

2. Every path in $\mathrm{PATHS}_{>}(a,b)$ goes through (0,1). So

$$|PATHS_{>}(a,b)| = |PATHS((0,1) \to (a,b))| - |\mathrm{PATHS}_{\not>}((0,1) \to (a,b))|.$$

Now

$$|PATHS((0,1) \to (a,b))| = \binom{a+b-1}{a}$$

and

$$|PATHS_{\not>}((0,1) \to (a,b))| = |PATHS((1,0) \to (a,b))| = \binom{a+b-1}{a-1}.$$

We explain the first equality momentarily. Thus

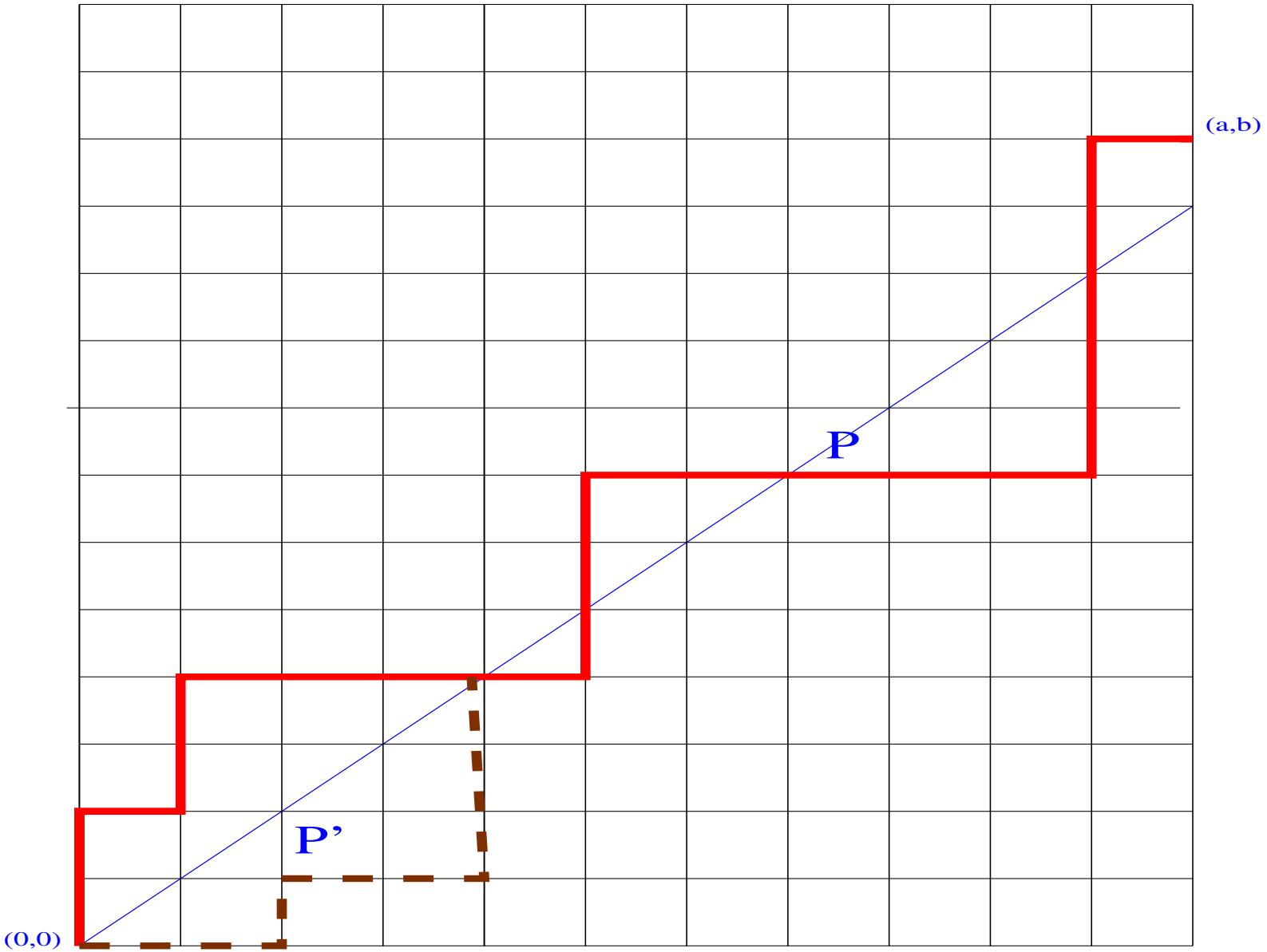
$$\begin{aligned} |\mathrm{PATHS}_{>}(a,b)| &= \binom{a+b-1}{a} - \binom{a+b-1}{a-1} \\ &= \frac{b-a}{a+b}\binom{a+b}{a}. \end{aligned}$$

Suppose $P \in \mathrm{PATHS}_{\not>}((0,1) \to (a,b))$. We define $P' \in \mathrm{PATHS}((1,0) \to (a,b))$ in such a way that $P \to P'$ is a bijection.

Let (c,c) be the first point of P, which lies on the line $L = \{x = y\}$ and let S denote the initial segment of P going from $(0,1)$ to (c,c).

P' is obtained from P by deleting S and replacing it by its reflection S' in L.

To show that this defines a bijection, observe that if $P' \in \mathrm{PATHS}((1,0) \to (a,b))$ then a similarly defined *reverse reflection* yields a $P \in \mathrm{PATHS}_{\not>}((0,1) \to (a,b))$.


(a,b)
P
P'
(0,0)

3. Suppose $P \in \mathrm{PATHS}_{\geq}(a,b)$. We define $P'' \in \mathrm{PATHS}_{>}(a,b+1)$ in such a way that $P \to P''$ is a bijection.

Thus

$$|PATHS_{\geq}(a,b)| = \frac{b-a+1}{a+b+1}\binom{a+b+1}{a}.$$

In particular

$$\begin{aligned} |\mathrm{PATHS}_{\geq}(a,a)| &= \frac{1}{2a+1}\binom{2a+1}{a} \\ &= \frac{1}{a+1}\binom{2a}{a}. \end{aligned}$$

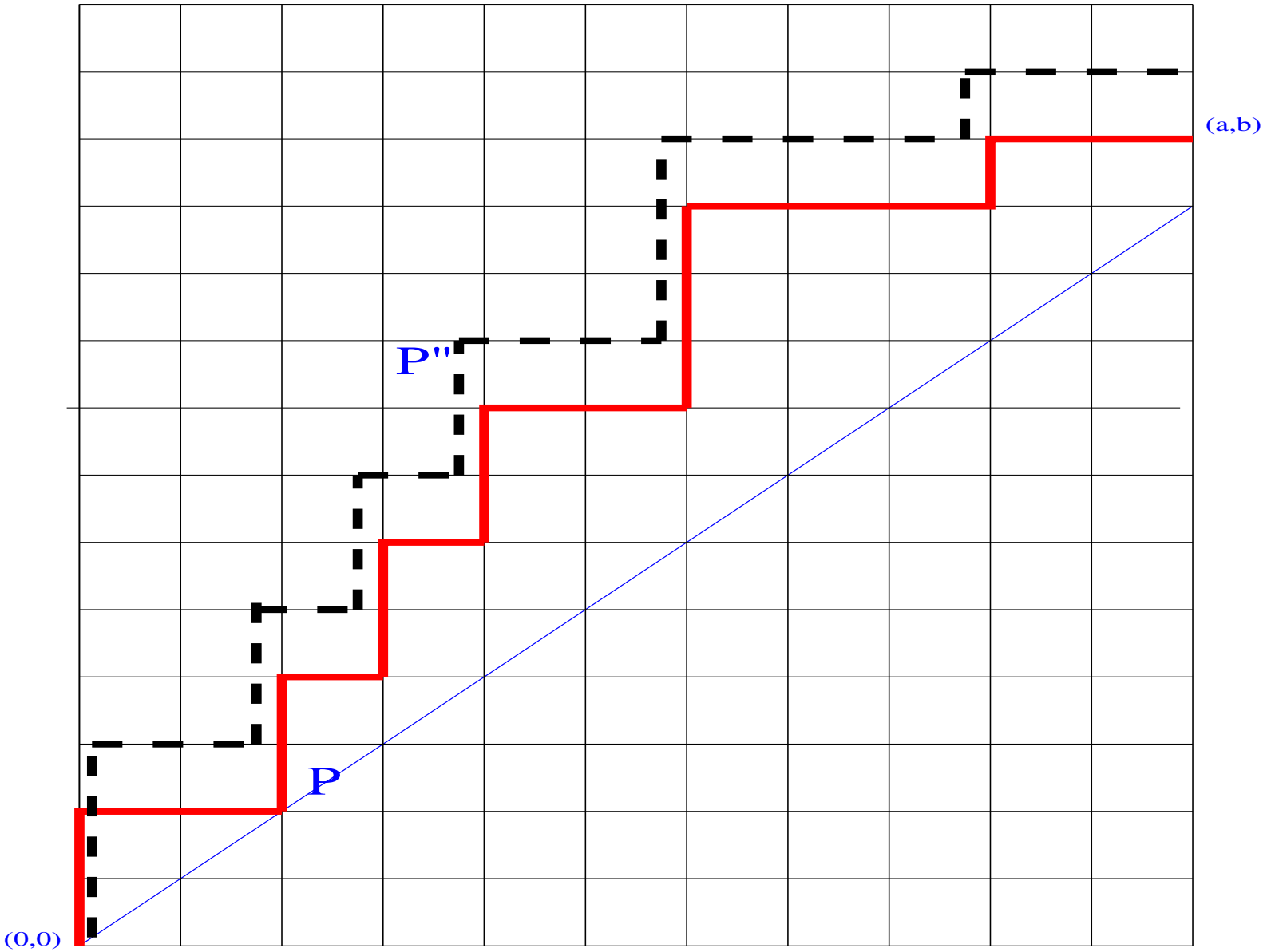
The final expression is called a *Catalan Number*.

The bijection

Given P we obtain P'' by *raising it vertically one position and then adding the segment* $(0,0) \to (0,1)$.

More precisely, if $P = (0,0), (x_1, y_1), (x_2, y_2), \ldots, (a,b)$ then $P'' = (0,0), (0,1), (x_1, y_1 + 1), \ldots, (a, b+1)$.

This is clearly a $1-1$ onto function between $\mathrm{PATHS}_{\geq}(a,b)$ and $\mathrm{PATHS}_{>}(a, b+1)$.


(a,b)
P''
P
(0,0)

FRIDAY, AUGUST 29, 2025.

Inclusion-Exclusion

2 sets:

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

So if $A_1, A_2 \subseteq A$ and $\overline{A}_i = A \setminus A_i$, $i = 1, 2$ then

$$|\overline{A}_1 \cap \overline{A}_2| = |A| - |A_1| - |A_2| + |A_1 \cap A_2|$$

3 sets:

$$\begin{aligned} |\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3| \quad = \quad & |A| - |A_1| - |A_2| - |A_3| \\ & + |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| \\ & - |A_1 \cap A_2 \cap A_3|. \end{aligned}$$

General Case

$A_1, A_2, \ldots, A_N \subseteq A$ and each $x \in A$ has a weight w_x. (In our examples $w_x = 1$ for all x and so $w(X) = |X|$.)

For $S \subseteq [N]$, $A_S = \bigcap_{i \in S} A_i$ and $w(S) = \sum_{x \in S} w_x$.

E.g. $A_{\{4,7,18\}} = A_4 \cap A_7 \cap A_{18}$.

$A_\emptyset = A$.

Inclusion-Exclusion Formula:

$$w\left(\bigcap_{i=1}^{N} \overline{A}_i\right) = \sum_{S \subseteq [N]} (-1)^{|S|} w(A_S).$$

Simple example. How many integers in [1000] are not divisible by 5,6 or 8 i.e. what is the size of $\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3$ below? Here we take $w_x = 1$ for all x.

$$\begin{aligned}
A = A_\emptyset &= \{1, 2, 3, \ldots, \} & |A| &= 1000 \\
A_1 &= \{5, 10, 15, \ldots, \} & |A_1| &= 200 \\
A_2 &= \{6, 12, 18, \ldots, \} & |A_2| &= 166 \\
A_3 &= \{8, 16, 24, \ldots, \} & |A_2| &= 125 \\
A_{\{1,2\}} &= \{30, 60, 90, \ldots, \} & |A_{\{1,2\}}| &= 33 \\
A_{\{1,3\}} &= \{40, 80, 120, \ldots, \} & |A_{\{1,3\}}| &= 25 \\
A_{\{2,3\}} &= \{24, 48, 72, \ldots, \} & |A_{\{2,3\}}| &= 41 \\
A_{\{1,2,3\}} &= \{120, 240, 360, \ldots, \} & |A_{\{1,2,3\}}| &= 8
\end{aligned}$$

$$\begin{aligned}
|\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3| &= 1000 - (200 + 166 + 125) \\
&\qquad + (33 + 25 + 41) - 8 \\
&= 600.
\end{aligned}$$

Derangements

A derangement of $[n]$ is a permutation π such that

$$\pi(i) \neq i : \ i = 1, 2, \ldots, n.$$

We must express the set of derangements D_n of $[n]$ as the intersection of the complements of sets.
We let $A_i = \{\text{permutations } \pi : \ \pi(i) = i\}$ and then

$$|D_n| = \left| \bigcap_{i=1}^{n} \overline{A}_i \right|.$$

We must now compute $|A_S|$ for $S \subseteq [n]$.

$|A_1| = (n-1)!$: after fixing $\pi(1) = 1$ there are $(n-1)!$ ways of permuting $2, 3, \ldots, n$.

$|A_{\{1,2\}}| = (n-2)!$: after fixing $\pi(1) = 1, \pi(2) = 2$ there are $(n-2)!$ ways of permuting $3, 4, \ldots, n$.

In general

$$|A_S| = (n - |S|)!$$

$$
\begin{aligned}
|D_n| &= \sum_{S \subseteq [n]} (-1)^{|S|} (n - |S|)! \\
&= \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)! \\
&= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!} \\
&= n! \sum_{k=0}^{n} (-1)^k \frac{1}{k!}.
\end{aligned}
$$

When n is large,

$$
\sum_{k=0}^{n} (-1)^k \frac{1}{k!} \approx e^{-1}.
$$

Proof of inclusion-exclusion formula

$$\theta_{x,i} = \begin{cases} 1 & x \in A_i \\ 0 & x \notin A_i \end{cases}$$

$$(1-\theta_{x,1})(1-\theta_{x,2})\cdots(1-\theta_{x,N}) = \begin{cases} 1 & x \in \bigcap_{i=1}^{N} \overline{A}_i \\ 0 & \text{otherwise} \end{cases}$$

So

$$\begin{aligned} w\left(\bigcap_{i=1}^{N} \overline{A}_i\right) &= \sum_{x\in A} w_x(1-\theta_{x,1})(1-\theta_{x,2})\cdots(1-\theta_{x,N}) \\ &= \sum_{x\in A} w_x \sum_{S\subseteq[N]} (-1)^{|S|} \prod_{i\in S} \theta_{x,i} \\ &= \sum_{S\subseteq[N]} (-1)^{|S|} \sum_{x\in A} w_x \prod_{i\in S} \theta_{x,i} \\ &= \sum_{S\subseteq[N]} (-1)^{|S|} w(A_S). \end{aligned}$$

Euler's Function $\phi(n)$.

Let $\phi(n)$ be the number of positive integers $x \leq n$ which are mutually prime to n i.e. have no common factors with n, other than 1.

$\phi(12) = 4$.

Let $n = p_1^{\alpha_1} p_2^{\alpha_2} p_1^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime factorisation of n.

$$A_i = \{x \in [n] : \ p_i \text{ divides } x\}, \qquad 1 \leq i \leq k.$$

$$\phi(n) = \left| \bigcap_{i=1}^{k} \overline{A}_i \right|$$

$$|A_S| = \frac{n}{\prod_{i \in S} p_i} \qquad S \subseteq [k].$$

$$\begin{aligned} \phi(n) &= \sum_{S \subseteq [k]} (-1)^{|S|} \frac{n}{\prod_{i \in S} p_i} \\ &= n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right) \end{aligned}$$

WEDNESDAY, SEPTEMBER 3, 2025.

Surjections

Fix n, m. Let

$$A = \{f : [n] \to [m]\}$$

Thus $|A| = m^n$. Let

$$F(n, m) = \{f \in A : \ f \text{ is onto } [m]\}.$$

How big is $F(n, m)$?
Let

$$A_i = \{f \in F : \ f(x) \neq i, \forall x \in [n]\}.$$

Then

$$F(n, m) = \bigcap_{i=1}^{m} \overline{A}_i.$$

For $S \subseteq [m]$

$$\begin{aligned} A_S &= \{f \in A : \ f(x) \notin S, \forall x \in [n]\}. \\ &= \{f : [n] \to [m] \setminus S\}. \end{aligned}$$

So

$$|A_S| = (m - |S|)^n.$$

Hence

$$\begin{aligned} F(n,m) &= \sum_{S \subseteq [m]} (-1)^{|S|} (m - |S|)^n \\ &= \sum_{k=0}^{m} (-1)^k \binom{m}{k} (m-k)^n. \end{aligned}$$

Scrambled Allocations

We have n boxes $B_1, B_2, \ldots, B_n$ and $2n$ distinguishable balls $b_1, b_2, \ldots, b_{2n}$.
An allocation of balls to boxes, two balls to a box, is said to be *scrambled* if there does **not** exist i such that box B_i contains balls b_{2i-1}, b_{2i}. Let σ_n be the number of scrambled allocations.

Let A_i be the set of allocations in which box B_i contains b_{2i-1}, b_{2i}. We show that

$$|A_S| = \frac{(2(n-|S|))!}{2^{n-|S|}}.$$

Inclusion-Exclusion then gives

$$\sigma_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(2(n-k))!}{2^{n-k}}.$$

First consider $A_\emptyset$:

Each permutation π of $[2n]$ yields an allocation of balls, placing $b_{\pi(2i-1)}, b_{\pi(2i)}$ into box B_i, for $i = 1, 2, \ldots, n$. The order of balls in the boxes is immaterial and so each allocation comes from exactly 2^n distinct permutations, giving

$$|A_\emptyset| = \frac{(2n)!}{2^n}.$$

To get the formula for $|A_S|$ observe that the contents of $2|S|$ boxes are fixed and so we are in essence dealing with $n - |S|$ boxes and $2(n - |S|)$ balls.

The weight of elements in exactly k sets:
Observe that

$$\prod_{i\in S}\theta_{x,i}\prod_{i\notin S}(1-\theta_{x,i})=1 \text{ iff } x\in A_i, i\in S \text{ and } x\notin A_i, i\notin S.$$

W_k is the total weight of elements in exactly k of the A_i:

$$\begin{aligned}
W_k &= \sum_{x\in A} w_x \sum_{|S|=k}\prod_{i\in S}\theta_{x,i}\prod_{i\notin S}(1-\theta_{x,i}) \\
&= \sum_{|S|=k}\sum_{x\in A} w_x \prod_{i\in S}\theta_{x,i}\prod_{i\notin S}(1-\theta_{x,i}) \\
&= \sum_{|S|=k}\sum_{T\supseteq S}\sum_{x\in A} w_x(-1)^{|T\setminus S|}\prod_{i\in T}\theta_{x,i} \\
&= \sum_{|S|=k}\sum_{T\supseteq S}(-1)^{|T\setminus S|}w(A_T).
\end{aligned}$$

As an example. Let $D_{n,k}$ denote the number of permutations π of $[n]$ for which there are exactly k indices i for which $\pi(i) = i$. Then

$$\begin{aligned}
D_{n,k} &= \sum_{\ell=k}^{n} \binom{n}{\ell} (-1)^{\ell-k} \binom{\ell}{k} (n-\ell)! \\
&= \sum_{\ell=k}^{n} \frac{n!}{\ell!(n-\ell)!} (-1)^{\ell-k} \frac{\ell!}{k!(\ell-k)!} (n-\ell)! \\
&= \frac{n!}{k!} \sum_{\ell=k}^{n} \frac{(-1)^{\ell-k}}{(\ell-k)!} \\
&= \frac{n!}{k!} \sum_{r=0}^{n-k} \frac{(-1)^{r}}{r!} \\
&\approx \frac{n!}{ek!}
\end{aligned}$$

when n is large and k is constant.

FRIDAY, SEPTEMBER 5, 2025.

Probléme des Ménages

In how many ways M_n can n male-female couples be seated around a table, with $2n$ numbered chairs, alternating male-female, so that no person is seated next to their partner, on either side?

Simplified version: before tackling this problem, we consider a simpler version where we only ban the case where if man i is seated in chair j then woman i cannot be seated in chair $j+1$. She can sit in chair $j-1$, although this is also forbidden in the Probléme des Ménage.

Let A_i be the set of seatings in which couple i sits together. Then in this version

$$|A_S| = 2\binom{n}{k} \times k! \times (n-k)!^2.$$

$$|A_S| = 2\binom{n}{k} \times k! \times (n-k)!^2.$$

Explanation: 2 choices for which seats are occupied by the men or women. $\binom{n}{k}$ choices for where to place the couples from S. $k!$ ways of assigning the couples to the positions; $(n-k)!^2$ ways of assigning the rest of the people.

Full version: let A_i be the set of seatings in which couple i sits together.

If $|S| = k$ then

$$|A_S| = 2k!(n-k)!^2 \times d_k.$$

d_k is the number of ways of placing k 1's on a cycle of length $2n$ so that no two 1's are adjacent. (Explanation below.)

2 choices for which seats are occupied by the men or women. $k!$ ways of assigning the couples to the positions; $(n-k)!^2$ ways of assigning the rest of the people.

Assume that the odd positions in the table are reserved for women. Given k 1's we put one from each of the k selected couples at the 1 (man or woman depending on the parity of the position) and then put their partner in the succeeding position.

Conversely, given an assignment in A_S we start at position 1 and we go round the table until we find the first of the k couples and put a 1 followed by a 0. We repeat from there until we have placed k 1's.

$$d_k = \frac{2n}{k}\binom{2n-k-1}{k-1} = \frac{2n}{2n-k}\binom{2n-k}{k}.$$

(See slides 11 and 12).

$$
\begin{aligned}
M_n &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \times 2k!(n-k)!^2 \times \frac{2n}{2n-k} \binom{2n-k}{k} \\
&= 2n! \sum_{k=0}^{n} (-1)^k \frac{2n}{2n-k} \binom{2n-k}{k} (n-k)!.
\end{aligned}
$$

Recurrence Relations and Generating Functions

Suppose $a_0, a_1, a_2, \ldots, a_n, \ldots,$is an infinite sequence.
A recurrence recurrence relation is a set of equations

$$a_n = f_n(a_{n-1}, a_{n-2}, \ldots, a_{n-k}). \tag{3}$$

The whole sequence is determined by (20) and the values of $a_0, a_1, \ldots, a_{k-1}$.

Linear Recurrence

Fibonacci Sequence

$$a_n = a_{n-1} + a_{n-2} \qquad n \geq 2.$$

$a_0 = a_1 = 1.$

$b_n = |B_n| = |\{x \in \{a, b, c\}^n : aa \text{ does not occur in } x\}|.$

$b_1 = 3 : \ a \ b \ c$

$b_2 = 8 : \ ab \ ac \ ba \ bb \ bc \ ca \ cb \ cc$

$b_n = 2b_{n-1} + 2b_{n-2} \qquad n \geq 2.$

$$b_n = 2b_{n-1} + 2b_{n-2} \qquad n \geq 2.$$

Let

$$B_n = B_n^{(b)} \cup B_n^{(c)} \cup B_n^{(a)}$$

where $B_n^{(\alpha)} = \{x \in B_n : \ x_1 = \alpha\}$ for $\alpha = a, b, c$.

Now $|B_n^{(b)}| = |B_n^{(c)}| = |B_{n-1}|$. The map $f : B_n^{(b)} \to B_{n-1}$,

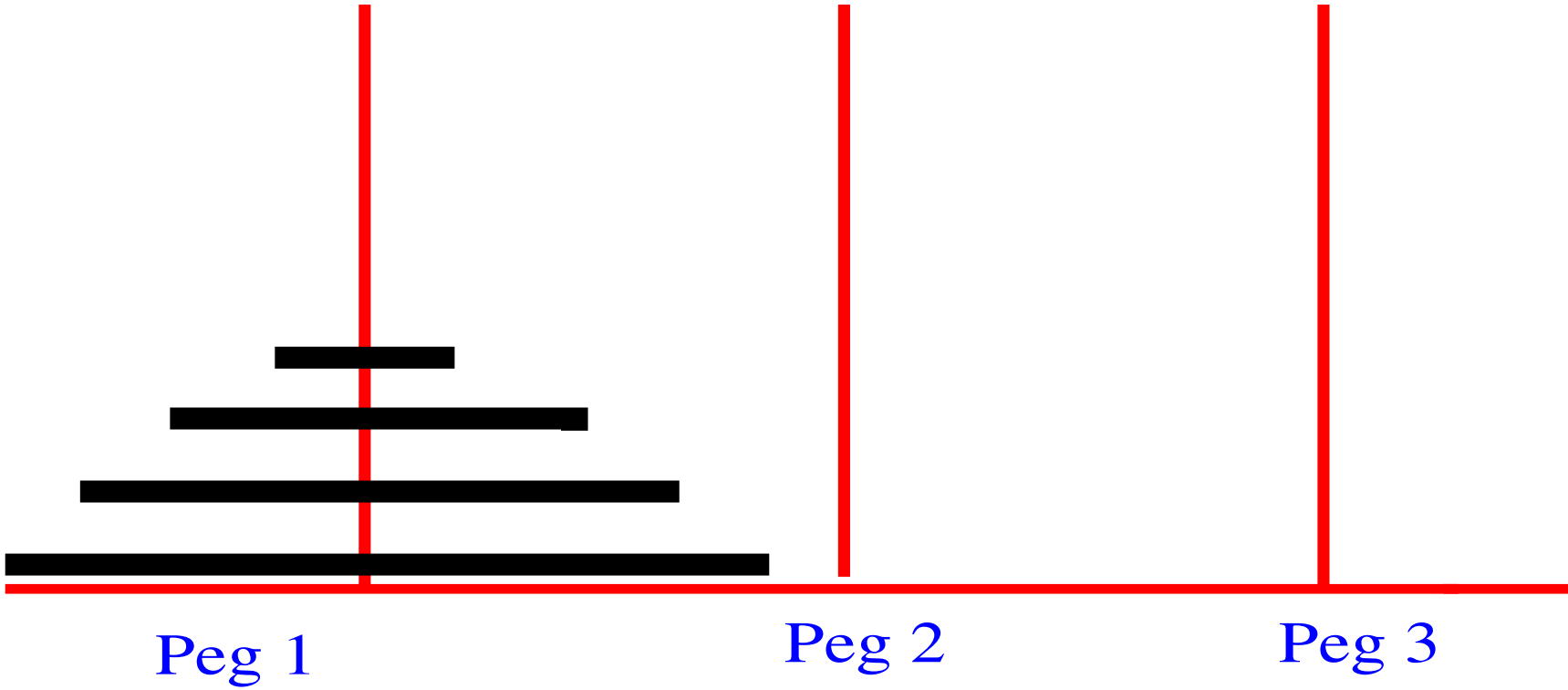
$$f(bx_2x_3 \ldots x_n) = x_2x_3 \ldots x_n \text{ is a bijection.}$$

$B_n^{(a)} = \{x \in B_n : \ x_1 = a \text{ and } x_2 = b \text{ or } c\}$. The map $g : B_n^{(a)} \to B_{n-1}^{(b)} \cup B_{n-1}^{(c)}$,

$$g(ax_2x_3 \ldots x_n) = x_2x_3 \ldots x_n \text{ is a bijection.}$$

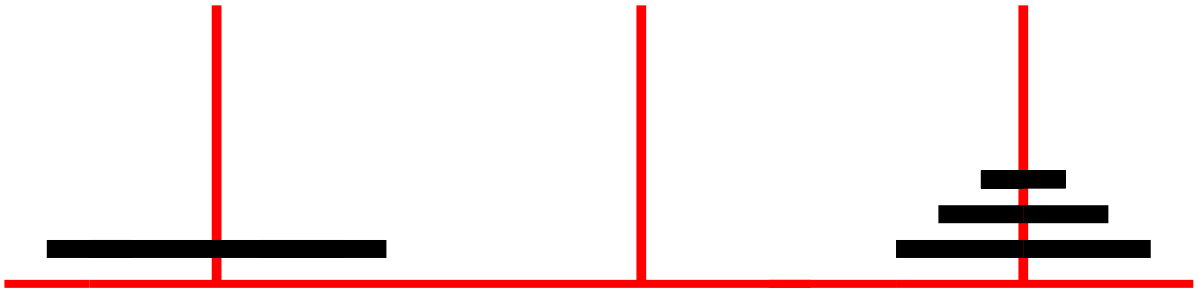
Hence, $|B_n^{(a)}| = 2|B_{n-2}|$.

Towers of Hanoi

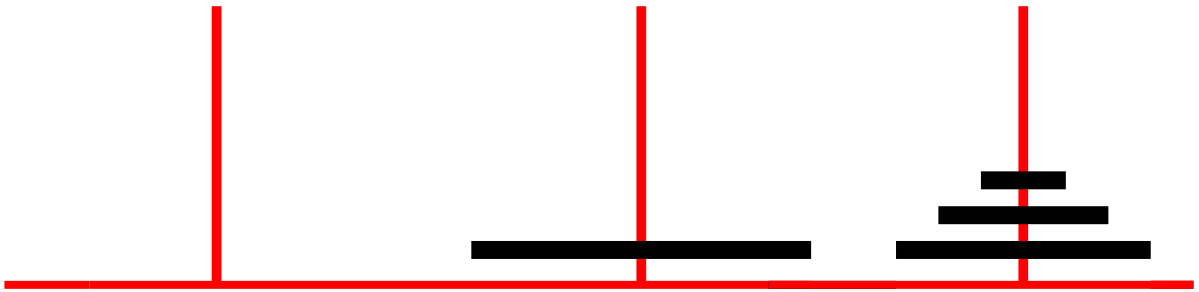


H_n is the minimum number of moves needed to shift n rings from Peg 1 to Peg 2. One is not allowed to place a larger ring on top of a smaller ring.

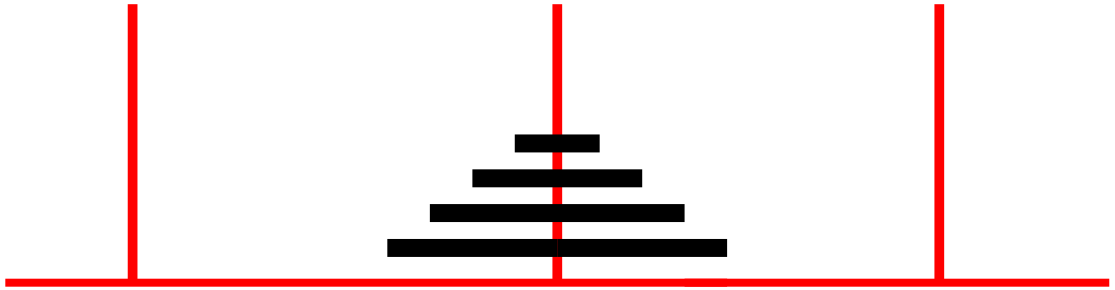
xxx



H_{n-1} moves



1 move



H_{n-1} moves

We see that $H_1 = 1$ and $H_n = 2H_{n-1} + 1$ for $n \geq 2$.

So,

$$\frac{H_n}{2^n} - \frac{H_{n-1}}{2^{n-1}} = \frac{1}{2^n}.$$

Summing these equations give

$$\frac{H_n}{2^n} - \frac{H_1}{2} = \frac{1}{2^n} + \frac{1}{2^{n-1}} + \cdots + \frac{1}{4} = \frac{1}{2} - \frac{1}{2^n}.$$

So

$$H_n = 2^n - 1.$$

General view.

Given a recurrence relation for the sequence (a_n), we

(a) Deduce from it, an equation satisfied by the generating function $a(x) = \sum_n a_n x^n$.

(b) Solve this equation to get an explicit expression for the generating function.

(c) Extract the coefficient a_n of x^n from $a(x)$, by expanding $a(x)$ as a power series.

Solution of linear recurrences

$$a_n - 6a_{n-1} + 9a_{n-2} = 0 \qquad n \geq 2.$$

$a_0 = 1, a_1 = 9.$

$$\sum_{n=2}^{\infty} (a_n - 6a_{n-1} + 9a_{n-2})x^n = 0. \tag{4}$$

$$\begin{aligned}
\sum_{n=2}^{\infty} a_n x^n &= a(x) - a_0 - a_1 x \\
&= a(x) - 1 - 9x. \\
\sum_{n=2}^{\infty} 6a_{n-1} x^n &= 6x \sum_{n=2}^{\infty} a_{n-1} x^{n-1} \\
&= 6x(a(x) - a_0) \\
&= 6x(a(x) - 1). \\
\sum_{n=2}^{\infty} 9a_{n-2} x^n &= 9x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2} \\
&= 9x^2 a(x).
\end{aligned}$$

$$a(x) - 1 - 9x - 6x(a(x) - 1) + 9x^2 a(x) = 0$$

or

$$a(x)(1 - 6x + 9x^2) - (1 + 3x) = 0.$$

$$\begin{aligned} a(x) &= \frac{1 + 3x}{1 - 6x + 9x^2} = \frac{1 + 3x}{(1 - 3x)^2} \\ &= \sum_{n=0}^{\infty} (n+1)3^n x^n + 3x \sum_{n=0}^{\infty} (n+1)3^n x^n \\ &= \sum_{n=0}^{\infty} (n+1)3^n x^n + \sum_{n=0}^{\infty} n3^n x^n \\ &= \sum_{n=0}^{\infty} (2n+1)3^n x^n. \end{aligned}$$

$$a_n = (2n+1)3^n.$$

MONDAY, SEPTEMBER 8, 2025.

Products of generating functions

$$a(x) = \sum_{n=0}^{\infty} a_n x^n,\ b(x)) = \sum_{n=0}^{\infty} b_n x^n.$$

$$\begin{aligned} a(x)b(x) &= (a_0 + a_1 x + a_2 x^2 + \cdots) \times \\ &\qquad (b_0 + b_1 x + b_2 x^2 + \cdots) \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0)x + \\ &\qquad (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \cdots \\ &= \sum_{n=0}^{\infty} c_n x^n \end{aligned}$$

where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

Derangements

$$n! = \sum_{k=0}^{n} \binom{n}{k} d_{n-k}.$$

Explanation: $\binom{n}{k} d_{n-k}$ is the number of permutations with exactly k cycles of length 1. Choose k elements ($\binom{n}{k}$ ways) for which $\pi(i) = i$ and then choose a derangement of the remaining $n - k$ elements.
So

$$\begin{aligned} 1 &= \sum_{k=0}^{n} \frac{1}{k!} \frac{d_{n-k}}{(n-k)!} \\ \sum_{n=0}^{\infty} x^n &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{1}{k!} \frac{d_{n-k}}{(n-k)!} \right) x^n. \end{aligned} \tag{5}$$

Let

$$d(x) = \sum_{m=0}^{\infty} \frac{d_m}{m!} x^m.$$

From (5) we have

$$\begin{aligned} \frac{1}{1-x} &= e^x d(x) \\ d(x) &= \frac{e^{-x}}{1-x} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left(\frac{(-1)^k}{k!} \right) x^n. \end{aligned}$$

So

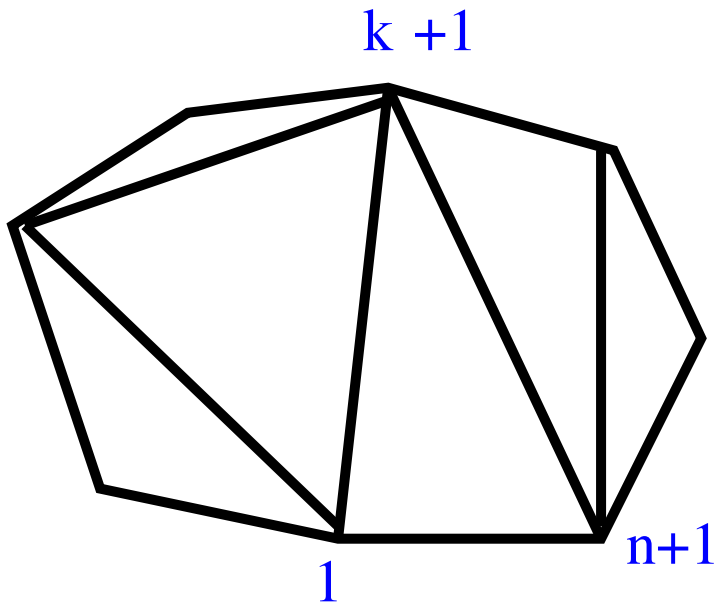
$$\frac{d_n}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

Triangulation of n-gon

Let

$$\begin{aligned} a_n &= \text{number of triangulations of } P_{n+1} \\ &= \sum_{k=0}^{n} a_k a_{n-k} \qquad n \geq 2 \end{aligned} \tag{6}$$

$a_0 = 0$, $a_1 = a_2 = 1$.



Explanation of (6):
$a_k a_{n-k}$ counts the number of triangulations in which edge $1, n+1$ is contained in triangle $1, k+1, n+1$.
There are a_k ways of triangulating $1, 2, \ldots, k+1, 1$ and for each such there are a_{n-k} ways of triangulating $k+1, k+2, \ldots, n+1, k+1$.

$$x + \sum_{n=2}^{\infty} a_n x^n = x + \sum_{n=2}^{\infty} \left(\sum_{k=0}^{n} a_k a_{n-k} \right) x^n.$$

But,

$$x + \sum_{n=2}^{\infty} a_n x^n = a(x)$$

since $a_0 = 0, a_1 = 1$.

$$\begin{aligned} \sum_{n=2}^{\infty} \left(\sum_{k=0}^{n} a_k a_{n-k} \right) x^n &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k a_{n-k} \right) x^n \\ &= a(x)^2. \end{aligned}$$

WEDNESDAY, SEPTEMBER 10, 2025.

So

$$a(x) = x + a(x)^2$$

and hence

$$a(x) = \frac{1+\sqrt{1-4x}}{2} \text{ or } \frac{1-\sqrt{1-4x}}{2}.$$

But $a(0) = 0$ and so

$$\begin{aligned} a(x) &= \frac{1-\sqrt{1-4x}}{2} \\ &= \frac{1}{2} - \frac{1}{2}\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n2^{2n-1}} \binom{2n-2}{n-1} (-4x)^n\right) \\ &= \sum_{n=1}^{\infty} \frac{1}{n}\binom{2n-2}{n-1} x^n. \end{aligned}$$

So

$$a_n = \frac{1}{n}\binom{2n-2}{n-1}.$$

$$
\begin{aligned}
\frac{1-\sqrt{1-4x}}{2} &= -\frac{1}{2}\sum_{n=1}^{\infty}\binom{\frac{1}{2}}{n}(-4x)^n \\
&= -\frac{1}{2}\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-n+1\right)}{n!}(-4x)^n \\
&= \sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^{n+1}n!}(4x)^n \\
&= \sum_{n=1}^{\infty}\frac{(2n-2)!}{n!(n-1)!}x^n \\
&= \sum_{n=1}^{\infty}\frac{1}{n}\binom{2n-2}{n-1}x^n.
\end{aligned}
$$

Probabilistic Method

Coloring problem:

Theorem

Let $A_1, A_2, \ldots, A_n$ be subsets of A and $|A_i| = k$ for $1 \leq i \leq n$. If $n < 2^{k-1}$ then there exists a partition $A = R \cup B$ such that

$$A_i \cap R \neq \emptyset \text{ and } A_i \cap B \neq \emptyset \qquad 1 \leq i \leq n.$$

[R = Red elements and B= Blue elements.]

Proof Randomly colour A.
$\Omega = \{R, B\}^A = \{f : A \to \{R, B\}\}$, uniform distribution.

$$BAD = \{\exists i : \ A_i \subseteq R \text{ or } A_i \subseteq B\}.$$

Claim: $\Pi(BAD) < 1$.
Thus $\Omega \setminus BAD \neq \emptyset$ and this proves the theorem.

$$BAD(i) = \{A_i \subseteq R \text{ or } A_i \subseteq B\} \text{ and } BAD = \bigcup_{i=1}^{n} BAD(i).$$

Boole's Inequality: if $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_N$ are a collection of events, then

$$\mathbf{Pr}\left(\bigcup_{i=1}^{N} \mathcal{A}_i\right) \leq \sum_{i=1}^{N} \mathbf{Pr}(\mathcal{A}_i).$$

This easily proved by induction on N. When $N = 2$ we use

$$\mathbf{Pr}(\mathcal{A}_1 \cup \mathcal{A}_2) = \mathbf{Pr}(\mathcal{A}_1) + \mathbf{Pr}(\mathcal{A}_2) - \mathbf{Pr}(\mathcal{A}_1 \cap \mathcal{A}_2) \leq \mathbf{Pr}(\mathcal{A}_1 \cup \mathcal{A}_2).$$

In general,

$$\mathbf{Pr}\left(\bigcup_{i=1}^{N} \mathcal{A}_i\right) \leq \mathbf{Pr}\left(\bigcup_{i=1}^{N-1} \mathcal{A}_i\right) + \mathbf{Pr}(\mathcal{A}_N) \leq \sum_{i=1}^{N-1} \mathbf{Pr}(\mathcal{A}_i) + \mathbf{Pr}(\mathcal{A}_N).$$

The first inequality is the two event case and the second is by induction on N.

So,

$$\begin{aligned}
\Pi(BAD) &\leq \sum_{i=1}^{n} \Pi(BAD(i)) \\
&= \sum_{i=1}^{n} \left(\frac{1}{2}\right)^{k-1} \\
&= n/2^{k-1} \\
&< 1.
\end{aligned}$$

Example of system which is not 2-colorable.

Let $n = \binom{2k-1}{k}$ and $A = [2k-1]$ and

$$\{A_1, A_2, \ldots, A_n\} = \binom{[2k-1]}{k}.$$

Then in any 2-coloring of $A_1, A_2, \ldots, A_n$ there is a set A_i all of whose elements are of one color.

Suppose A is partitioned into 2 sets R, B. At least one of these two sets is of size at least k (since $(k-1)+(k-1) < 2k-1$). Suppose then that $R \geq k$ and let S be any k-subset of R. Then there exists i such that $A_i = S \subseteq R$.

Tournaments

n players in a tournament each play each other i.e. there are $\binom{n}{2}$ games.

Fix some k. Is it possible that for every set S of k players there is a person w_S who beats everyone in S?

Suppose that the results of the tournament are decided by a random coin toss.

Fix S, $|S| = k$ and let $\mathcal{E}_S$ be the event that nobody beats everyone in S.

The event

$$\mathcal{E} = \bigcup_S \mathcal{E}_S$$

is that there is a set S for which w_S does not exist.

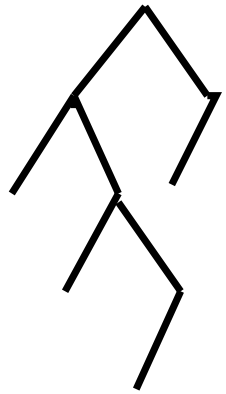
We only have to show that $\mathbf{Pr}(\mathcal{E}) < 1$.

$$
\begin{aligned}
\mathbf{Pr}(\mathcal{E}) &\leq \sum_{|S|=k} \mathbf{Pr}(\mathcal{E}_S) \\
&= \binom{n}{k}(1-2^{-k})^{n-k} \\
&< n^k e^{-(n-k)2^{-k}} \\
&= \exp\{k \ln n - (n-k)2^{-k}\} \\
&\to 0
\end{aligned}
$$

since we are assuming here that k is fixed independent of n.

FRIDAY, SEPTEMBER 12, 2025.

Random Binary Search Trees



A binary tree consists of a set of *nodes*, one of which is the *root*. Each node is connected to 0,1 or 2 nodes below it and every node other than the root is connected to exactly one node above it. The root is the highest node.
The depth of a node is the number of edges in its path to the root.
The depth of a tree is the maximum over the depths of its nodes.

Starting with a tree T_0 consisting of a single root r, we grow a tree T_n as follows:

The n'th *particle* starts at r and flips a fair coin. It goes left (L) with probability 1/2 and right (R) with probability 1/2.

It tries to move along the tree in the chosen direction. If there is a node below it in this direction then it goes there and continues its random moves. Otherwise it creates a new node where it wanted to move and stops.

Let D_n be the depth of this tree.
Claim: for any $t \geq 0$,

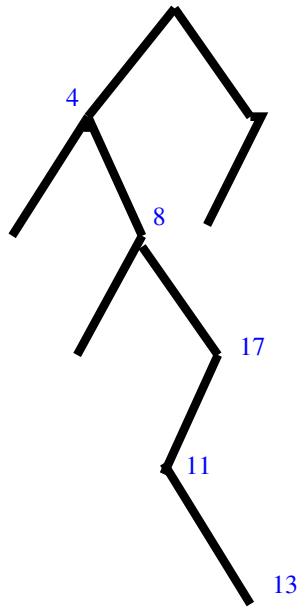
$$\Pi(D_n \geq t) \leq (n2^{-(t-1)/2})^t.$$

Proof The process requires at most n^2 coin flips and so we let $\Omega = \{L, R\}^{n^2}$ – most coin flips will not be needed most of the time.

$$DEEP = \{D_n \geq t\}.$$

For $P \in \{L, R\}^t$ and $S \subseteq [n]$, $|S| = t$ let $DEEP(P, S)$ = {the particles $S = \{s_1, s_2, \ldots, s_t\}$ follow P in the tree i.e. the first i moves of s_i are along P, $1 \leq i \leq t$}.

$$DEEP = \bigcup_P \bigcup_S DEEP(P, S).$$



S={4,8,11,13,17}

t=5 and DEEP(P,S) occurs if
4 goes L...
8 goes LR...
17 goes LRR...
11 goes LRRL...
13 goes LRRLR...

$$\begin{aligned}
\Pi(DEEP) &\leq \sum_P \sum_S \Pi(DEEP(P,S)) \\
&= \sum_P \sum_S 2^{-(1+2+\cdots+t)} \\
&= \sum_P \sum_S 2^{-t(t+1)/2} \\
&= 2^t \binom{n}{t} 2^{-t(t+1)/2} \\
&\leq 2^t n^t 2^{-t(t+1)/2} \\
&= (n 2^{-(t-1)/2})^t.
\end{aligned}$$

So if we put $t = A \log_2 n$ then

$$\Pi(D_n \geq A \log_2 n) \leq (2n^{1-A/2})^{A \log_2 n}$$

which is very small, for $A > 2$.

A problem with hats

There are n people standing a circle. They are blind-folded and someone places a hat on each person's head. The hat has been randomly colored Red or Blue.

They take off their blind-folds and everyone can see everyone else's hat. Each person then simultaneously declares (i) my hat is red or (ii) my hat is blue or (iii) or I pass.

They win a big prize if the people who opt for (i) or (ii) are all correct. They pay a big penalty if there is a person who incorrectly guesses the color of their hat.

Is there a strategy which means they will win with probability better than 1/2?

Suppose that we partition $Q_n = \{0,1\}^n$ into 2 sets W, L which have the property that L is a cover i.e. if $x = x_1 x_2 \cdots x_n \in W = Q_n \setminus L$ then there is $y_1 y_2 \cdots y_n \in L$ such that $h(x,y) = 1$ where

$$h(x,y) = |\{j : \ x_j \neq y_j\}|.$$

Hamming distance between x and y.

Assume that $0 \equiv \textit{Red}$ and $1 \equiv \textit{Blue}$. Person i knows x_j for $j \neq i$ (color of hat j) and if there is a unique value ξ of x_i which places x in W then person i will declare that their hat has color ξ.

The people assume that $x \in W$ and if indeed $x \in W$ then there is at least one person who will be in this situation and any such person will guess correctly.

Is there a small cover L?

Let $p = \frac{\ln n}{n}$. Choose L_1 randomly by placing $y \in Q_n$ into L_1 with probability p.

Then let L_2 be those $z \in Q_n$ which are not at Hamming distance ≤ 1 from some member of L_1.

Clearly $L = L_1 \cup L_2$ is a cover and
$\mathbf{E}(|L|) = 2^n p + 2^n(1-p)^{n+1} \leq 2^n(p + e^{-np}) \leq 2^n \frac{2\ln n}{n}$.

So there must exist a cover of size at most $2^n \frac{2\ln n}{n}$ and the players can win with probability at least $1 - \frac{2\ln n}{n}$.

MONDAY, SEPTEMBER 15, 2025.

Intersection Safe Families

Let $\mathcal{A}$ be a family of sub-sets of $[n]$. We say that $\mathcal{A}$ is *Intersection Safe* if for distinct $A, B, C \in \mathcal{A}$ we have $C \not\supseteq A \cap B$.
We use the probabilistic method to show the existence of an Intersection Safe family of exponential size.
Suppose that $\mathcal{A}$ consists of p randomly and independently chosen sets $X_1, X_2, \ldots, X_p$. Let Z denote the number of 3-tples i, j, k such that $X_i \cap X_j \subseteq X_k$. Then

$$\mathbf{E}(Z) = p(p-1)(p-2)\mathbf{Pr}(X_i \cap X_j \subseteq X_k) = p(p-1)(p-2)\left(\frac{7}{8}\right)^n.$$

(Observe that $\mathbf{Pr}(x \in (X_i \cap X_j) \setminus X_k = 1/8$.)

So if $p \leq (8/7)^{n/3}$ then

$$\mathbf{Pr}(Z \geq 1) \leq \mathbf{E}(Z) < p^3 \left(\frac{7}{8}\right)^n \leq 1$$

implying that there exists a union free family of size p.

There is a small problem here in that we might have repetitions $X_i = X_j$ for $i \neq j$. Then our set will not be of size p.

But if Z_1 denotes the number of pairs i, j such that $X_i = X_j$ then

$$\mathbf{Pr}(Z_1 \neq 0) \leq \mathbf{E}(Z_1) = \binom{p}{2} 2^{-n}$$

and so we should really choose p so that
$\mathbf{Pr}(Z + Z_1 \neq 0) \leq \mathbf{E}(Z) + \mathbf{E}(Z_1) < p^3 \left(\frac{7}{8}\right)^n + p^2 \left(\frac{1}{2}\right)^n \leq 1.$

Application: Suppose that we have a central storage containing n keys $\{k_1, k_2, \ldots, k_n\}$.

We must distribute sets of keys to p people. Person i will get the set $K_i = \{k_j : j \in X_i\}$. The sets $X_1, X_2, \ldots, X_p$ are public knowledge.

If person r wishes to communicate with person s then he/she will send them $\{k_j : j \in X_r \cap X_s\}$ as a means of proving their identity.

If the sets $X_1, X_2, \ldots, X_p$ are intersection safe, then person t cannot pretend to be person r.

It is possible therefore to have a "secure" system with p people that requires each person to get $O(\ln p)$ keys.

Graph Crossing Number

The crossing number of a graph G is the minimum number of edge crossings of a drawing of G in the plane.

Euler's forula implies that a planar graph with n vertices has at most $3n$ edges.

This implies that a graph $G = (V, E)$ requires at least $|E| - 3|V|$ crossings.

Theorem

If $|E| > 4|V|$ then G has crossing number $\Omega(|E|^3/|V|^2)$.

If $|E| \approx |V|^{3/2}$ then this gives $\Omega(|V|^{5/2})$ whereas $|E| - 3|V| = O(|V|^{3/2})$.

Proof

Suppose that G has a drawing with k crossings and let $0 < p < 1$.

Let $G_p = (V_p, E_p)$ denote the subgraph of G obtained by including each vertex in V_p independently with probability p.

E_p is then the set of edges $\{x, y\}$ such that $x, y \in V_p$.

$$\mathbf{E}(|V_p|) = p|V| \text{ and } \mathbf{E}(|E_p|) = p^2|E|.$$

Also,

$$\mathbf{E}(\text{number of crossings in the drawing of } G_p) = p^4 k.$$

So,

$$p^4 k \geq \mathbf{E}(|E_p| - 3|V_p|) = p^2|E| - 3p|V|.$$

So

$$k \geq \frac{p^2|E| - 3p|V|}{p^4}.$$

Maximising the RHS over $p \leq 1$ gives $p = 4|V|/|E|$ and

$$k \geq \frac{|E|^3}{64|V|^2}.$$

□

Connectivity of a Random Graph

The random graph $G_{n,p}$ is defined as follows: each edge of the complete graph K_n is included independently with probability p.

Thus if $G = (V, E)$ has vertex set $V = [n]$ then

$$\mathbf{Pr}(G_{n,p} = G) = p^{|E|}(1-p)^{\binom{n}{2}-|E|}.$$

Theorem

Let $p = \frac{\log n + c_n}{n}$. Then

$$\lim_{n\to\infty} \mathbf{Pr}(G_{n,p} \text{ is connected}) = \begin{cases} 0 & c_n \to -\infty. \\ e^{-e^{-c}} & c_n \to c. \\ 1 & c_n \to \infty. \end{cases}$$

FRIDAY, SEPTEMBER 19, 2025.

Proof

A vertex of $G_{n,p}$ is said to isolated if it has degree zero. If $G_{n,p}$ has isolated vertices then it is not connected. Let X_1 denote the number of isolated vertices in $G_{n,p}$. Therefore

$$\mathbf{Pr}(X_1 > 0) \leq \mathbf{Pr}(G_{n,p} \text{ is not connected})$$

$$\leq \mathbf{Pr}(X_1 > 0) + \mathbf{Pr}\left(G_{n,p} \text{ has component with } 2 \leq k \leq \frac{n}{2} \text{ vertices}\right).$$

It will suffice to prove that if $c_n \to c$ then

$$\lim_{n\to\infty} \mathbf{Pr}(G_{n,p} \text{ is connected}) = e^{-e^{-c}}.$$

This is because if $p' \geq p$ then

$$\mathbf{Pr}(G_{n,p} \text{ is connected}) \leq \mathbf{Pr}(G_{n,p'} \text{ is connected}).$$

$e^{-e^{-c_n}} \to 0$ if $c_n \to -\infty$ and $e^{-e^{-c_n}} \to 1$ if $c_n \to \infty$.

Let X_k denote the number of components in $G_{n,p}$ with k vertices. Then

$$\mathbf{Pr}\left(G_{n,p} \text{ has component with } 2 \le k \le \frac{n}{2} \text{ vertices}\right) = \mathbf{Pr}\left(\exists 2 \le k \le \frac{n}{2} : X_k > 0\right).$$

So, using Boole's inequality and the first moment method,

$$\mathbf{Pr}\left(\exists 2 \le k \le \frac{n}{2} : X_k > 0\right) \le \sum_{k=2}^{n/2} \Pi(X_k > 0) \le \sum_{k=2}^{n/2} \mathbf{E}(X_k)$$

$$\le \sum_{k=2}^{n/2} \binom{n}{k} k^{k-2} p^{k-1} (1-p)^{k(n-k)} = \sum_{k=2}^{n/2} u_k.$$

The factor k^{k-2} in the second line is the number of spanning trees of K_k. In effect we are bounding the number of connected components by the number of spanning trees in such components.

Now, for $2 \le k \le 10$,

$$u_k \le e^k n^k \left(\frac{\log n + c}{n}\right)^{k-1} e^{-k(n-10)(\log n + c)/n}$$

$$\le (1 + o(1)) e^{k(1-c)} \left(\frac{\log n}{n}\right)^{k-1},$$

and for $k > 10$

$$u_k \le \left(\frac{ne}{k}\right)^k k^{k-2} \left(\frac{\log n + c}{n}\right)^{k-1} e^{-k(\log n + c)/2}$$

$$\le n \left(\frac{e^{1-c/2+o(1)} \log n}{n^{1/2}}\right)^k.$$

So

$$\sum_{k=2}^{n/2} u_k \le (1 + o(1)) \frac{e^{2(1-c)} \log n}{n} + \sum_{k=10}^{n/2} n^{1+o(1)-k/2}$$

$$= O\left(n^{o(1)-1}\right).$$

Inequalities

Markov Inequality: let $X : \Omega \to \{0, 1, 2, \ldots, \}$ be a random variable. For any $t \geq 1$

$$\Pi(X \geq t) \leq \frac{\mathbf{E}(X)}{t}.$$

Proof

$$\begin{aligned}
\mathbf{E}(X) &= \sum_{k=0}^{\infty} k\Pi(X = k) \\
&\geq \sum_{k=t}^{\infty} k\Pi(X = k) \\
&\geq \sum_{k=t}^{\infty} t\Pi(X = k) \\
&= t\Pi(X \geq t).
\end{aligned}$$

In particular, if $t = 1$ then $\Pi(X \neq 0) \leq \mathbf{E}(X)$.

Chebycheff Inequality

Now let $\sigma = \sqrt{\mathbf{Var}(Z)}$.

$$\begin{aligned} \Pi(|Z-\mu| \geq t\sigma) &= \Pi((Z-\mu)^2 \geq t^2\sigma^2) \\ &\leq \frac{\mathbf{E}((Z-\mu)^2)}{t^2\sigma^2} \qquad (7) \\ &= \frac{1}{t^2}. \end{aligned}$$

(7) comes from the Markov inequality applied to the random variable $(Z-\mu)^2$.

Back to Binomial: $\sigma = \sqrt{np(1-p)}$.

$$\Pi(|B_{n,p} - np| \geq t\sqrt{np(1-p)}) \leq \frac{1}{t^2}$$

which implies

$$\Pi(|B_{n,p} - np| \geq \epsilon np) \leq \frac{1}{\epsilon^2 np}$$

[Law of large numbers.]

MONDAY, SEPTEMBER 22, 2025.

The Local Lemma

We go back to the coloring problem at the beginning of these slides. We now place a different restriction on the sets involved.

Theorem

Let $A_1, A_2, \ldots, A_n$ be subsets of A where $|A_i| \geq k$ for $1 \leq i \leq n$. If each A_i intersects at most 2^{k-3} other sets then there exists a partition $A = R \cup B$ such that

$$A_i \cap R \neq \emptyset \text{ and } A_i \cap B \neq \emptyset \qquad 1 \leq i \leq n.$$

Symmetric Local Lemma: We consider the following situation. $X = \{x_1, x_2, \ldots, x_N\}$ is a collection of independent random variables. Suppose that we have events $\mathcal{E}_i, i = 1, 2, \ldots, m$ where $\mathcal{E}_i$ depends only on the set $X_i \subseteq X$. Thus if $X_i \cap X_j = \emptyset$ then $\mathcal{E}_i$ and $\mathcal{E}_j$ are independent.
The dependency graph Γ has vertex set $[m]$ and an edge (i, j) iff $X_i \cap X_j \neq \emptyset$.

Theorem

Let

$$p = \max_i \Pi(\mathcal{E}_i) \text{ and let } d \text{ be the maximum degree of } \Gamma.$$

$$4dp \leq 1 \text{ implies that } \Pi\left(\bigcap_{i=1}^{m} \bar{\mathcal{E}}_i\right) \geq (1-2p)^m > 0.$$

Proof: We prove by induction on $|S|$ that for any i,

$$\Pi\left(\mathcal{E}_i \middle| \bigcap_{j\in S} \bar{\mathcal{E}}_j\right) \leq 2p. \tag{8}$$

Notice that this suffices, since

$$\Pi\left(\bigcap_{i=1}^{m} \bar{\mathcal{E}}_i\right) = \prod_{i=1}^{m} \Pi\left(\bar{\mathcal{E}}_i \middle| \bigcap_{j=1}^{i-1} \bar{\mathcal{E}}_j\right)$$

The base case $|S| = 0$ for (8) is trivial.

Inductive Step: Renumber for convenience so that $i = n, S = [s]$ and $(i, x) \notin \Gamma$ for $x > d$. Now

$$\begin{aligned}
\Pi\left(\mathcal{E}_n \middle| \bigcap_{i=1}^{s} \bar{\mathcal{E}}_i\right) &= \frac{\Pi\left(\mathcal{E}_n \cap \bigcap_{i=1}^{d} \bar{\mathcal{E}}_i \middle| \bigcap_{i=d+1}^{s} \bar{\mathcal{E}}_i\right)}{\Pi\left(\bigcap_{i=1}^{d} \bar{\mathcal{E}}_i \middle| \bigcap_{i=d+1}^{s} \bar{\mathcal{E}}_i\right)}, \\
&\leq \frac{\Pi\left(\mathcal{E}_n \middle| \bigcap_{i=d+1}^{s} \bar{\mathcal{E}}_i\right)}{\Pi\left(\bigcap_{i=1}^{d} \bar{\mathcal{E}}_i \middle| \bigcap_{i=d+1}^{s} \bar{\mathcal{E}}_i\right)}, \\
&\leq \frac{\Pi\left(\mathcal{E}_n \middle| \bigcap_{i=d+1}^{s} \bar{\mathcal{E}}_i\right)}{1 - \sum_{i=1}^{d} \Pi\left(\mathcal{E}_i \middle| \bigcap_{i=d+1}^{s} \bar{\mathcal{E}}_i\right)}.
\end{aligned}$$

Now

$$\Pi\left(\mathcal{E}_n \middle| \bigcap_{i=d+1}^{s} \bar{\mathcal{E}}_i\right) = \Pi(\mathcal{E}_n) \leq p, \tag{9}$$

since $\mathcal{E}_n$ is independent of $\mathcal{E}_{d+1}, \ldots, \mathcal{E}_s$.

Furthermore, we can assume that $d > 0$, else the events $\mathcal{E}_1, \ldots, \mathcal{E}_m$ are independent and the result is trivial. So, by induction, we have that

$$1 - \sum_{i=1}^{d} \Pi\left(\mathcal{E}_i \middle| \bigcap_{i=d+1}^{s} \bar{\mathcal{E}}_i\right) \geq 1 - 2dp \geq \frac{1}{2}. \tag{10}$$

The induction is now completed by using (9) and (10) in (**??**).

Proof of Theorem 15: We randomly color the elements of A Red and Blue. Let $\mathcal{E}_i$ be the event that A_i is mono-colored. Clearly, $\Pi(\mathcal{E}_i) \leq 2^{-(k-1)}$. Thus,

$$p \leq 2^{-(k-1)}.$$

The degree of vertex i of Γ is the number of j such that $A_i \cap A_j \neq \emptyset$. So, by assumption,

$$d \leq 2^{k-3}.$$

Theorem 16 implies that $\Pi\left(\bigcap_{i=1}^{n} \bar{\mathcal{E}}_i\right) > 0$ and so the required coloring exists.

WEDNESDAY, SEPTEMBER 24, 2025.

For the next application, let $D = (V, E)$ be a k-regular digraph. By this we mean that each vertex has exactly k in-neighbors and k out-neighbors.

Theorem

Every k-regular digraph has a collection of $\lfloor k/(4 \log k) \rfloor$ vertex disjoint cycles.

Proof: Let $r = \lfloor k/(4 \log k) \rfloor$ and color the vertices of D with colors $[r]$. For $v \in V$, let $\mathcal{E}_v$ be the event that there is a color missing at the out-neighbors of v. We will show that $\Pi\left(\bigcap_{v \in V} \bar{\mathcal{E}}_v\right) > 0$.
Suppose then that none of the events $\mathcal{E}_v, v \in V$ occur. Consider the graph D_j induced by a single color $j \in [r]$. Note that D_j is not the empty graph. Let $P_j = (v_1, v_2, \ldots, v_m)$ be a longest directed path in D_j. Let w be an out-neighbor of v_m of color j. We must have $w \in \{v_1, \ldots, v_m\}$, else P_j is not a longest path in D_j. Thus each $D_j, j \in [r]$ contains a cycle and these cycles are vertex disjoint.

We first estimate

$$\Pi(\mathcal{E}_v) \leq r\left(1-\frac{1}{r}\right)^k \leq ke^{-k/r} \leq ke^{-4\log k} = k^{-3}.$$

On the other hand, if $N^+(v)$ denotes the out-neighbors of v plus v then $\mathcal{E}_v$ is independent of all events $\mathcal{E}_w$ for which $N^+(v) \cap N^+(w) = \emptyset$. It follows that

$$d \leq k^2.$$

To apply Theorem 16 we need to have $4k^{-3}k^2 \leq 1$. This is true for $k \geq 4$. For $k \leq 3$ we have $r = 1$ and the local lemma is not needed.

Some extremal problems

Let $\mathcal{P}_n = \{A : A \subseteq [n]\}$ denote the *power set* of $[n]$. $\mathcal{A} \subseteq \mathcal{P}_n$ is a *Sperner* family if $A, B \in \mathcal{A}$ implies that $A \not\subseteq B$ and $B \not\subseteq A$

Theorem

If $\mathcal{A} \subseteq \mathcal{P}_n$ is a Sperner family $|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$.

Proof We will show that

$$\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq 1. \tag{11}$$

Now $\binom{n}{k} \leq \binom{n}{\lfloor n/2 \rfloor}$ for all k and so

$$1 \geq \sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{\lfloor n/2 \rfloor}} = \frac{|\mathcal{A}|}{\binom{n}{\lfloor n/2 \rfloor}}.$$

The set of all sets of size $\lfloor n/2 \rfloor$ is a Sperner family and so the bound in the above theorem is best possible.

Inequality (20) can be generalised as follows: Let $s \geq 1$ be fixed. Let $\mathcal{A}$ be a family of subsets of $[n]$ such that **there do not exist** distinct $A_1, A_2, \ldots, A_{s+1} \in \mathcal{A}$ such that $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_{s+1}$.

Theorem

$$\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq s.$$

Proof Let π be a random permutation of $[n]$.

Let $\mathcal{E}(A)$ be the event $\{\pi(1), \pi(2), \ldots, \pi(|A|) = A\}\}$.

Let

$$Z_i = \begin{cases} 1 & \mathcal{E}(A_i) \text{ occurs.} \\ 0 & \text{otherwise.} \end{cases}$$

and let $Z = \sum_i Z_i$ be the number of events $\mathcal{E}(A_i)$ that occur.

Now our family is such that $Z \leq s$ for all π and so

$$E(Z) = \sum_i E(Z_i) = \sum_i \mathbf{Pr}(\mathcal{E}(A_i)) \leq s.$$

On the other hand, $A \in \mathcal{A}$ implies that $\mathbf{Pr}(\mathcal{E}(A)) = \frac{1}{\binom{n}{|A|}}$ and the required inequality follows. □

Intersecting Families

A family $\mathcal{A} \subseteq \mathcal{P}_n$ is an *intersecting* family if $A, B \in \mathcal{A}$ implies $A \cap B \neq \emptyset$.

Theorem

If $\mathcal{A}$ is an intersecting family then $|\mathcal{A}| \leq 2^{n-1}$.

Proof Pair up each $A \in \mathcal{P}_n$ with its complement $A^c = [n] \setminus A$. This gives us 2^{n-1} pairs altogether. Since $\mathcal{A}$ is intersecting it can contain at most one member of each pair. □

If $\mathcal{A} = \{A \subseteq [n] : \ 1 \in A\}$ then $\mathcal{A}$ is intersecting and $|\mathcal{A}| = 2^{n-1}$ and so the above theorem is best possible.

Theorem

If $\mathcal{A}$ is an intersecting family and $A \in \mathcal{A}$ implies that $|A| = k \leq \lfloor n/2 \rfloor$ then

$$|\mathcal{A}| \leq \binom{n-1}{k-1}$$

Proof If π is a permutation of $[n]$ and $A \subseteq [n]$ let

$$\theta(\pi, A) = \begin{cases} 1 & \exists s: \ \{\pi(s), \pi(s+1), \ldots, \pi(s+k-1)\} = A \\ 0 & \textit{otherwise} \end{cases}$$

where $\pi(i) = \pi(i-n)$ if $i > n$.

We will show that for any permutation π,

$$\sum_{A \in \mathcal{A}} \theta(\pi, A) \leq k. \tag{12}$$

Assume (12). We first observe that if π is a random permutation then

$$\mathbf{E}(\theta(\pi, A)) = n\frac{k!(n-k)!}{n!} = \frac{k}{\binom{n-1}{k-1}}$$

and so, from (12),

$$k \geq \mathbf{E}(\sum_{A\in\mathcal{A}} \theta(\pi, A)) = \sum_{A\in\mathcal{A}} \frac{k}{\binom{n-1}{|A|-1}}$$

Hence

$$|\mathcal{A}| \leq \binom{n-1}{k-1}$$

Assume w.l.o.g. that π is the identity permutation.

Let $A_t = \{t, t+1, \ldots, t+k-1\}$ and suppose that $A_s \in \mathcal{A}$.

All of the other sets A_t that intersect A_s can be partitioned into pairs A_{s-i}, A_{s+k-i}, $1 \leq i \leq k-1$ and the members of each pair are disjoint. Thus $\mathcal{A}$ can contain at most one from each pair. This verifies (12).

FRIDAY, SEPTEMBER 26, 2025.

Sunflowers

A sunflower of size r is a family of sets $A_1, A_2, \ldots, A_r$ such that every element that belongs to more than one of the sets belongs to all of them.

Let $f(k, r)$ be the maximum size of a family of k-sets without a sunflower of size r.

Theorem

$f(k, r) \leq (r-1)^k k!$.

Proof Let $\mathcal{F}$ be a family of k-sets without a sunflower of size r. Let $A_1, A_2, \ldots, A_t$ be a maximum subfamily of pairwise disjoint subsets in $\mathcal{F}$.

Since a family of pairwise disjoint is a sunflower, we must have $t < r$.

Now let $A = \bigcup_{i=1}^{t} A_i$. For every $a \in A$ consider the family $\mathcal{F}_a = \{S \setminus \{a\} : S \in \mathcal{F}, a \in S\}$.

Now the size of A is at most $(r-1)k$.

The size of each $\mathcal{F}_a$ is at most $f(k-1, r)$. This is because a sunflower in $\mathcal{F}_a$ is a sunflower in $\mathcal{F}$.

So,

$$f(k, r) \leq (r-1)k \times f(k-1, r) \leq (r-1)k \times (r-1)^{k-1}(k-1)!,$$

by induction. □

Distinct Distances

Suppose that $X_1, X_2, \ldots, X_n$ are n points in the plane. We put bounds on the number of distinct distances among $|X_i X_j|$.

Let $f(n)$ denote the minimum among all sets of n points.

Lower bound: $f(n) \geq (n - 3/4)^{1/2} - 1/2$.

Assume that X_1 is a vertex of the least (in y value) convex polygon contained in the points. Let K be the number of distinct values among $\{|X_1 X_i| : i \geq 2\}$.

If N is the maximum number of times the same distance occurs then $KN \geq n - 1$.

If r is a distance that occurs N times then there are N points on the circle with center X_1 and radius r. They all lie on a semi-circle.

Going round the circle, let these points be $Q_1, Q_2, \ldots, Q_N$. Then $|Q_1Q_2| < |Q_1Q_3| \cdots < |Q_1Q_N|$.

Thus $f(n) \geq \max\{(n-1)/N, N-1\}$. $N(N-1)$ minimises this lower bound and gives us what we claim.

Upper bound: we consider the integer points $\{(x, y)\}$ where $0 \leq x, y \leq n^{1/2}$. These have distance of the form $(u^2 + v^2)^{1/2}$ and $cn/\log^{1/2} n$ is a bound on the number of integers of the form $0 \leq u^2 + v^2 \leq 2n$.

MONDAY, SEPTEMBER 29, 2025.

Oddtown

In order to cut down the number of committees a town of n people has instituted the following rules:

(a) Each club shall have an odd number of members.

(b) Each pair of clubs shall share an even number of members.

Theorem

With these rules, there are at most n clubs.

Proof Suppose that the clubs are $C_1, C_2, \ldots, C_m \subseteq [n]$.

Let $\bar{\nu}_i = (v_{i,1}, v_{i,2}, \ldots, v_{i,n})$ denote the incidence vector of C_i for $1 \leq i \leq m$ i.e. $v_{i,j} = 1$ iff $j \in C_i$. We treat these vectors as being over the two element field $\mathbb{F}_2$.

We claim that $\bar{\nu}_1, \bar{\nu}_2, \ldots, \bar{\nu}_m$ are linearly independent and the theorem will follow.

The rules imply that (i) $\bar{\nu}_i \cdot \bar{\nu}_i = 1$ and (ii) $\bar{\nu}_i \cdot \bar{\nu}_j = 0$ for $1 \leq i \neq j \leq m$.
(Remember that we are working over $\mathbb{F}_2$.)

Suppose then that

$$c_1\bar{\nu}_1 + c_2\bar{\nu}_2 + \cdots + c_m\bar{\nu}_m = 0.$$

We show that $c_1 = c_2 = \cdots = c_m = 0$.

Indeed, we have

$$\begin{aligned} 0 &= \bar{\nu}_j \cdot (c_1\bar{\nu}_1 + c_2\bar{\nu}_2 + \cdots + c_m\bar{\nu}_m) \\ &= c_1\bar{\nu}_1 \cdot \bar{\nu}_j + c_2\bar{\nu}_2 \cdot \bar{\nu}_j + \cdots + c_m\bar{\nu}_m \cdot \bar{\nu}_j \\ &= c_j, \end{aligned}$$

for $j = 1, 2, \ldots, m$. □

Decomposing K_n into bipartite subgraphs

Here we show

Theorem

If $G_k, k = 1, 2, \ldots, m$ is a collection of complete bipartite graphs with vertex partitions A_k, B_k, such that every edge of K_n is in exactly one subgraph, then $m \geq n - 1$. (Note that $A_k \cap B_k = \emptyset$ here.)

Proof This is tight since we can take $A_k = \{k\}, B_k = \{k + 1, \ldots, n\}$ for $k = 1, 2, \ldots, n - 1$.

Define $n \times n$ matrices M_k where $M_k(i, j) = 1$ if $i \in A_k, j \in B_k$ and $M_k(i, j) = 0$ otherwise.

Let $S = M_1 + M_2 + \cdots + M_m$. Then $S + S^T = J_n - I_n$ where I_n is the identity matrix and J_n is the all ones matrix.

Decomposing K_n into bipartite subgraphs

We show next that $rank(S) \geq n-1$ and then the theorem follows from

$$rank(S) \leq rank(M_1) + rank(M_2) + \cdots + rank(M_m) \leq m.$$

Suppose then that $rank(S) \leq n-2$ so that there exists a non-zero solution $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ to the system of equations

$$S\mathbf{x} = 0, \sum_{i=1}^{n} x_i = 0.$$

But then, $J_n\mathbf{x} = 0$ and $S^T\mathbf{x} = -\mathbf{x}$ and $-|\mathbf{x}|^2 = -\mathbf{x}^T S^T \mathbf{x} = 0$, contradiction. □

Lighting problem

Let $G = (V, E)$ be an arbitrary graph. Suppose that each vertex contains a light bulb and $\ell(v) = 1$ indicates that the light bulb on v is on and $\ell(v) = 0$ indicates that it is off.

Suppose that for $v \in V$, the transformation $T(v)$ flips the values at v and all of its neighbors. I.e. $T(v)$ switches on a neighboring light bulb if it is off and turns it off if it is on.

Suppose that initially, $\ell(v) = 0$ for all $v \in V$, i.e. all light bulbs are off. We show that there exists a set $S \subseteq V$ such that applying $T(v), v \in S$ in any order makes $\ell(v) = 1$ for $v \in V$.

Lighting problem

Observe first that applying $T(v)$ and then $T(w)$ achieves the same effect as applying $T(w)$ and then $T(v)$ i.e. the order of application of the transformations does not matter.
(The value of $\ell(u)$ is flipped by the two transformations iff it is adjacent to exactly one of $\{v, w\}$.)

Let A be the 0-1 adjacency matrix of G i.e. let $A(v, w) = 1$ iff $w \in N(v)$. In addition put $A(v, v) = 1$ for $v \in V$.

The set of transformations corresponding to S will turn on all of the lights iff $A\mathbf{1}_S = \mathbf{1}_V$ where $\mathbf{1}_S$ is the 0-1 vector indexed by V such that there is a 1 in component v iff $v \in S$.

Lighting problem

Our claim amounts to saying that there exists S such that $A\mathbf{1}_S = \mathbf{1}_V$ where calculations are done in the binary field.

If there is no such $\mathbf{1}_S$ then basic linear algebra theory tells us that there exists x such that $x^T A = 0$ and $x^T 1_V \neq 0$.

Since A is symmetric, this means that $Ax = 0$ as well. Let $x = 1_S$. Then S has the following properties:

(a) $|S \cap N(v)|$ is odd for all $v \in V$. This is a consequence of $Ax = 0$.

(b) $|S|$ is odd. This is a consequence of $x^T 1_V \neq 0$.

Lighting problem

Now consider the sub-graph of G induced by S.

Every vertex has odd degree by (a). But in any graph, the number of odd vertices is even. Contradiction.

WEDNESDAY, OCTOBER 1, 2025.

Pigeon Hole Principle

If $f : [m] \to [n]$ then there exists $i \in [n]$ such that

$$|f^{-1}(i)| \geq \lceil m/n \rceil.$$

Informally: If m pigeons are to be placed in n pigeon-holes, at least one hole will end up with at leat $\lceil m/n \rceil$ pigeons.

We have two disks, each partitioned into 200 sectors of the same size. 100 of the sectors of Disk 1 are coloured Red and 100 are colored Blue. The 200 sectors of Disk 2 are arbitrarily coloured Red and Blue.

It is always possible to place Disk 2 on top of Disk 1 so that the centres coincide, the sectors line up and at least 100 sectors of Disk 2 have the same colour as the sector underneath them.

Fix the position of Disk 1. There are 200 positions for Disk 2 and let q_i denote the number of matches if Disk 2 is placed in position i. Now for each sector of Disk 2 there are 100 positions i in which the colour of the sector underneath it coincides with its own.

Therefore

$$q_1 + q_2 + \cdots + q_{200} = 200 \times 100 \tag{13}$$

and so there is an i such that $q_i \geq 100$.

Explanation of (20).
Consider 0-1 200×200 matrix $A(i,j)$ where $A(i,j) = 1$ iff sector j lies on top of a sector with the same colour when in position i. Row i of A has q_i 1's and column j of A has 100 1's. The LHS of (20) counts the number of 1's by adding rows and the RHS counts the number of 1's by adding columns.

Alternative solution: Place Disk 2 randomly on Disk 1 so that the sectors align. For $i = 1, 2, \ldots, 200$ let

$$X_i = \begin{cases} 1 & \textit{sector } i \textit{ of disk 2 is on sector of disk 1 of same color} \\ 0 & \textit{otherwise} \end{cases}$$

We have

$$\mathbf{E}(X_i) = 1/2 \qquad \textit{for } i = 1, 2, \ldots, 200.$$

So if $X = X_1 + \cdots + X_{200}$ is the number of sectors sitting above sectors of the same color, then $\mathbf{E}(X) = 100$ and there must exist at least one way to achieve 100.

Theorem

(Erdős-Szekeres) An arbitrary sequence of integers $(a_1, a_2, \ldots, a_{k^2+1})$ *contains a monotone subsequence of length* $k+1$.

Proof. Let $(a_i, a_i^1, a_i^2, \ldots, a_i^{\ell-1})$ be the longest *monotone increasing* subsequence of $(a_1, \ldots, a_{k^2+1})$ that starts with $a_i, (1 \leq i \leq k^2+1)$, and let $\ell(a_i)$ be its length.

If for some $1 \leq i \leq k^2+1$, $\ell(a_i) \geq k+1$, then $(a_i, a_i^1, a_i^2, \ldots, a_i^{l-1})$ is a monotone increasing subsequence of length $\geq k+1$.

So assume that $\ell(a_i) \leq k$ holds for every $1 \leq i \leq k^2+1$.

Consider k holes $1, 2, \ldots, k$ and place i into hole $\ell(a_i)$.

There are $k^2 + 1$ subsequences and $\leq k$ non-empty holes (different lengths), so by the pigeon-hole principle there will exist ℓ^* such that there are (at least) $k + 1$ indices $i_1 < i_2 < \cdots < i_{k+1}$ such that $\ell(a_{i_t}) = \ell^*$ for $1 \leq t \leq k + 1$.

Then we must have $a_{i_1} \geq a_{i_2} \geq \cdots \geq a_{i_{k+1}}$.

Indeed, assume to the contrary that $a_{i_m} < a_{i_n}$ for some $1 \leq m < n \leq k + 1$. Then $a_{i_m} \leq a_{i_n} \leq a^1_{i_n} \leq a^2_{i_n} \leq \cdots \leq a^{\ell^*-1}_{i_n}$, i.e., $\ell(a_{i_m}) \geq \ell^* + 1$, a contradiction. □

The sequence

$$n, n-1, \ldots, 1, 2n, 2n-1, \ldots, n+1, \ldots, n^2, n^2-1, \ldots, n^2-n+1$$

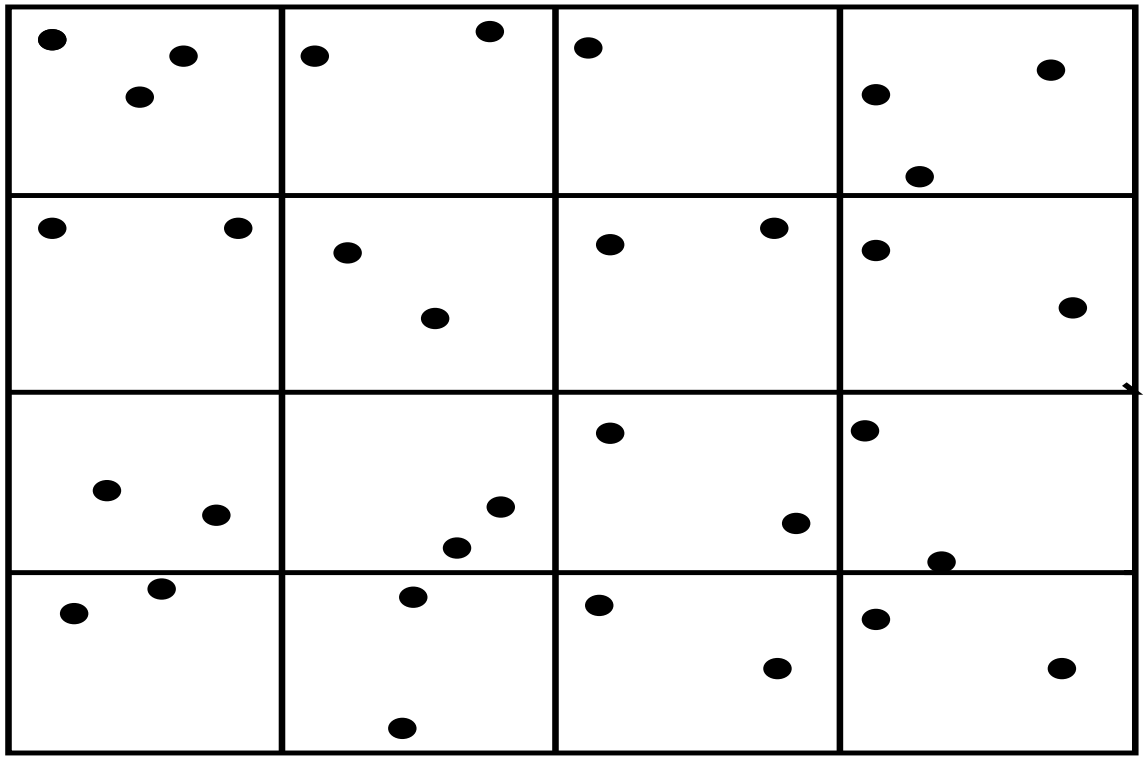
has no monotone subsequence of length $n+1$ and so the Erdős-Szekerés result is best possible.

Let $P_1, P_2, \ldots, P_n$ be n points in the unit square $[0,1]^2$. We will show that there exist $i, j, k \in [n]$ such that the triangle $P_i P_j P_k$ has area

$$\leq \frac{1}{2(\left\lfloor \sqrt{(n-1)/2} \right\rfloor)^2} \sim \frac{1}{n}$$

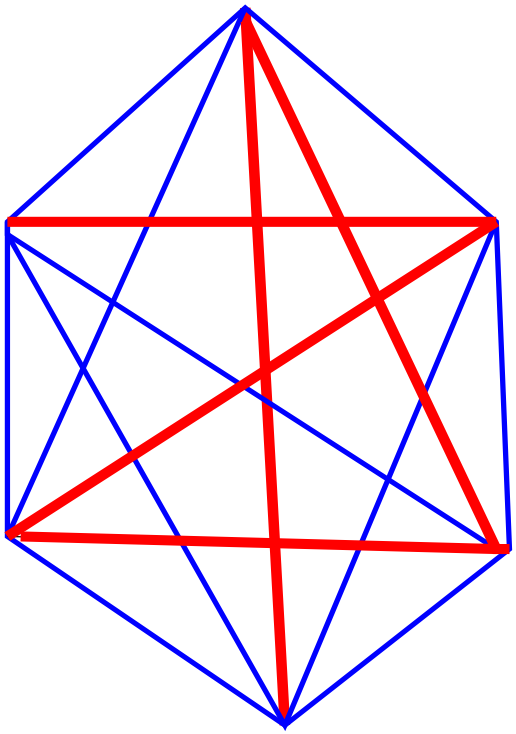
for large n.

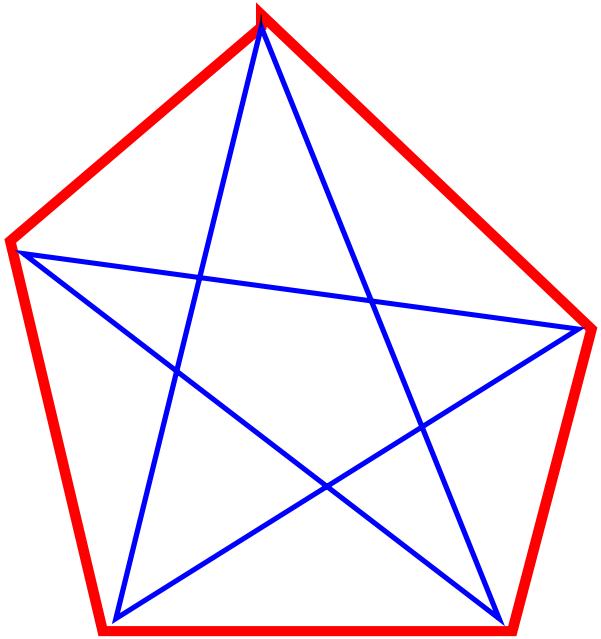
Let $m = \left\lfloor \sqrt{(n-1)/2} \right\rfloor$ and divide the square up into $m^2 < \frac{n}{2}$ subsquares. By the pigeonhole principle, there must be a square containing ≥ 3 points. Let 3 of these points be $P_i P_j P_k$. The area of the corresponding triangle is at most one half of the area of an individual square.



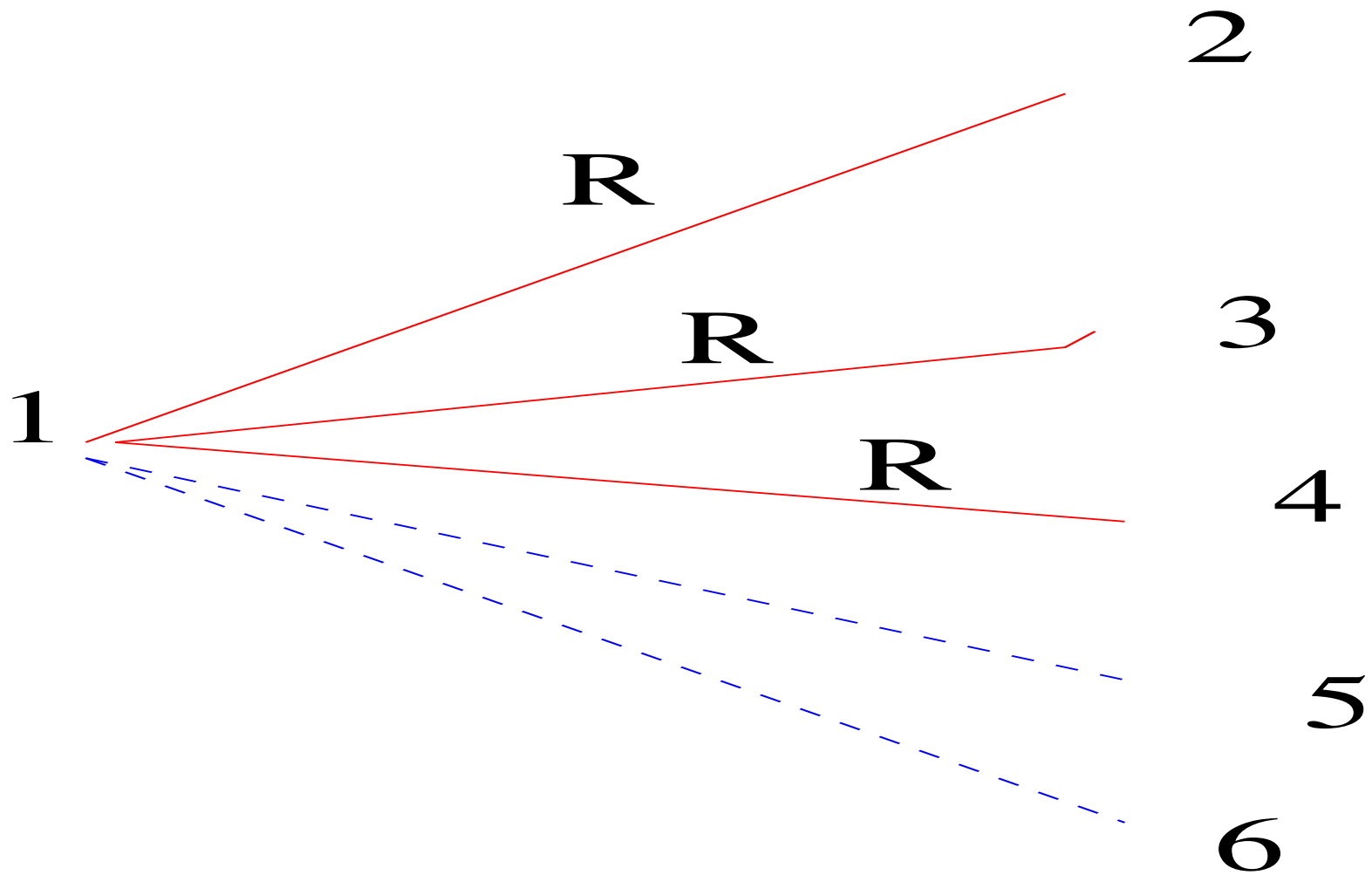
Ramsey's Theorem

Suppose we 2-colour the edges of K_6 of Red and Blue. There *must* be either a Red triangle or a Blue triangle.





This is not true for K_5.



There are 3 edges of the same colour incident with vertex 1, say (1,2), (1,3), (1,4) are Red. Either (2,3,4) is a blue triangle or one of the edges of (2,3,4) is Red, say (2,3). But the latter implies (1,2,3) is a Red triangle.

FRIDAY, OCTOBER 3, 2025.

Ramsey's Theorem

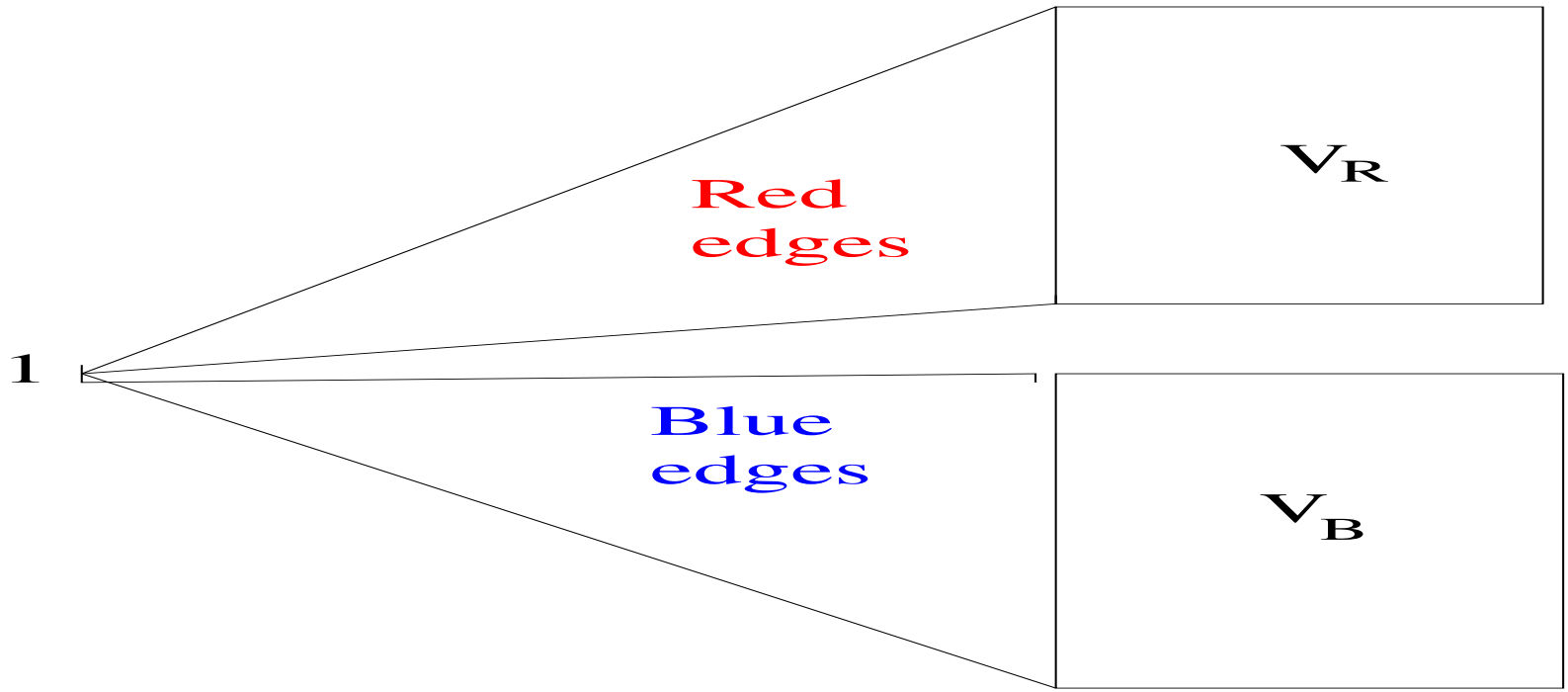
For all positive integers k, ℓ there exists $R(k, \ell)$ such that if $N \geq R(k, \ell)$ and the edges of K_N are coloured Red or Blue then then either there is a "Red k-clique" or there is a "Blue ℓ-clique. A clique is a complete subgraph and it is Red if all of its edges are coloured red etc.

$$\begin{aligned} R(1,k) &= R(k,1) = 1 \\ R(2,k) &= R(k,2) = k \end{aligned}$$

Theorem

$$R(k,\ell) \leq R(k,\ell-1) + R(k-1,\ell).$$

Proof Let $N = R(k,\ell-1) + R(k-1,\ell)$.



$V_R = \{(x :\ (1,x)$ is coloured Red$\}$ and $V_B = \{(x :\ (1,x)$ is coloured Blue$\}$.

$$|V_R| \geq R(k-1,\ell) \text{ or } |V_B| \geq R(k,\ell-1).$$

Since

$$\begin{aligned} |V_R| + |V_B| &= N-1 \\ &= R(k,\ell-1) + R(k-1,\ell) - 1. \end{aligned}$$

Suppose for example that $|V_R| \geq R(k-1,\ell)$. Then either V_R contains a Blue ℓ-clique – done, or it contains a Red $k-1$-clique K. But then $K \cup \{1\}$ is a Red k-clique. Similarly, if $|V_B| \geq R(k,\ell-1)$ then either V_B contains a Red k-clique – done, or it contains a Blue $\ell-1$-clique L and then $L \cup \{1\}$ is a Blue ℓ-clique. □

Theorem

$$R(k,\ell) \leq \binom{k+\ell-2}{k-1}.$$

Proof Induction on $k+\ell$. True for $k+\ell \leq 5$ say. Then

$$\begin{aligned} R(k,\ell) &\leq R(k,\ell-1) + R(k-1,\ell) \\ &\leq \binom{k+\ell-3}{k-1} + \binom{k+\ell-3}{k-2} \\ &= \binom{k+\ell-2}{k-1}. \end{aligned}$$

□

So, for example,

$$\begin{aligned} R(k,k) &\leq \binom{2k-2}{k-1} \\ &\leq 4^k \end{aligned}$$

Theorem

$$R(k,k) > 2^{k/2}$$

Proof We must prove that if $n \leq 2^{k/2}$ then there exists a Red-Blue colouring of the edges of K_n which contains no Red k-clique and no Blue k-clique. We can assume $k \geq 4$ since we know $R(3,3) = 6$.

We show that this is true with positive probability in a *random* Red-Blue colouring. So let Ω be the set of all Red-Blue edge colourings of K_n with uniform distribution. Equivalently we independently colour each edge Red with probability 1/2 and Blue with probability 1/2.

Let
$\mathcal{E}_R$ be the event: {There is a Red k-clique} and
$\mathcal{E}_B$ be the event: {There is a Blue k-clique}.
We show

$$\mathbf{Pr}(\mathcal{E}_R \cup \mathcal{E}_B) < 1.$$

Let $C_1, C_2, \ldots, C_N$, $N = \binom{n}{k}$ be the vertices of the N k-cliques of K_n.

Let $\mathcal{E}_{R,j}$ be the event: {C_j is Red} and let $\mathcal{E}_{B,j}$ be the event: {C_j is Blue}.

$$
\begin{aligned}
\mathbf{Pr}(\mathcal{E}_R \cup \mathcal{E}_B) &\leq \mathbf{Pr}(\mathcal{E}_R) + \mathbf{Pr}(\mathcal{E}_B) = 2\mathbf{Pr}(\mathcal{E}_R) \\
&= 2\mathbf{Pr}\left(\bigcup_{j=1}^{N} \mathcal{E}_{R,j}\right) \leq 2\sum_{j=1}^{N} \mathbf{Pr}(\mathcal{E}_{R,j}) \\
&= 2\sum_{j=1}^{N} \left(\frac{1}{2}\right)^{\binom{k}{2}} = 2\binom{n}{k}\left(\frac{1}{2}\right)^{\binom{k}{2}} \\
&\leq 2\frac{n^k}{k!}\left(\frac{1}{2}\right)^{\binom{k}{2}} \\
&\leq 2\frac{2^{k^2/2}}{k!}\left(\frac{1}{2}\right)^{\binom{k}{2}} \\
&= \frac{2^{1+k/2}}{k!} \\
&< 1.
\end{aligned}
$$

Very few of the Ramsey numbers are known exactly. Here are a few known values.

$$\begin{array}{rcrcl} & & R(3,3) & = & 6 \\ & & R(3,4) & = & 9 \\ & & R(4,4) & = & 18 \\ & & R(4,5) & = & 25 \\ 43 & \leq & R(5,5) & \leq & 49 \end{array}$$

MONDAY, October 6, 2025.

Ramsey's Theorem in general

Remember that the elements of $\binom{S}{r}$ are the r-subsets of S

Theorem

Let $r, s \geq 1$, $q_i \geq r$, $1 \leq i \leq s$ be given. Then there exists $N = N(q_1, q_2, \ldots, q_s; r)$ with the following property: Suppose that S is a set with $n \geq N$ elements. Let each of the elements of $\binom{S}{r}$ be given one of s colors. .

Then there exists i and a q_i-subset T of S such that all of the elements of $\binom{T}{r}$ are colored with the ith color.

Proof First assume that $s = 2$ i.e. two colors, Red, Blue.

Ramsey's Theorem in general

Note that

$$\begin{aligned} (a)\ N(p,q;1) &= p+q-1 \\ (b)\ N(p,r;r) &= p(\geq r) \\ N(r,q;r) &= q(\geq r) \end{aligned}$$

We proceed by induction on r. It is true for $r = 1$ and so assume $r \geq 2$ and it is true for $r-1$ and arbitrary p, q. Now we further proceed by induction on $p+q$. It is true for $p+q = 2r$ and so assume it is true for r and all p', q' with $p'+q' < p+q$.
Let

$$\begin{aligned} p_1 &= N(p-1,q;r) \\ p_2 &= N(p,q-1;r) \end{aligned}$$

These exist by induction.

Ramsey's Theorem in general

Now we prove that

$$N(p, q; r) \leq 1 + N(p_1, q_1; r - 1)$$

where the RHS exists by induction.

Suppose that $n \geq 1 + N(p_1, q_1; r - 1)$ and we color $\binom{[n]}{r}$ with 2 colors. Call this coloring σ.

From this we define a coloring τ of $\binom{[n-1]}{r-1}$ as follows: If $X \in \binom{[n-1]}{r-1}$ then give it the color of $X \cup \{n\}$ under σ.

Now either (i) there exists $A \subseteq [n-1]$, $|A| = p_1$ such that (under τ) all members of $\binom{A}{r-1}$ are Red or (ii) there exists $B \subseteq [n-1]$, $|A| = q_1$ such that (under τ) all members of $\binom{B}{r-1}$ are Blue.

Ramsey's Theorem in general

Assume w.l.o.g. that (i) holds.

$$|A| = p_1 = N(p-1, q; r).$$

Then either

(a) $\exists B \subseteq A$ such that $|B| = q$ and under σ all of $\binom{B}{r}$ is Blue,

or

(b) $\exists A' \subseteq A$ such that $|A'| = p-1$ and all of $\binom{A'}{r}$ is Red. But then all of $\binom{A' \cup \{n\}}{r}$ is Red. If $X \subseteq A'$, $|X| = r-1$ then τ colors X Red, since $A' \subseteq A$. But then σ will color $X \cup \{n\}$ Red.

Ramsey's Theorem in general

Now consider the case of s colors. We show that

$$N(q_1, q_2, \ldots, q_s; r) \leq N(Q_1, Q_2; r)$$

where

$$\begin{aligned} Q_1 &= N(q_1, q_2, \ldots, q_{\lfloor s/2 \rfloor}; r) \\ Q_2 &= N(q_{\lfloor s/2 \rfloor + 1}, q_{\lfloor s/2 \rfloor + 2}, \ldots, q_s; r) \end{aligned}$$

Let $n = N(Q_1, Q_2; r)$ and assume we are given an s-coloring of $\binom{[n]}{r}$.

First temporarily re-color Red, any r-set colored with $i \leq \lfloor s/2 \rfloor$ and re-color Blue any r-set colored with $i > \lfloor s/2 \rfloor$.

Ramsey's Theorem in general

Then either (a) there exists a Q_1-subset A of $[n]$ with $\binom{A}{r}$ colored Red or (b) there exists a Q_2-subset B of $[n]$ with $\binom{A}{r}$ colored Blue.

W.l.o.g. assume the first case. Now replace the colors of the r-sets of A by there original colors. We have a $\lfloor s/2 \rfloor$-coloring of $\binom{A}{r}$. Since $|A| = N(q_1, q_2, \ldots, q_{\lfloor s/2 \rfloor}; r)$ there must exist some $i \leq \lfloor s/2 \rfloor$ and a q_i-subset S of A such that all of $\binom{S}{r}$ has color i.

□

Schur's Theorem

Let $r_k = N(3, 3, \ldots, 3; 2)$ be the smallest n such that if we k-color the edges of K_n then there is a mono-chromatic triangle.

Theorem

For all partitions $S_1, S_2, \ldots, S_k$ of $[r_k]$, there exist i and $x, y, z \in S_i$ such that $x + y = z$.

Proof Given a partition $S_1, S_2, \ldots, S_k$ of $[n]$ where $n \geq r_k$ we define a coloring of the edges of K_n by coloring (u, v) with color j where $|u - v| \in S_j$.

There will be a mono-chromatic triangle i.e. there exist j and $x < y < z$ such that $u = y - x$, $v = z - x$, $w = z - y \in S_j$. But $u + v = w$. □

Convex Polygons

A set of points X in the plane is in general position if no 3 points of X are collinear.

Theorem

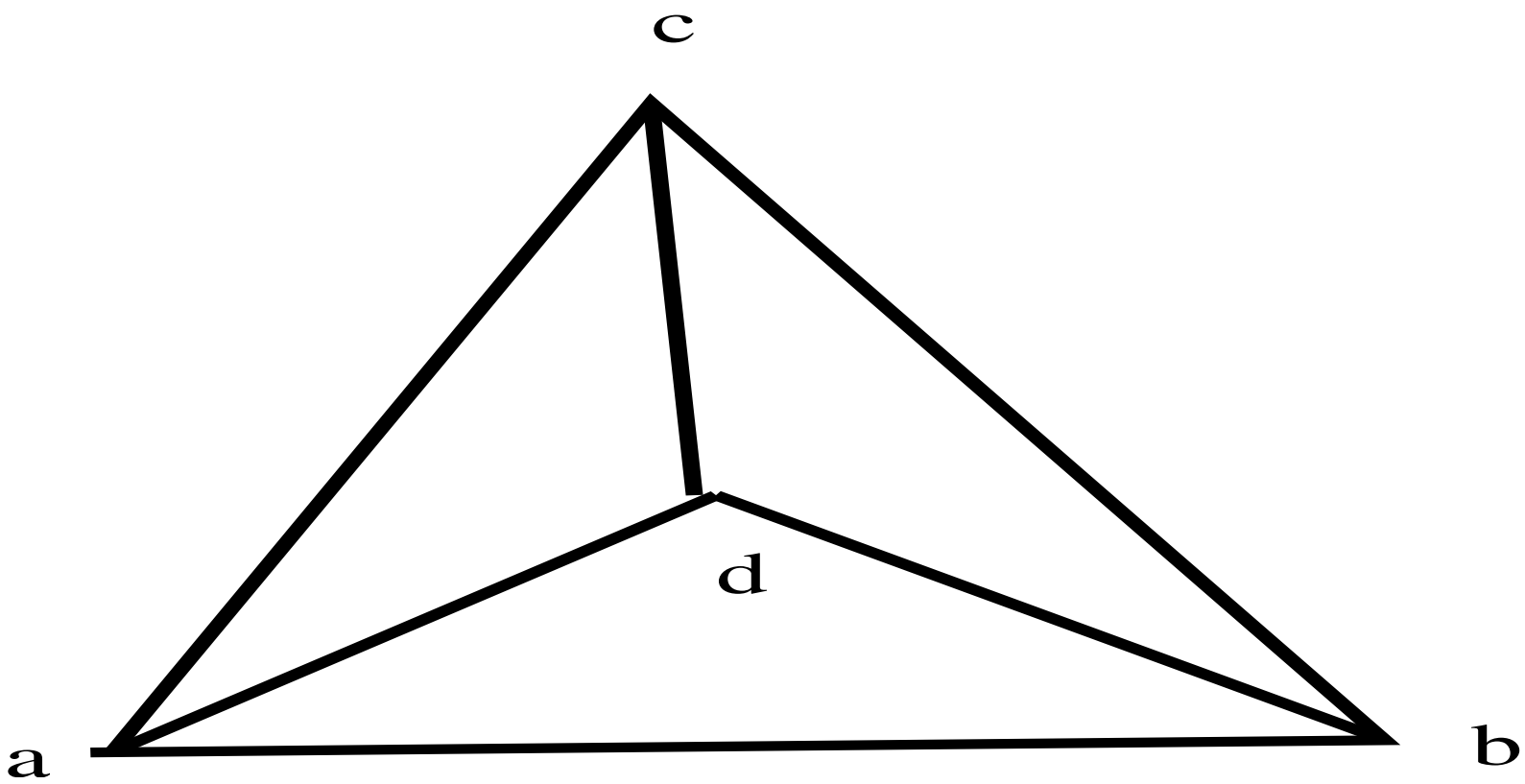
If $n \geq N(k, k; 3)$ and X is a set of n points in the plane which are in general position then X contains a k-subset Y which form the vertices of a convex polygon.

Proof We first observe that if every 4-subset of $Y \subseteq X$ forms a convex quadrilateral then Y itself induces a convex polygon.

Now label the points in S from X_1 to X_n and then color each triangle $T = \{X_i, X_j, X_k\}$, $i < j < k$ as follows: If traversing triangle $X_iX_jX_k$ in this order goes round it clockwise, color T Red, otherwise color T Blue.

Convex Polygons

Now there must exist a k-set T such that all triangles formed from T have the same color. All we have to show is that T does not contain the following configuration:

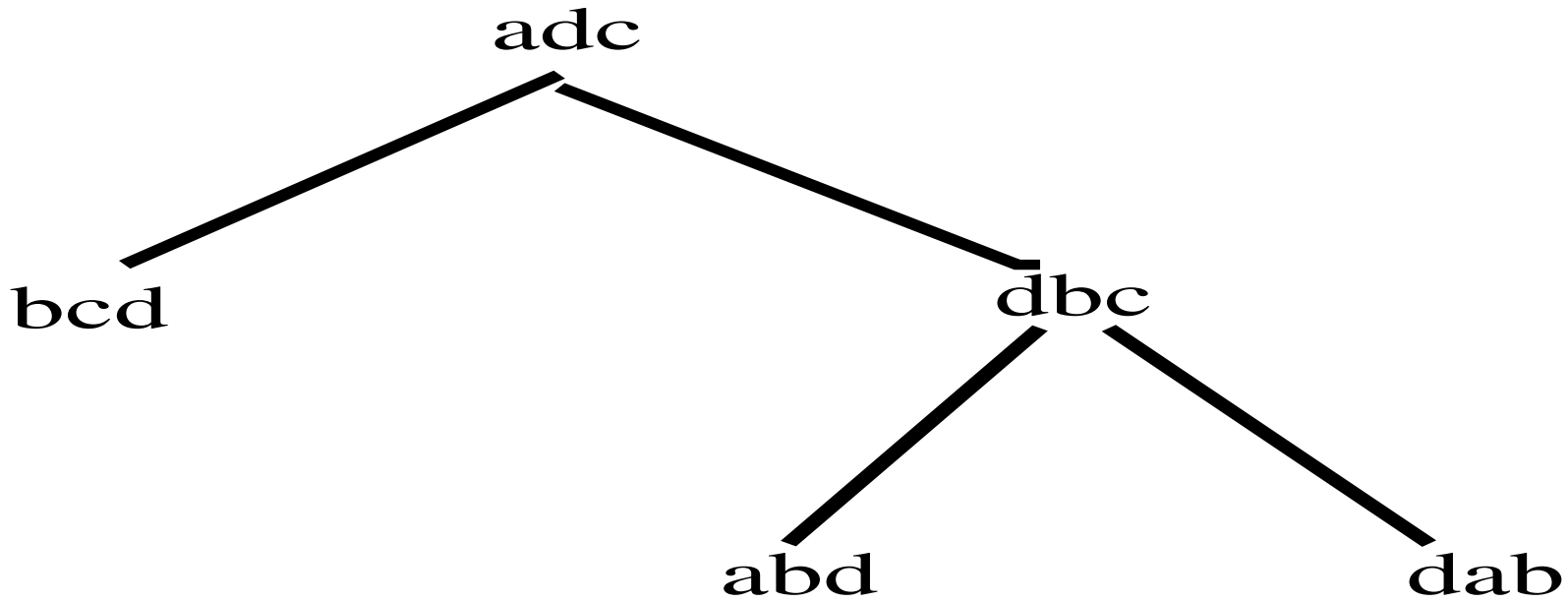


Convex Polygons

Assume w.l.o.g. that $a < b < c$ which implies that $X_i X_j X_k$ is colored Blue.

All triangles in the previous picture are colored Blue.

So the possibilities are



and all are impossible.

Ramsey $R(H_1, H_2)$

We define $r(H_1, H_2)$ to be the minimum n such that in in Red-Blue coloring of the edges of K_n there is eithere (i) a Red copy of H_1 or (ii) a Blue copy of H_2.

As an example, consider $r(P_3, P_3)$ where P_t denotes a path with t edges.

We show that

$$r(P_3, P_3) = 5.$$

$R(P_3, P_3) > 4$: We color edges incident with 1 Red and the remaining edges $\{(2,3),(3,4),(4,1)\}$ Blue. There is no mono-chromatic P_3. $R(P_3, P_3) \leq 5$: There must be two edges of the same color incident with 1.

Ramsey $R(H_1, H_2)$

$R(P_3, P_3) \leq 5$: There must be two edges of the same color incident with 1. Assume then that $(1,2), (1,3)$ are both Red.

If any of $(2,4), (2,5), (3,4), (3,5)$ are Red then we have a Red P_3.

If all four of these edges are Blue then $(4,2,5,3)$ is Blue.

Ramsey $R(H_1, H_2)$

We show next that $r(K_{1,s}, P_t) \leq s + t$. Here $K_{1,s}$ is a star: i.e. a vertex v and s incident edges.

Let $n = s + t$. If there is no vertex of Red degree s then the minimum degree in the graph induced by the Blue edges is at least t.

We then note that a graph of minimum degree δ contains a path of length δ.

MONDAY, October 20, 2025.

Partially Ordered Sets

A **partially ordered set** or **poset** is a set P and a binary relation $\preceq$ such that for all $a, b, c \in P$

1. $a \preceq a$ (reflexivity).
2. $a \preceq b$ and $b \preceq c$ implies $a \preceq c$ (transitivity).
3. $a \preceq b$ and $b \preceq a$ implies $a = b$. (anti-symmetry).

Examples

1. $P = \{1, 2, \ldots, \}$ and $a \leq b$ has the usual meaning.
2. $P = \{1, 2, \ldots, \}$ and $a \preceq b$ if a divides b.
3. $P = \{A_1, A_2, \ldots, A_m\}$ where the A_i are sets and $\preceq = \subseteq$.

Partially Ordered Sets

A pair of elements a, b are **comparable** if $a \preceq b$ or $b \preceq a$. Otherwise they are **incomparable**.

A poset without incomparable elements (Example 1) is a linear or total order.

We write $a < b$ if $a \preceq b$ and $a \neq b$.

A **chain** is a sequence $a_1 < a_2 < \cdots < a_s$.

A set A is an **anti-chain** if every pair of elements in A are incomparable.

Thus a Sperner family is an anti-chain in our third example.

Partially Ordered Sets

Theorem

Let P be a finite poset, then
$\min\{m : \exists \text{ anti-chains } A_1, A_2, \ldots, A_\mu \text{ with } P = \bigcup_{i=1}^{\mu} A_i\} =$
$\max\{|C| : A \text{ is a chain}\}$.

The minimum number of anti-chains needed to cover P is at least the size of any chain, since a chain can contain at most one element from each anti-chain.

Partially Ordered Sets

We prove the converse by induction on the maximum length μ of a chain. We have to show that P can be partitioned into μ anti-chains.

If $\mu = 1$ then P itself is an anti-chain and this provides the basis of the induction.

So now suppose that $C = x_1 < x_2 < \cdots < x_\mu$ is a maximum length chain and let A be the set of maximal elements of P.

(An element is x *maximal* if $\nexists y$ such that $y > x$.)

A is an anti-chain.

Now consider $P' = P \setminus A$. P' contains no chain of length μ. If it contained $y_1 < y_2 < \cdots < y_\mu$ then since $y_\mu \notin A$, there exists $a \in A$ such that P contains the chain $y_1 < y_2 < \cdots < y_\mu < a$, contradiction.

Thus the maximum length of a chain in P' is $\mu - 1$ and so it can be partitioned into anti-chains $A_1 \cup A_2 \cup \cdots A_{\mu-1}$. Putting $A_\mu = A$ completes the proof. □

Partially Ordered Sets

Suppose that $C_1, C_2, \ldots, C_m$ are a collection of chains such that $P = \bigcup_{i=1}^{m} C_i$.

Suppose that A is an anti-chain. Then $m \geq |A|$ because if $m < |A|$ then by the pigeon-hole principle there will be two elements of A in some chain.

Theorem

(Dilworth) Let P be a finite poset, then
$\min\{m : \exists$ *chains* $C_1, C_2, \ldots, C_m$ *with* $P = \bigcup_{i=1}^{m} C_i\} =$
$\max\{|A| :$ *A is an anti-chain*$\}$.

Partially Ordered Sets

We have already argued that $\max\{|A|\} \leq \min\{m\}$.

We will prove there is equality here by induction on $|P|$.

The result is trivial if $|P| = 0$.

Now assume that $|P| > 0$ and that μ is the maximum size of an anti-chain in P. We show that P can be partitioned into μ chains.

Let $C = x_1 < x_2 < \cdots < x_p$ be a *maximal* chain in P i.e. we cannot add elements to it and keep it a chain.

Case 1 Every anti-chain in $P \setminus C$ has $\leq \mu - 1$ elements. Then by induction $P \setminus C = \bigcup_{i=1}^{\mu-1} C_i$ and then $P = C \cup \bigcup_{i=1}^{\mu-1} C_i$ and we are done.

Partially Ordered Sets

Case 2

There exists an anti-chain $A = \{a_1, a_2, \ldots, a_\mu\}$ in $P \setminus C$. Let

- $P^- = \{x \in P : \ x \preceq a_i \text{ for some } i\}$.
- $P^+ = \{x \in P : \ x \succeq a_i \text{ for some } i\}$.

Note that

1. $P = P^- \cup P^+$. Otherwise there is an element x of P which is incomparable with every element of A and so μ is not the maximum size of an anti-chain.
2. $P^- \cap P^+ = A$. Otherwise there exists x, i, j such that $a_i < x < a_j$ and so A is not an anti-chain.
3. $x_p \notin P^-$. Otherwise $x_p < a_i$ for some i and the chain C is not maximal.

Partially Ordered Sets

Applying the inductive hypothesis to P^- ($|P^-| < |P|$ follows from 3) we see that P^- can be partitioned into μ chains $C_1^-, C_2^-, \ldots, C_\mu^-$.

Now the elements of A must be distributed one to a chain and so we can assume that $a_i \in C_i^-$ for $i = 1, 2, \ldots, \mu$.

a_i must be the maximum element of chain C_i^-, else the maximum of C_i^- is in $(P^- \cap P^+) \setminus A$, which contradicts 2.

Applying the same argument to P^+ we get chains $C_1^+, C_2^+, \ldots, C_\mu^+$ with a_i as the minimum element of C_i^+ for $i = 1, 2, \ldots, \mu$.

Then from 2 we see that $P = C_1 \cup C_2 \cup \cdots \cup C_\mu$ where $C_i = C_i^- \cup C_i^+$ is a chain for $i = 1, 2, \ldots, \mu$.

(i) Another proof of

Theorem

Erdős and Szekerés
$a_1, a_2, \ldots, a_{n^2+1}$ *contains a monotone subsequence of length* $n+1$.

Let $P = \{(i, a_i) : \ 1 \leq i \leq n^2 + 1\}$ and let say $(i, a_i) \preceq (j, a_j)$ if $i < j$ and $a_i \leq a_j$.

A chain in P corresponds to a monotone increasing subsequence. So, suppose that there are no monotone increasing sequences of length $n+1$. Then any cover of P by chains requires at least $n+1$ chains and so, by Dilworths theorem, there exists an anti-chain A of size $n+1$.

Let $A = \{(i_t, a_{i_t}) : \ 1 \leq t \leq n+1\}$ where $i_1 < i_2 \leq \cdots < i_{n+1}$.

Observe that $a_{i_t} > a_{i_{t+1}}$ for $1 \leq t \leq n$, for otherwise $(i_t, a_{i_t}) \preceq (i_{t+1}, a_{i_{t+1}})$ and A is not an anti-chain.

Thus A defines a monotone decreasing sequence of length $n+1$. □

FRIDAY, October 24, 2025.

Intervals Problem

$I_1, I_2, \ldots, I_{mn+1}$ are closed intervals on the real line i.e. $I_j = [a_j, b_j]$ where $a_j \leq b_j$ for $1 \leq j \leq mn + 1$.

Theorem

Either (i) there are $m + 1$ intervals that are pair-wise disjoint or (ii) there are $n + 1$ intervals with a non-empty intersection

Define a partial ordering $\preceq$ on the intervals by $I_r \preceq I_s$ iff $b_r \leq a_s$. Suppose that $I_{i_1}, I_{i_2}, \ldots, I_{i_t}$ is a collection of pair-wise disjoint intervals. Assume that $a_{i_1} < a_{i_2} \cdots < a_{i_t}$. Then $I_{i_1} < I_{i_2} \cdots < I_{i_t}$ form a chain and conversely a chain of length t comes from a set of t pair-wise disjoint intervals.
So if (i) does not hold, then the maximum length of a chain is m.

This means that the minimum number of chains needed to cover the poset is at least $\left\lceil \frac{mn+1}{m} \right\rceil = n+1$.

Dilworth's theorem implies that there must exist an anti-chain $\{I_{j_1}, I_{j_2}, \ldots, I_{j_{n+1}}\}$.

Let $a = \max\{a_{j_1}, a_{j_2}, \ldots, a_{j_{n+1}}\}$ and $b = \min\{b_{j_1}, b_{j_2}, \ldots, b_{j_{n+1}}\}$.

We must have $a \leq b$ else the two intervals giving a, b are disjoint.

But then every interval of the anti-chain contains $[a, b]$.

A *network* consists of a loopless digraph $D = (V, A)$ plus a function $c : A \to \boldsymbol{R}_+$. Here $c(x, y)$ for $(x, y) \in A$ is the *capacity* of the edge (x, y).

We use the following notation: if $\phi : A \to \boldsymbol{R}$ and S, T are (not necessarily disjoint) subsets of V then

$$\phi(S, T) = \sum_{\substack{x \in S \\ y \in T}} \phi(x, y).$$

Let s, t be distinct vertices. An $s - t$ flow is a function $f : A \to \boldsymbol{R}$ such that

$$f(v, V \setminus \{v\}) = f(V \setminus \{v\}, v) \quad \text{for all } v \neq s, t.$$

In words: flow into v equals flow out of v.

An $s-t$ flow is *feasible* if

$$0 \leq f(x,y) \leq c(x,y) \quad \text{for all } (x,y) \in A.$$

An $s-t$ *cut* is a partition of V into two sets $S, \bar{S}$ such that $s \in S$ and $t \in \bar{S}$.

The *value* v_f of the flow f is given by

$$v_f = f(s, V \setminus \{s\}) - f(V \setminus \{s\}, s).$$

Thus v_f is the net flow leaving s.

The *capacity* of the cut $S : \bar{S}$ is equal to $c(S, \bar{S})$.

Max-Flow Min-Cut Theorem

Theorem

$$\max v_f = \min c(S, \bar{S})$$

where the maximum is over feasible $s - t$ flows and the minimum is over $s - t$ cuts.

Proof We observe first that

$$\begin{aligned} f(S, \bar{S}) - f(\bar{S}, S) &= (f(S, V) - f(S, S)) - (f(V, S) - f(S, S)) \\ &= f(S, V) - f(V, S) \\ &= v_f + \sum_{v \in S \setminus \{s\}} (f(v, V) - f(V, v)) \\ &= v_f. \end{aligned}$$

So,

$$v_f \leq f(S, \bar{S}) \leq c(S, \bar{S}).$$

This implies that

$$\max v_f \leq \min c(S, \bar{S}). \tag{14}$$

Given a flow f we define a *flow augmenting path* P to be a sequence of distinct vertices $x_0 = s, x_1, x_2, \ldots, x_k = t$ such that for all i, either

- F1 $(x_i, x_{i+1}) \in A$ and $f(x_i, x_{i+1}) < c(x_i, x_{i+1})$, or
- F2 $(x_{i+1}, x_i) \in A$ and $f(x_{i+1}, x_i) > 0$.

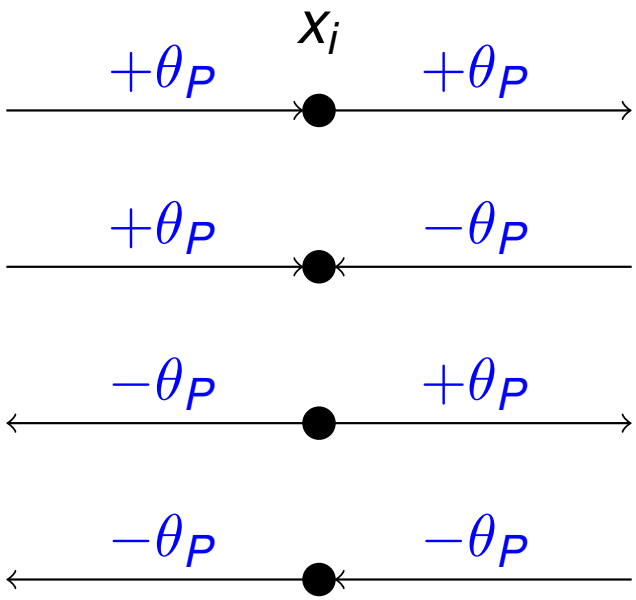
If P is such a sequence, then we define $\theta_P > 0$ to be the minimum over i of $c(x_i, x_{i+1}) - f(x_i, x_{i+1})$ (Case (F1)) and $f(x_{i+1}, x_i)$ (Case (F2)).

Claim 1: f is a maximum value flow, iff there are no flow augmenting paths.

Proof If P is flow augmenting then define a new flow f' as follows:

1. $f'(x_i, x_{i+1}) = f(x_i, x_{i+1}) + \theta_P$ or
2. $f'(x_{i+1}, x_i) = f(x_{i+1}, x_i) - \theta_P$
3. For all other edges, (x, y), we have $f'(x, y) = f(x, y)$.

We can see that the flow stays balanced at x_i.



We can see then that if there is a flow augmenting path then the new flow satisfies

$$v_{f'} = v_f + \theta_P > v_f.$$

Let S_f denote the set of vertices v for which there is a sequence $x_0 = s, x_1, x_2, \ldots, x_k = v$ which satisfies F1, F2 of the definition of flow augmenting paths.

If $t \in S_f$ then the associated sequence defines a flow augmenting path. So, assume that $t \notin S_f$. Then we have,

1. $s \in S_f$.
2. If $x \in S_f, y \in \bar{S}_f, (x, y) \in A$ then $f(x, y) = c(x, y)$, else we would have $y \in S_f$.
3. If $x \in S_f, y \in \bar{S}_f, (y, x) \in A$ then $f(y, x) = 0$, else we would have $y \in S_f$.

We therefore have

$$\begin{aligned} v_f &= f(S_f, \bar{S}_f) - f(\bar{S}_f, S) \\ &= c(S, \bar{S}_f). \end{aligned}$$

We see from this and (14) that f is a flow of maximum value and that the cut $S_f : \bar{S}_f$ is of minimum capacity.

This finishes the proof of Claim 1 and the Max-Flow Min-Cut theorem.

Note also that we can construct S_f by beginning with $S_f = \{s\}$ and then repeatedly adding any vertex $y \notin S_f$ for which there is $x \in S_f$ such that F1 or F2 holds. (A simple inductive argument based on sequence length shows that all of S_f is constructed in this way.)

MONDAY, October 27, 2025.

Hall's Theorem.

Let $G = (A, B, E)$ be a bipartite graph with $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$. A matching M is a set of edges that meets each vertex at most once. A matching is perfect if it meets each vertex.
Hall's theorem:

Theorem

G contains a perfect matching iff $|N(S)| \geq |S|$ for all $S \subseteq A$.

Here $N(S) = \{b \in B : \exists a \in A \text{ s.t. } \{a, b\} \in E\}$.

Define a digraph Γ by adding vertices $s, t \notin A \cup B$. Then add edges (s, a_i) and (b_i, t) of capacity 1 for $i = 1, 2, \ldots, n$. Orient the edges E for A to B and give them capacity ∞.

G has a matching of size m iff there is an $s-t$ flow of value m. An $s-t$ cut $X:\bar{X}$ has capacity

$$|A\setminus X|+|B\cap X|+|\{a\in X\cap A, b\in B\setminus X:\{a,b\}\in E\}|\times\infty.$$

It follows that to find a minimum cut, we need only consider X such that

$$\{a\in X\cap A, b\in B\setminus X:\{a,b\}\in E\}=\emptyset. \tag{15}$$

For such a set, we let $S=A\cap X$ and $T=X\cap B$. Condition (15) means that $T\supseteq N(S)$. The capacity of $X:\bar{X}$ is now $(n-|S|)+|T|$ and for a fixed S this is minimised for $T=N(S)$.

Thus, by the Max-Flow Min-Cut theorem

$$\max\{|M|\}=\min_X\{c(X:\bar{X})\}=\min_S\{n-|S|+|N(S)\}.$$

This implies Hall's theorem.

Graph orientation problem

Let $G = (V, E)$ be a graph. When is it possible to orient the edges of G to create a digraph $\Gamma = (V, A)$ so that every vertex has out-degree at least d. We say that G is d-orientable.

Theorem

G is d-orientable iff

$$|\{e \in E : e \cap S \neq \emptyset\}| \geq d|S| \text{ for all } S \subseteq V. \tag{16}$$

Proof If G is d-orientable then

$$|\{e \in E : e \cap S \neq \emptyset\}| \geq |\{(x, y) \in A : x \in S\}| \geq d|S|.$$

Suppose now that (16) holds. Define a network D as follows; the vertices are s, t, V, E – yes, D has a vertex for each edge of G.

There is an edge of capacity d from s to each $v \in V$ and an edge of capacity one from each $e \in E$ to t. There is an edge of infinite capacity from $v \in V$ to each edge e that contains v.

Consider an integer flow f. Suppose that $e = \{v, w\} \in E$ and $f(e, t) = 1$. Then either $f(v, e) = 1$ or $f(w, e) = 1$. In the former we interpret this as orienting the edge e from v to w and in the latter from w to v.

Under this interpretation, G is d-orientable iff D has a flow of value $d|V|$.

Let $X : \bar{X}$ be an $s - t$ cut in N. Let $S = X \cap V$ and $T = X \cap E$.

To have a finite capacity, there must be no $x \in S$ and $e \in E \setminus T$ such that $x \in e$.

So, the capacity of a finite capacity cut is at least

$$d(|V| - |S|) + |\{e \in E : e \cap S \neq \emptyset\}|$$

And this is at least $d|V|$ if (16) holds.

Game 1

Start with n chips. Players A,B alternately take 1,2,3 or 4 chips until there are none left. The winner is the person who takes the last chip:

Example

	A	B	A	B	A	
$n = 10$	3	2	4	1		B wins
$n = 11$	1	2	3	4	1	A wins

What is the optimal strategy for playing this game?

Game 2

Chip placed at point (m, n). Players can move chip to (m', n) or (m, n') where $0 \leq m' < m$ and $0 \leq n' < n$. The player who makes the last move and puts the chip onto $(0, 0)$ wins.

What is the optimal strategy for this game?

Game 2a Chip placed at point (m, n). Players can move chip to (m', n) or (m, n') or to $(m - a, n - a)$ where $0 \leq m' < m$ and $0 \leq n' < n$ and $0 \leq a \leq \min\{m, n\}$. The player who makes the last move and puts the chip onto $(0, 0)$ wins.

What is the optimal strategy for this game?

Game 3

W is a set of words. A and B alternately remove words $w_1, w_2, \ldots,$ from W. The rule is that the first letter of w_{i+1} must be the same as the last letter of w_i. The player who makes the last legal move wins.

Example

$W = \{England, France, Germany, Russia, Bulgaria, \ldots\}$

What is the optimal strategy for this game?

WEDNESDAY, October 29, 2025.

Abstraction

In Game 1, $P = \{5k : \ k \geq 0\}$.

In Game 2, $P = \{(x, x) : \ x \geq 0\}$.

Lemma

The partition into N, P satisfying $x \in N$ iff $N^+(x) \cap P \neq \emptyset$ is unique.

Proof If there were two partitions $N_i, P_i, \ i = 1, 2$, let x_i be the vertex of highest topological number which is not in $(N_1 \cap N_2) \cup (P_1 \cap P_2)$. Suppose that $x_i \in N_1 \setminus N_2$.

But then $x_i \in N_1$ implies $N^+(x_i) \cap P_1 \cap \{x_{i+1}, \ldots, x_n\} \neq \emptyset$ and $x_i \in P_2$ implies $N^+(x_i) \cap P_2 \cap \{x_{i+1}, \ldots, x_n\} = \emptyset$.

But $P_1 \cap \{x_{i+1}, \ldots, x_n\} = P_2 \cap \{x_{i+1}, \ldots, x_n\}$. □

Sums of games

Suppose that we have p games $G_1, G_2, \ldots, G_p$ with digraphs $D_i = (X_i, A_i)$, $i = 1, 2, \ldots, p$.
The sum $G_1 \oplus G_2 \oplus \cdots \oplus G_p$ of these games is played as follows. A position is a vector $(x_1, x_2, \ldots, x_p) \in X = X_1 \times X_2 \times \cdots \times X_p$. To make a move, a player chooses i such that x_i is not a sink of D_i and then replaces x_i by $y \in N_i^+(x_i)$. The game ends when each x_i is a sink of D_i for $i = 1, 2, \ldots, n$.

Knowing the partitions N_i, P_i for game $i = 1, 2, \ldots, p$ does not seem to be enough to determine how to play the sum of the games.

We need more information. This will be provided by the Sprague-Grundy Numbering

Example

Nim In a one pile game, we start with $a \geq 0$ chips and while there is a positive number x of chips, a move consists of deleting $y \leq x$ chips. In this game the N-positions are the positive integers and the unique P-position is 0.

In general, Nim consists of the sum of n single pile games starting with $a_1, a_2, \ldots, a_n > 0$. A move consists of deleting some chips from a non-empty pile.

Example 2 is Nim with 2 piles.

Sprague-Grundy (SG) Numbering

For $S \subseteq \{0, 1, 2, \ldots, \}$ let

$$mex(S) = \min\{x \geq 0 : \ x \notin S\}.$$

Now given an acyclic digraph $D = X, A$ with topological ordering $x_1, x_2, \ldots, x_n$ define g iteratively by

1. $i \leftarrow n$, $g(x_n) = 0$.
2. $i \leftarrow i - 1$. If $i = 0$ STOP.
3. $g(x_i) = mex(\{g(x) : \ x \in N^+(x_i)\})$.
4. goto 2.

Lemma

$$x \in P \leftrightarrow g(x) = 0.$$

Proof Because

$$x \in N \text{ iff } N^+(x) \cap P \neq \emptyset$$

all we have to show is that

$$g(x) > 0 \text{ iff } \exists y \in N^+(y) \text{ such that } g(y) = 0.$$

But this is immediate from $g(x) = mex(\{g(y) : \ y \in N^+(x)\})$ $\square$

Sums of games

Another one pile subtraction game.

- A player can remove any even number of chips, but not the whole pile.
- A player can remove the whole pile if it is odd.

The terminal positions are 0 or 2.

Lemma

$g(0) = 0$, $g(2k) = k - 1$ *and* $g(2k-1) = k$ *for* $k \geq 1$.

Proof 0,2 are terminal postions and so $g(0) = g(2) = 0$. $g(1) = 1$ because the only position one can move to from 1 is 0. We prove the remainder by induction on k.

Assume that $k > 1$.

$$\begin{aligned} g(2k) &= mex\{g(2k-2), g(2k-4), \ldots, g(2)\} \\ &= mex\{k-2, k-3, \ldots, 0\} \\ &= k-1. \\ g(2k-1) &= mex\{g(2k-3), g(2k-5), \ldots, g(1), g(0)\} \\ &= mex\{k-1, k-2, \ldots, 0\} \\ &= k. \end{aligned}$$

□

Sums of games

We now show how to compute the SG numbering for a sum of games.

For binary integers $a = a_m a_{m-1} \cdots a_1 a_0$ and $b = b_m b_{m-1} \cdots b_1 b_0$ we define $a \oplus b = c_m c_{m-1} \cdots c_1 c_0$ by

$$c_i = \begin{cases} 1 & \text{if } a_i \neq b_i \\ 0 & \text{if } a_i = b_i \end{cases}$$

for $i = 1, 2, \ldots, m$.

So $11 \oplus 5 = 14$.

FRIDAY, October 31, 2025.

Sums of games

Theorem

If g_i is the SG function for game G_i, $i = 1, 2, \ldots, p$ then the SG function g for the sum of the games $G = G_1 \oplus G_2 \oplus \cdots \oplus G_p$ is defined by

$$g(x) = g_1(x_1) \oplus g_2(x_2) \oplus \cdots \oplus g_p(x_p)$$

where $x = (x_1, x_2, \ldots, x_p)$.

For example if in a game of Nim, the pile sizes are $x_1, x_2, \ldots, x_p$ then the *SG* value of the position is

$$x_1 \oplus x_2 \oplus \cdots \oplus x_p$$

Proof It is enough to show this for $p = 2$ and then use induction on p.

Write $G = H \oplus G_p$ where $H = G_1 \oplus G_2 \oplus \cdots \oplus G_{p-1}$. Let h be the SG numbering for H. Then, if $y = (x_1, x_2, \ldots, x_{p-1})$,

$$\begin{aligned} g(x) &= h(y) \oplus g_p(x_p) \qquad \textit{assuming theorem for } p = 2 \\ &= g_1(x_1) \oplus g_2(x_2) \oplus \cdots \oplus g_{p-1}(x_{p-1}) \oplus g_p(x_p) \end{aligned}$$

by induction.
It is enough now to show, for $p = 2$, that

A1 If $x \in X$ and $g(x) = b > a$ then there exists $x' \in N^+(x)$ such that $g(x') = a$.

A2 If $x \in X$ and $g(x) = b$ and $x' \in N^+(x)$ then $g(x') \neq g(x)$.

A1. Write $d = a \oplus b$. Then

$$a = d \oplus b = d \oplus g_1(x_1) \oplus g_2(x_2). \tag{17}$$

Now suppose that we can show that either

$$(i)\ d \oplus g_1(x_1) < g_1(x_1)\ \textit{or}\ (ii)\ d \oplus g_2(x_2) < g_2(x_2)\ \textit{or both}. \tag{18}$$

Assume that (i) holds.

Then since $g_1(x_1) = mex(N_1^+(x_1))$ there must exist $x_1' \in N_1^+(x_1)$ such that $g_1(x_1') = d \oplus g_1(x_1)$.

Then from (20) we have

$$a = g_1(x_1') \oplus g_2(x_2) = g(x_1', x_2).$$

Furthermore, $(x_1', x_2) \in N^+(x)$ and so we will have verified A1.

Sums of games

Let us verify (18).

Suppose that $2^{k-1} \leq d < 2^k$.

Then d has a 1 in position k and no higher.

Since $d_k = a_k \oplus b_k$ and $a < b$ we must have $a_k = 0$ and $b_k = 1$.

So either (i) $g_1(x_1)$ has a 1 in position k or (ii) $g_2(x_2)$ has a 1 in position k. Assume (i).

But then $d \oplus g_1(x_1) < g_1(x_1)$ since d "destroys" the kth bit of $g_1(x_1)$ and does not change any higher bit.

A2. Suppose without loss of generality that $g(x_1', x_2) = g(x_1, x_2)$ where $x_1' \in N^+(x_1)$.

Then $g_1(x_1') \oplus g_2(x_2) = g_1(x_1) \oplus g_2(x_2)$ implies that $g_1(x_1') = g_1(x_1)$, contradition. □

If we apply this theorem to the game of Nim then if the position x consists of piles of x_i chips for $i = 1, 2, \ldots, p$ then $g(x) = x_1 \oplus x_2 \oplus \cdots \oplus x_p$.

In our first example, $g(x) = x \mod 5$ and so for the sum of p such games we have

$$g(x_1, x_2, \ldots, x_p) = (x_1 \mod 5) \oplus (x_2 \mod 5) \oplus \cdots \oplus (x_p \mod 5).$$

A more complicated one pile game

Start with n chips. First player can remove up to $n-1$ chips.

In general, if the previous player took x chips, then the next player can take $y \leq x$ chips.

Thus a games position can be represented by (n, x) where n is the current size of the pile and x is the maximum number of chips that can be removed in this round.

Theorem

Suppose that the position is (n, x) where $n = m2^k$ and m is odd. Then,

(a) This is an N-position if $x \geq 2^k$.

(b) This is a P-position if $m = 1$ and $x < n$.

A more complicated one pile game

Proof For a non-negative integer $n = m2^k$, let $ones(n)$ denote the number of ones in the binary expansion of n and let $k = \rho(n)$ determine the position of the right-most one in this expansion.

We claim that the following strategy is a win for the player in a postion described in (a):

Remove $y = 2^k$ chips.

Suppose this player is A.

If $m = 1$ then $x \geq n$ and A wins.

A more complicated one pile game

Otherwise, after such a move the position is (n', y) where $\rho(n') > \rho(n)$.

Note first that $ones(n') = ones(n) - 1 > 0$ and $\rho(n') > k$. B cannot remove more than 2^k chips and so B cannot win at this point.

If B moves the position to (n'', x'') then $ones(n'') > ones(n')$ and furthermore, $x'' \geq 2^{\rho(n'')}$, since x'' must have a 1 in position $\rho(n'')$. ($\rho(n'')$ is the least significant bit of x''.)

Thus, by induction, A is in an N-position and wins the game.

To prove (b), note that after the first move, the position satisfies the conditions of (a). $\square$.

MONDAY, November 3, 2025.

Wythoff's Nim

This is Game 2a.

Theorem

The set of P-positions is $\mathcal{A} = ((a_i, b_i),\ i = 0, 1, 2, \ldots)$ *where* $a_i < b_i$, $i \neq 0$ *can be generated as follows:* $a_0 = b_0 = 0$ *and*

- a_i *is the smallest integer not appearing in* $a_0, b_0, \ldots, a_{i-1}, b_{i-1}$
- $b_i = a_i + i$.

The sequence $\mathcal{A}$ starts

0	0	1	2	3	5
4	7	6	10	8	13
9	15	11	18	12	20
14	23	16	26	17	28
19	31	21	34	22	36
24	39	25	41	27	44

Proof We first prove that each positive integer appears exactly once either as a_i or b_i.

We cannot have $a_i = a_j$ for $i < j$ because a_j is the smallest integer that has not previously appeared. Similarly, we cannot have $a_i < a_{i-1}$, else a_{i-1} was too large.

Since $b_i = a_i + i$ we see that both of the sequences $a_0, a_1, \ldots,$ and $b_0, b_1, \ldots,$ are monotone increasing.

Suppose then that $x = a_i = b_j$. Since $a_i < b_i < b_j$ for $i < j$, we must have $i > j$ here. But then a_i is not an integer that has not appeared before.

Thus each positive integer appears exactly once either as a_i or b_i.

Wythoff's Nim

Now suppose that $(a_i, b_i) \in \mathcal{A}$. We consider the possible positions we can move to and check that we cannot move to $\mathcal{A}$:

1. $(a_i - x, b_i) = (a_j, b_j)$ where $x > 0$.
   We must have $j < i$ and $b_j = b_i$. Not possible.

2. $(a_i, b_i - x) = (a_j, b_j)$ where $x > 0$.
   We must have $j < i$ and $a_j = a_i$. Not possible.

3. $(a_i - x, b_i - x) = (a_j, b_j)$ where $x > 0$.
   We must have $j < i$ and $i = b_i - a_i = b_j - a_j = j$. Not possible.

Wythoff's Nim

Now suppose that $(c, d) \notin \mathcal{A}$. We see that we can move to a pair in $\mathcal{A}$.

1. $c = a_i$ and $d > b_i$.
   We can move to (a_i, b_i) by removing $d - b_i$ from the b pile.
2. $c = a_i$ and $d < b_i$.
   Let $j = d - c$. We can move to (a_j, b_j) by deleting $c - a_j = d - b_j$ from each pile.
3. $d = b_i$ and $c > a_i$.
   We can move to (a_i, b_i) by removing $c - a_i$ from the c pile.
4. $d = b_i$ and $c < a_i$ and we are not in Case 1 (with i replaced by i').
   Thus, $c = b_j$ for some $j < i$. We can move to (a_j, b_j) by removing $d - a_j$ from the d pile.

We have therefore verified that the sequence $\mathcal{A}$ does indeed define the set of P positions.

We can give the following description of the sequence $\mathcal{A}$.

Theorem

$$a_k = \left\lfloor \frac{k}{2}(1+\sqrt{5}) \right\rfloor \text{ and } b_k = \left\lfloor \frac{k}{2}(3+\sqrt{5}) \right\rfloor$$

for $k = 0, 1, 2, \ldots$.

Proof It will be enough to show that each non-negative integer appears exactly once in the sequence $(x_k, y_k) = \left(\left\lfloor \frac{k}{2}(1+\sqrt{5}) \right\rfloor, \left\lfloor \frac{k}{2}(3+\sqrt{5}) \right\rfloor\right)$ $(*)$.

Given (*) we assume inductively that $(a_i, b_i) = (x_i, y_i)$ for $0 \leq i \leq k$. This is true for $k = 0$.

Using (*) we see that a_{k+1} appears in some pair x_j, y_j. We must have $j > k$ else a_{k+1} will appear in $a_0, \ldots, b_k$.

Now x_{k+1} is the smallest integer that that does not appear in $(x_0, \ldots, y_k) = (a_0, \ldots, b_k)$ and so $x_{k+1} = a_{k+1}$ and then $y_{k+1} = x_{k+1} + k = b_{k+1}$, completing the induction.

Beatty Sequences: let $r > 1$ be irrational and let $s = r/(r-1)$. Let $B_r = \{\lfloor nr \rfloor : n \geq 1\}$ and $B_s = \{\lfloor ns \rfloor : n \geq 1\}$.

Theorem

B_r, B_s partitions $1, 2, \ldots,$.

Proof

We order the set $\{n/r : n \geq 1\} \cup \{n/s : n \geq 1\}$ as a list L in increasing order.

Suppose first that $j/r = k/s$ for positive integers j, k. Then, $r/s = j/k$, which is rational and $r/s = r - 1$ which is irrational, contradiction. So, $j/r \neq k/s$ for all $j, k \geq 1$.

Wythoff's Nim

For any j/r there are j positive integers i such that $i/r \leq j/r$ and $\lfloor js/r \rfloor$ positive integers k such that $k/s \leq j/r$, so that the position of j/r in L is $j + \lfloor js/r \rfloor$. The equation $1/r + 1/s = 1$ implies that

$$j + \lfloor js/r \rfloor = j + \lfloor j(s-1) \rfloor = \lfloor js \rfloor.$$

Likewise, the position of k/s in L is $\lfloor kr \rfloor$.

This implies (*).

WEDNESDAY, November 5, 2025.

Geography

Start with a chip sitting on a vertex v of a graph or digraph G. A move consists of moving the chip to a neighbouring vertex.

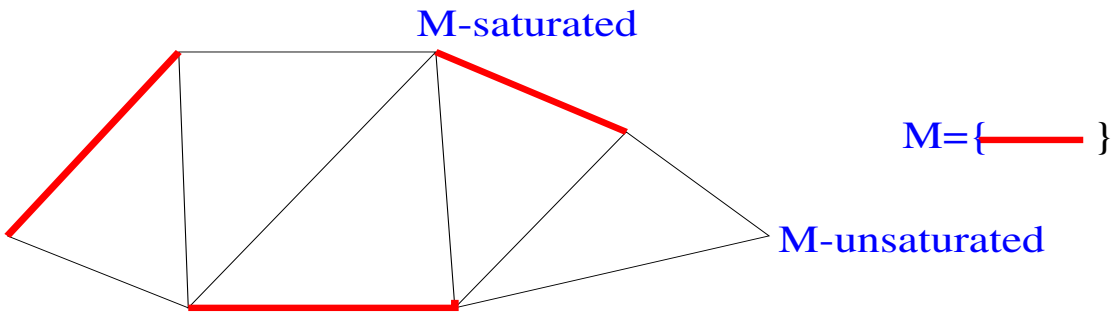
In edge geography, moving the chip from x to y deletes the edge (x, y). In vertex geography, moving the chip from x to y deletes the vertex x.

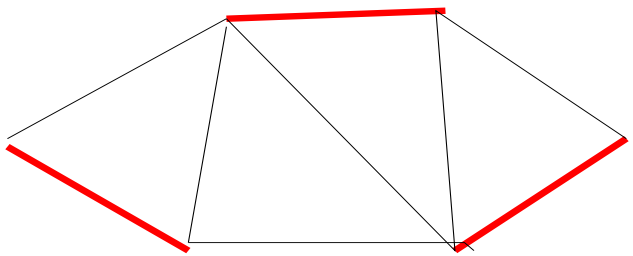
The problem is given a position (G, v), to determine whether this is a P or N position.

Complexity Both edge and vertex geography are Pspace-hard on digraphs. Edge geography is Pspace-hard on an undirected graph. Only vertex geography on a graph is polynomial time solvable.

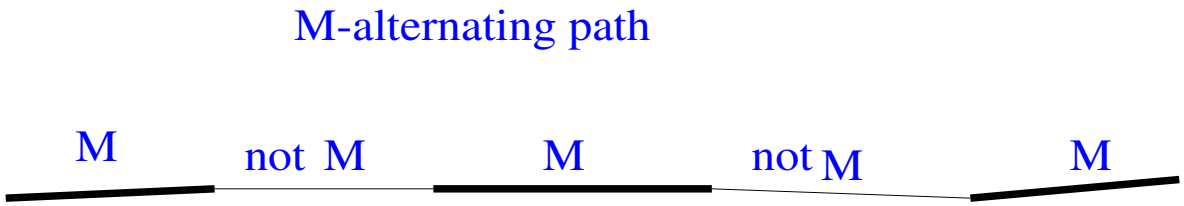
Undirected Vertex Geography

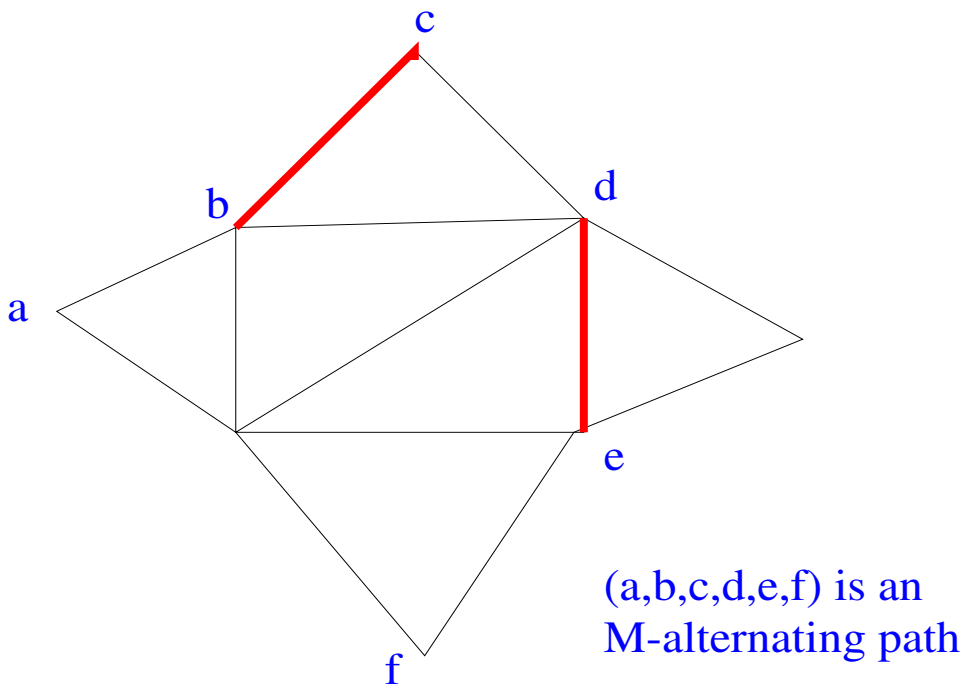
We need some simple results from the theory of matchings on graphs.
A *matching* M of a graph $G = (V, E)$ is a set of edges, no two of which are incident to a common vertex.





Undirected Vertex Geography





An M-alternating path joining 2 M-unsaturated vertices is called an M-augmenting path.

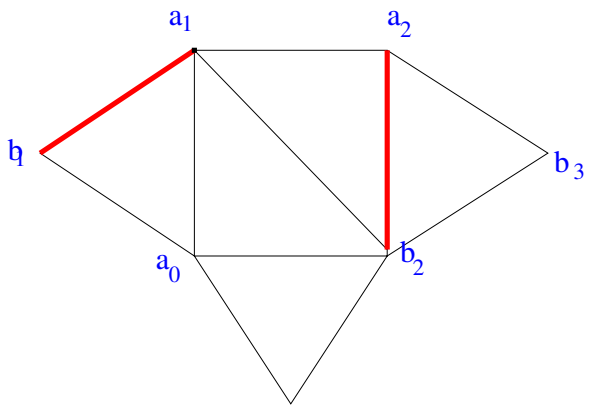
Undirected Vertex Geography

M is a *maximum* matching of G if no matching M' has more edges.

Theorem

M is a maximum matching iff M admits no M-augmenting paths.

Proof Suppose M has an augmenting path $P = (a_0, b_1, a_1, \ldots, a_k, b_{k+1})$ where $e_i = (a_{i-1}, b_i) \notin M$, $1 \leq i \leq k+1$ and $f_i = (b_i, a_i) \in M$, $1 \leq i \leq k$.



Let $M' = M - \{f_1, f_2, \ldots, f_k\} + \{e_1, e_2, \ldots, e_{k+1}\}$.

Undirected Vertex Geography

- $|M'| = |M| + 1$.
- M' is a matching

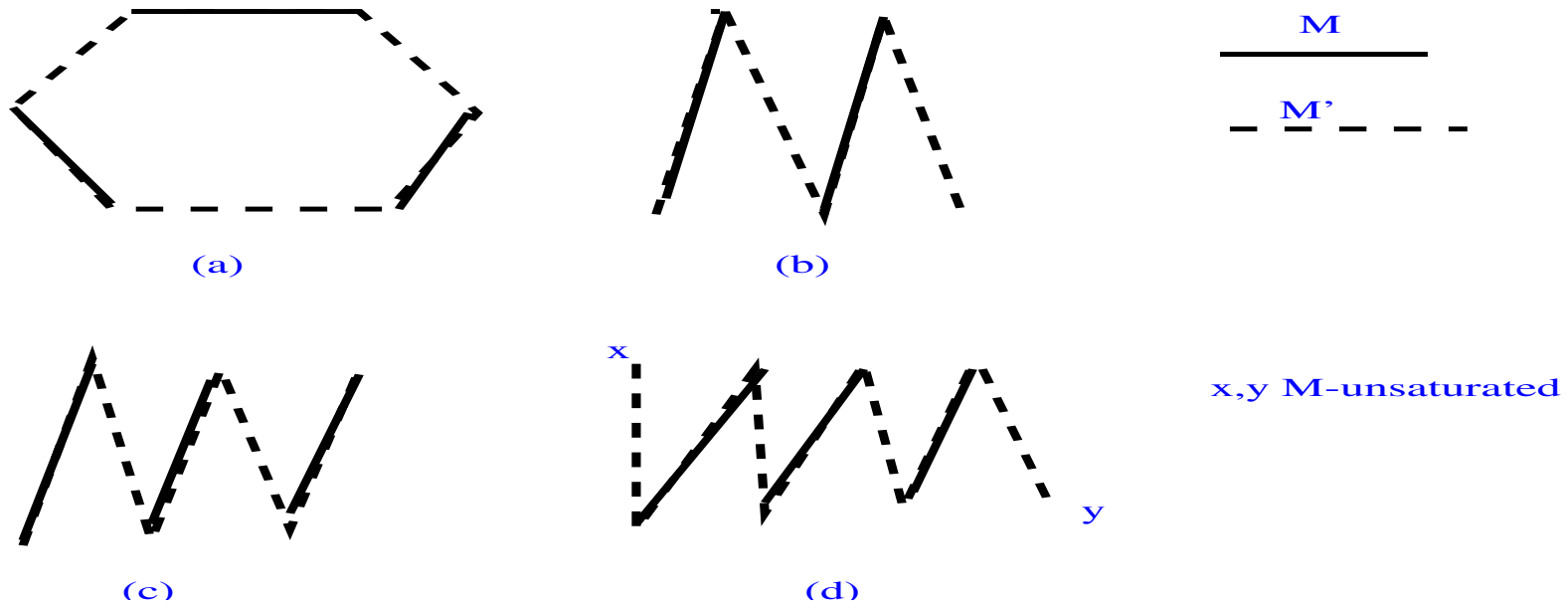
For $x \in V$ let $d_M(x)$ denote the degree of x in matching M, So $d_M(x)$ is 0 or 1.

$$d_{M'}(x) = \begin{cases} d_M(x) & x \notin \{a_0, b_1, \ldots, b_{k+1}\} \\ d_M(x) & x \in \{b_1, \ldots, a_k\} \\ d_M(x) + 1 & x \in \{a_0, b_{k+1}\} \end{cases}$$

So if M has an augmenting path it is not maximum.

Undirected Vertex Geography

Suppose M is not a maximum matching and $|M'| > |M|$.
Consider $H = G[M \nabla M']$ where $M \nabla M' = (M \setminus M') \cup (M' \setminus M)$ is the set of edges in *exactly* one of M, M'.
Maximum degree of H is 2 – ≤ 1 edge from M or M'. So H is a collection of vertex disjoint alternating paths and cycles.



$|M'| > |M|$ implies that there is at least one path of type (d).
Such a path is M-augmenting

□

Undirected Vertex Geography

Theorem

(G, v) is an N-position in UVG iff every maximum matching of G covers v.

Proof (i) Suppose that M is a maximum matching of G which covers v. Player 1's strategy is now: Move along the M-edge that contains the current vertex.

If Player 1 were to lose, then there would exist a sequence of edges $e_1, f_1, \ldots, e_k, f_k$ such that $v \in e_1$, $e_1, e_2, \ldots, e_k \in M$, $f_1, f_2, \ldots, f_k \notin M$ and $f_k = (x, y)$ where y is the current vertex for Player 1 and y is not covered by M.

But then if $A = \{e_1, e_2, \ldots, e_k\}$ and $B = \{f_1, f_2, \ldots, f_k\}$ then $(M \setminus A) \cup B$ is a maximum matching (same size as M) which does not cover v, contradiction.

Undirected Vertex Geography

(ii) Suppose now that there is some maximum matching M which does not cover v. If (v, w) is Player 1's move,then w

must be covered by M, else M is not a maximum matching.

Player 2's strategy is now: Move along the M-edge that contains the current vertex. If Player 2 were to lose then there exists $e_1 = (v, w), f_1, \ldots, e_k, f_k, e_{k+1} = (x, y)$ where y is the current vertex for Player 2 and y is not covered by M.

But then we have defined an augmenting path from v to y and so M is not a maximum matching, contradiction. □

Undirected Vertex Geography

Note that we can determine whether or not v is covered by all maximum matchings as follows: Find the size σ of the maximum matching G.

This can be done in $O(n^3)$ time on an n-vertex graph. Find the size σ' of a maximum matching in $G - v$. Then v is covered by all maximum matchings of G iff $\sigma \neq \sigma'$.

Tic Tac Toe

We consider the following multi-dimensional version of Tic Tac Toe (Noughts and Crosses to the English).

The *board* consists of $[n]^d$. A point on the board is therefore a vector $(x_1, x_2, \ldots, x_d)$ where $1 \leq x_i \leq n$ for $1 \leq i \leq d$.

A *line* is a set points $(x_j^{(1)}, x_j^{(2)}, \ldots, x_j^{(d)})$, $j = 1, 2, \ldots, n$ where each sequence $x^{(i)}$ is either (i) of the form $k, k, \ldots, k$ for some $k \in [n]$ or is (ii) $1, 2, \ldots, n$ or is (iii) $n, n-1, \ldots, 1$. Finally, we cannot have Case (i) for all i.

Thus in the (familiar) 3×3 case, the top row is defined by $x^{(1)} = 1, 1, 1$ and $x^{(2)} = 1, 2, 3$ and the diagonal from the bottom left to the top right is defined by $x^{(1)} = 3, 2, 1$ and $x^{(2)} = 1, 2, 3$

Tic Tac Toe

The game is played by 2 players. The Red player (X player) goes first and colours a point red. Then the Blue player (0 player) colours a different point blue and so on.

A player wins if there is a line, all of whose points are that players colour. If neither player wins then the game is a draw. The second player does not have a wnning strategy:

Lemma

Player 1 can always get at least a draw.

Proof We prove this by considering *strategy stealing*.

Suppose that Player 2 did have a winning strategy. Then Player 1 can make an arbitrary first move x_1. Player 2 will then move with y_1. Player 1 will now win playing the winning strategy for Player 2 against a first move of y_1.

This can be carried out until the strategy calls for move x_1 (if at all). But then Player 1 can make an arbitrary move and continue, since x_1 has already been made. □

The Hales-Jewett Theorem of Ramsey Theory implies that there is a winner in the (n, d) game, when n is large enough with respect to d. The winner is of course Player 1.

Tic Tac Toe

$$\begin{bmatrix} 11 & 1 & 8 & 1 & 12 \\ 6 & 2 & 2 & 9 & 10 \\ 3 & 7 & * & 9 & 3 \\ 6 & 7 & 4 & 4 & 10 \\ 12 & 5 & 8 & 5 & 11 \end{bmatrix}$$

The above array gives a strategy for Player 2 in the 5×5 game ($d = 2, n = 5$).

For each of the 12 lines there is an associated pair of positions. If Player 1 chooses a position with a number i, then Player 2 responds by choosing the other cell with the number i.

This ensures that Player 1 cannot take line i. If Player 1 chooses the * then Player 2 can choose any cell with an unused number.

Tic Tac Toe

So, later in the game if Player 1 chooses a cell with j and Player 2 already has the other j, then Player 2 can choose an arbitrary cell.

Player 2's strategy is to ensure that after all cells have been chosen, he/she will have chosen one of the numbered cells asociated with each line. This prevents Player 1 from taking a whole line. This is called a *pairing* strategy.

Tic Tac Toe

We now generalise the game to the following: We have a family $\mathcal{F} = A_1, A_2, \ldots, A_N \subseteq A$. A move consists of one player, taking an uncoloured member of A and giving it his colour.

A player wins if one of the sets A_i is completely coloured with his colour.

A pairing strategy is a collection of distinct elements $X = \{x_1, x_2, \ldots, x_{2N-1}, x_{2N}\}$ such that $x_{2i-1}, x_{2i} \in A_i$ for $i \geq 1$.

This is called a *draw forcing pairing*. Player 2 responds to Player 1's choice of $x_{2i+\delta}, \delta = 0, 1$ by choosing $x_{2i+3-\delta}$. If Player 1 does not choose from X, then Player 2 can choose any uncoloured element of X.

Tic Tac Toe

In this way, Player 2 avoids defeat, because at the end of the game Player 2 will have coloured at least one of each of the pairs x_{2i-1}, x_{2i} and so Player 1 cannot have completely coloured A_i for $i = 1, 2, \ldots, N$.

FRIDAY, NOVEMBER 7, 2025.

Theorem

If

$$\left|\bigcup_{X\in\mathcal{G}} X\right| \geq 2|\mathcal{G}| \qquad \forall \mathcal{G} \subseteq \mathcal{F} \tag{19}$$

then there is a draw forcing pairing.

Proof We define a bipartite graph Γ. A will be one side of the bipartition and $B = \{b_1, b_2, \ldots, b_{2N}\}$. Here b_{2i-1} and b_{2i} both represent A_i in the sense that if $a \in A_i$ then there is an edge (a, b_{2i-1}) and an edge (a, b_{2i}).

A draw forcing pairing corresponds to a complete matching of B into A and the condition (19) implies that Hall's condition is satisfied. □

Corollary

If $|A_i| \geq n$ for $i = 1, 2, \ldots, n$ and every $x \in A$ is contained in at most $n/2$ sets of $\mathcal{F}$ then there is a draw forcing pairing.

Proof The degree of $a \in A$ is at most $2(n/2)$ in Γ and the degree of each $b \in B$ is at least n. This implies (via Hall's condition) that there is a complete matching of B into A. □

Tic Tac Toe

Consider Tic tac Toe when $d = 2$. If n is even then every array element is in at most 3 lines (one row, one column and at most one diagonal) and if n is odd then every array element is in at most 4 lines (one row, one column and at most two diagonals).

Thus there is a draw forcing pairing if $n \geq 6$, n even and if $n \geq 9$, n odd. (The cases $n = 4, 7$ have been settled as draws. $n = 7$ required the use of a computer to examine all possible strategies.)

Tic Tac Toe

In general we have

Lemma

If $n \geq 3^d - 1$ and n is odd or if $n \geq 2^d - 1$ and n is even, then there is a draw forcing pairing of (n, d) Tic tac Toe.

Proof We only have to estimate the number of lines through a fixed point $\mathbf{c} = (c_1, c_2, \ldots, c_d)$.

If n is odd then to choose a line L through $\mathbf{c}$ we specify, for each index i whether L is (i) constant on i, (ii) increasing on i or (iii) decreasing on i.

This gives 3^d choices. Subtract 1 to avoid the all constant case and divide by 2 because each line gets counted twice this way.

Tic Tac Toe

When n is even, we observe that once we have chosen in which positions L is constant, L is determined.

Suppose $c_1 = x$ and 1 is not a fixed position. Then every other non-fixed position is x or $n - x + 1$. Assuming w.l.o.g. that $x \leq n/2$ we see that $x < n - x + 1$ and the positions with x increase together at the same time as the positions with $n - x + 1$ decrease together.

Thus the number of lines through $\mathbf{c}$ in this case is bounded by $\sum_{i=0}^{d-1} \binom{d}{i} = 2^d - 1$. □

Quasi-probabilistic method

We now prove a theorem of Erdős and Selfridge.

Theorem

If $|A_i| \geq n$ for $i \in [N]$ and $N < 2^{n-1}$, then Player 2 can get a draw in the game defined by $\mathcal{F}$.

Proof At any point in the game, let C_j denote the set of elements in A which have been coloured with Player j's colour, $j = 1, 2$ and $U = A \setminus C_1 \cup C_2$. Let

$$\Phi = \sum_{i: A_i \cap C_2 = \emptyset} 2^{-|A_i \cap U|}.$$

Suppose that the players choices are $x_1, y_1, x_2, y_2, \ldots,$. Then we observe that immediately after Player 1's first move, $\Phi < N2^{-(n-1)} < 1$.

Quasi-probabilistic method

We will show that Player 2 can keep $\Phi < 1$ through out. Then at the end, when $U = \emptyset$, $\Phi = \sum_{i:A_i \cap C_2 = \emptyset} 1 < 1$ implies that $A_i \cap C_2 \neq \emptyset$ for all $i \in [N]$.

So, now let Φ_j be the value of Φ after the choice of $x_1, y_1, \ldots, x_j$. then if U, C_1, C_2 are defined at precisely this time,

$$\begin{aligned}
\Phi_{j+1} - \Phi_j &= -\sum_{\substack{i:A_i \cap C_2 = \emptyset \\ y_j \in A_i}} 2^{-|A_i \cap U|} + \sum_{\substack{i:A_i \cap C_2 = \emptyset \\ y_j \notin A_i, x_{j+1} \in A_i}} 2^{-|A_i \cap U|} \\
&\leq -\sum_{\substack{i:A_i \cap C_2 = \emptyset \\ y_j \in A_i}} 2^{-|A_i \cap U|} + \sum_{\substack{i:A_i \cap C_2 = \emptyset \\ x_{j+1} \in A_i}} 2^{-|A_i \cap U|}
\end{aligned}$$

Quasi-probabilistic method

We deduce that $\Phi_{j+1} - \Phi_j \leq 0$ if Player 2 chooses y_j to maximise $\sum_{\substack{i: A_i \cap C_2 = \emptyset \\ y \in A_i}} 2^{-|A_i \cap U|}$ over y.

In this way, Player 2 keeps $\Phi < 1$ and obtains a draw. □

In the case of (n, d) Tic Tac Toe, we see that Player 2 can force a draw if

$$\frac{(n+2)^d - n^d}{2} < 2^{n-1}$$

which is implied, for n large, by

$$n \geq (1+\epsilon) d \log_2 d$$

where $\epsilon > 0$ is a small positive constant.

WEDNESDAY, NOVEMBER 12, 2025.

Polya's Theory of Counting

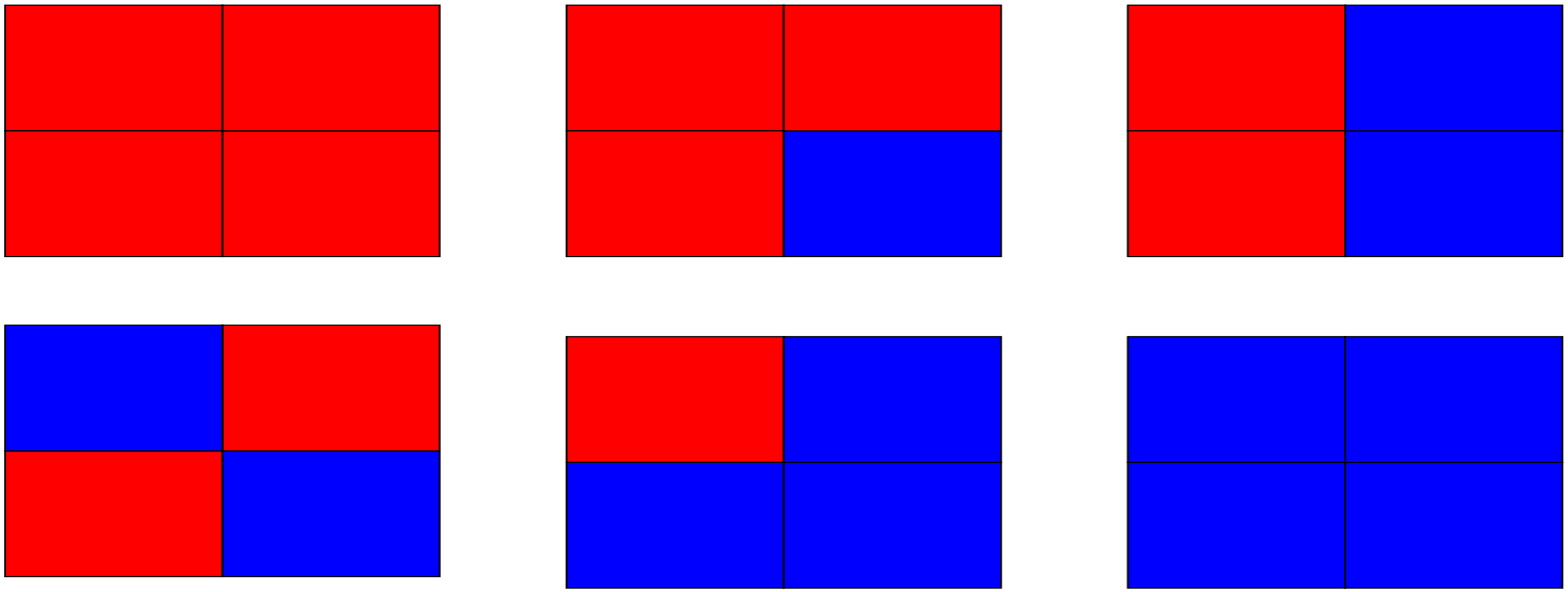
Example 1 A disc lies in a plane. Its centre is fixed but it is free to rotate. It has been divided into n sectors of angle $2\pi/n$. Each sector is to be colored Red or Blue. How many different colorings are there?

One could argue for 2^n.

On the other hand, what if we only distinguish colorings which cannot be obtained from one another by a rotation. For example if $n = 4$ and the sectors are numbered 0,1,2,3 in clockwise order around the disc, then there are only 6 ways of coloring the disc – 4R, 4B, 3R1B, 1R3B, RRBB and RBRB.

Example 2

Now consider an $n \times n$ "chessboard" where $n \geq 2$. Here we color the squares Red and Blue and two colorings are different only if one cannot be obtained from another by a rotation or a reflection. For $n = 2$ there are 6 colorings.



The general scenario that we consider is as follows: We have a set X which will stand for the set of colorings when transformations are not allowed. (In example 1, $|X| = 2^n$ and in example 2, $|X| = 2^{n^2}$).

In addition there is a set G of permutations of X. This set will have a group structure:

Given two members $g_1, g_2 \in G$ we can define their composition $g_1 \circ g_2$ by $g_1 \circ g_2(x) = g_1(g_2(x))$ for $x \in X$. We require that G is *closed* under composition i.e. $g_1 \circ g_2 \in G$ if $g_1, g_2 \in G$.

We also have the following:

- A1 The *identity* permutation $1_X \in G$.
- A2 $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$ (Composition is associative).
- A3 The inverse permutation $g^{-1} \in G$ for every $g \in G$.

(A set G with a binary relation $\circ$ which satisfies **A1,A2,A3** is called a **Group**).

In example 1 $D = \{0, 1, 2, \ldots, n-1\}$, $X = 2^D$ and the group is $G_1 = \{e_0, e_1, \ldots, e_{n-1}\}$ where $e_j * x = x + j \ mod \ n$ stands for rotation by $2j\pi/n$.

In example 2, $X = 2^{[n]^2}$. We number the squares 1,2,3,4 in clockwise order starting at the upper left and represent X as a sequence from $\{r, b\}^4$ where for example rrbr means color 1,2,4 Red and 3 Blue. $G_2 = \{e, a, b, c, p, q, r, s\}$ is in a sense independent of n. e, a, b, c represent a rotation through $0, 90, 180, 270$ degrees respectively. p, q represent reflections in the vertical and horizontal and r, s represent reflections in the diagonals 1,3 and 2,4 respectively.

	e	a	b	c	p	q	r	s
rrrr	rrrr	rrrr	rrrr	rrrr	rrrr	rrrr	rrrr	rrrr
brrr	brrr	rbrr	rrbr	rrrb	rbrr	rrrb	brrr	rrbr
rbrr	rbrr	rrbr	rrrb	brrr	brrr	rrbr	rrrb	rbrr
rrbr	rrbr	rrrb	brrr	rbrr	rrrb	rbrr	rrbr	brrr
rrrb	rrrb	brrr	rbrr	rrbr	rrbr	brrr	rbrr	rrrb
bbrr	bbrr	rbbr	rrbb	brrb	bbrr	rrbb	brrb	rbbr
rbbr	rbbr	rrbb	brrb	bbrr	brrb	rbbr	rrbb	bbrr
rrbb	rrbb	brrb	bbrr	rbbr	rrbb	bbrr	rbbr	brrb
brrb	brrb	bbrr	rbbr	rrbb	rbbr	brrb	bbrr	rrbb
rbrb	rbrb	brbr	rbrb	brbr	brbr	brbr	rbrb	rbrb
brbr	brbr	rbrb	brbr	rbrb	rbrb	rbrb	brbr	brbr
bbbr	bbbr	rbbb	brbb	bbrb	bbrb	rbbb	brbb	bbbr
bbrb	bbrb	bbbr	rbbb	brbb	bbbr	brbb	bbrb	rbbb
brbb	brbb	bbrb	bbbr	rbbb	brbb	bbrb	bbbr	brbb
rbbb	rbbb	brbb	bbrb	bbbr	brbb	bbbr	rbbb	bbrb
bbbb	bbbb	bbbb	bbbb	bbbb	bbbb	bbbb	bbbb	bbbb

From now on we will write $g * x$ in place of $g(x)$.
Orbits: If $x \in X$ then its orbit
$O_x = \{y \in X : \ \exists g \in G \text{ such that } g * x = y\}$.
Lemma 1 The orbits partition X.
Proof $x = 1_X * x$ and so $x \in O_x$ and so $X = \bigcup_{x \in X} O_x$.

Suppose now that $O_x \cap O_y \neq \emptyset$ i.e. $\exists g_1, g_2$ such that $g_1 * x = g_2 * y$. But then for any $g \in G$ we have

$$g * x = (g \circ (g_1^{-1} \circ g_2)) * y \in O_y$$

and so $O_x \subseteq O_y$. Similarly $O_y \subseteq O_x$. Thus $O_x = O_y$ whenever $O_x \cap O_y \neq \emptyset$. □

The two problems we started with are of the following form: Given a set X and a group of permutations *acting* on X, compute the number of orbits i.e. distinct colorings.

A subset H of G is called a *sub-group* of G if it satisfies *axioms* **A1,A2,A3** (with G replaced by H).

The *stabilizer* S_x of the element x is $\{g : \ g * x = x\}$. It is a sub-group of G.

- A1: $1_X * x = x$.
- A3: $g, h \in S_x$ implies $(g \circ h) * x = g * (h * x) = g * x = x$.

A2 holds for any subset.

Lemma 2
If $x \in X$ then $|O_x|\,|S_x| = |G|$.

Proof Fix $x \in X$ and define an equivalence relation $\sim$ on G by

$$g_1 \sim g_2 \text{ if } g_1 * x = g_2 * x.$$

Let the equivalence classes be $A_1, A_2, \ldots, A_m$. We first argue that

$$|A_i| = |S_x| \qquad i = 1, 2, \ldots, m. \tag{20}$$

Fix i and $g \in A_i$. Then

$$h \in A_i \leftrightarrow g * x = h * x \leftrightarrow (g^{-1} \circ h) * x = x$$
$$\leftrightarrow (g^{-1} \circ h) \in S_x \leftrightarrow h \in g \circ S_x$$

where $g \circ S_x = \{g \circ \sigma : \ \sigma \in S_x\}$.

Thus $|A_i| = |g \circ S_x|$. But $|g \circ S_x| = |S_x|$ since if $\sigma_1, \sigma_2 \in S_x$ and $g \circ \sigma_1 = g \circ \sigma_2$ then

$$g^{-1} \circ (g \circ \sigma_1) = (g^{-1} \circ g) \circ \sigma_1 = \sigma_1 = g^{-1} \circ (g \circ \sigma_2) = \sigma_2.$$

This proves (20).

Finally, $m = |O_x|$ since there is a distinct equivalence class for each distinct $g * x$. □

x	O_x	S_x	
rrrr	$\{rrrr\}$	G	
brrr	{brrr,rbrr,rrbr,rrrb}	$\{e_0\}$	E
rbrr	{brrr,rbrr,rrbr,rrrb}	$\{e_0\}$	x
rrbr	{brrr,rbrr,rrbr,rrrb}	$\{e_0\}$	a
rrrb	{brrr,rbrr,rrbr,rrrb}	$\{e_0\}$	m
bbrr	{bbrr,rbbr,rrbb,brrb}	$\{e_0\}$	p
rbbr	{bbrr,rbbr,rrbb,brrb}	$\{e_0\}$	l
rrbb	{bbrr,rbbr,rrbb,brrb}	$\{e_0\}$	e
brrb	{bbrr,rbbr,rrbb,brrb}	$\{e_0\}$	
rbrb	{rbrb,brbr}	$\{e_0, e_2\}$	1
brbr	{rbrb,brbr}	$\{e_0, e_2\}$	
bbbr	{bbbr,rbbb,brbb,bbrb}	$\{e_0\}$	$n = 4$
bbrb	{bbbr,rbbb,brbb,bbrb}	$\{e_0\}$	
brbb	{bbbr,rbbb,brbb,bbrb}	$\{e_0\}$	
rbbb	{bbbr,rbbb,brbb,bbrb}	$\{e_0\}$	
bbbb	{bbbb}	G	

x	O_x	S_x	
rrrr	$\{e\}$	G	
brrr	{brrr,rbrr,rrbr,rrrb}	{e,r}	E
rbrr	{brrr,rbrr,rrbr,rrrb}	{e,s}	x
rrbr	{brrr,rbrr,rrbr,rrrb}	{e,r}	a
rrrb	{brrr,rbrr,rrbr,rrrb}	{e,s}	m
bbrr	{bbrr,rbbr,rrbb,brrb}	{e,p}	p
rbbr	{bbrr,rbbr,rrbb,brrb}	{e,q}	l
rrbb	{bbrr,rbbr,rrbb,brrb}	{e,p}	e
brrb	{bbrr,rbbr,rrbb,brrb}	{e,q}	
rbrb	{rbrb,brbr}	{e,b,r,s}	2
brbr	{rbrb,brbr}	{e,b,r,s}	
bbbr	{bbbr,rbbb,brbb,bbrb}	{e,s}	
bbrb	{bbbr,rbbb,brbb,bbrb}	{e,r}	
brbb	{bbbr,rbbb,brbb,bbrb}	{e,s}	
rbbb	{bbbr,rbbb,brbb,bbrb}	{e,r}	
bbbb	{e}	G	

Let $\nu_{X,G}$ denote the number of orbits.

Theorem 1

$$\nu_{X,G} = \frac{1}{|G|} \sum_{x \in X} |S_x|.$$

Proof

$$\begin{aligned} \nu_{X,G} &= \sum_{x \in X} \frac{1}{|O_x|} \\ &= \sum_{x \in X} \frac{|S_x|}{|G|}, \end{aligned}$$

from Lemma 1. □

Thus in example 1 we have

$$\nu_{X,G} = \frac{1}{4}(4+1+1+1+1+1+1+1+1+2+2+1+1+1+1+4) = 6.$$

In example 2 we have

$$\nu_{X,G} = \frac{1}{8}(8+2+2+2+2+2+2+2+2+4+4+2+2+2+2+8) = 6.$$

Theorem 1 is hard to use if $|X|$ is large, even if $|G|$ is small.

For $g \in G$ let $Fix(g) = \{x \in X : \ g * x = x\}$.

FRIDAY, NOVEMBER 14, 2025.

Theorem 2

(Frobenius, Burnside)

$$\nu_{X,G} = \frac{1}{|G|} \sum_{g \in G} |Fix(g)|.$$

Proof Let $A(x, g) = 1_{g*x=x}$. Then

$$\begin{aligned}
\nu_{X,G} &= \frac{1}{|G|} \sum_{x \in X} |S_x| \\
&= \frac{1}{|G|} \sum_{x \in X} \sum_{g \in G} A(x, g) \\
&= \frac{1}{|G|} \sum_{g \in G} \sum_{x \in X} A(x, g) \\
&= \frac{1}{|G|} \sum_{g \in G} |Fix(g)|.
\end{aligned}$$



Let us consider example 1 with $n = 6$. We compute

g	e_0	e_1	e_2	e_3	e_4	e_5
$\lvert Fix(g)\rvert$	64	2	4	8	4	2

Applying Theorem 2 we obtain

$$\nu_{X,G} = \frac{1}{6}(64 + 2 + 4 + 8 + 4 + 2) = 14.$$

Cycles of a permutation

Let $\pi : D \to D$ be a permutation of the finite set D. Consider the digraph $\Gamma_\pi = (D, A)$ where $A = \{(i, \pi(i)) : \ i \in D\}$. Γ_π is a collection of vertex disjoint cycles. Each $x \in D$ being on a unique cycle. Here a cycle can consist of a loop i.e. when $\pi(x) = x$.
Example: $D = [10]$.

i	1	2	3	4	5	6	7	8	9	10
$\pi(i)$	6	2	7	10	3	8	9	1	5	4

The cycles are $(1, 6, 8), (2), (3, 7, 9, 5), (4, 10)$.

In general consider the sequence $i, \pi(i), \pi^2(i), \ldots,$.

Since D is finite, there exists a first pair $k < \ell$ such that $\pi^k(i) = \pi^\ell(i)$. Now we must have $k = 0$, since otherwise putting $x = \pi^{k-1}(i) \neq y = \pi^{\ell-1}(i)$ we see that $\pi(x) = \pi(y)$, contradicting the fact that π is a permutation.

So i lies on the cycle $C = (i, \pi(i), \pi^2(i), \ldots, \pi^{k-1}(i), i)$.

If j is not a vertex of C then $\pi(j)$ is not on C and so we can repeat the argument to show that the rest of D is partitioned into cycles.

Example 1

First consider $e_0, e_1, \ldots, e_{n-1}$ as permutations of D.

The cycles of e_0 are $(1), (2), \ldots, (n)$.

Now suppose that $0 < m < n$. Let $a_m = gcd(m, n)$ and $k_m = n/a_m$. The cycle C_i of e_m containing the element i is is $(i, i+m, i+2m, \ldots, i+(k_m-1)m)$ since n is a divisor $k_m m$ and not a divisor of $k'm$ for $k' < k_m$. In total, the cycles of e_m are $C_0, C_1, \ldots, C_{a_m-1}$.

This is because they are disjoint and together contain n elements. (If $i + rm = i' + r'm \mod n$ then $(r - r')m + (i - i') = \ell n$. But $|i - i'| < a_m$ and so dividing by a_m we see that we must have $i = i'$.)

Next observe that if coloring x is fixed by e_m then elements on the same cycle C_i must be colored the same. Suppose for example that the color of $i + bm$ is different from the color of $i + (b+1)m$, say Red versus Blue. Then in $e_m(x)$ the color of $i + (b+1)m$ will be Red and so $e_m(x) \neq x$. Conversely, if elements on the same cycle of e_m have the same color then in $x \in Fix(e_m)$. This property is not peculiar to this example, as we will see.

Thus in this example we see that $|Fix(e_m)| = 2^{a_m}$ and then applying Theorem 2 we see that

$$\nu_{X,G} = \frac{1}{n} \sum_{m=0}^{n-1} 2^{gcd(m,n)}.$$

Example 2

It is straightforward to check that when n is even, we have

g	e	a	b	c	p	q	r	s
$\lvert Fix(g)\rvert$	2^{n^2}	$2^{n^2/4}$	$2^{n^2/2}$	$2^{n^2/4}$	$2^{n^2/2}$	$2^{n^2/2}$	$2^{n(n+1)/2}$	$2^{n(n+1)/2}$

For example, if we divide the chessboard into 4 $n/2 \times n/2$ sub-squares, numbered 1,2,3,4 then a coloring is in $Fix(a)$ iff each of these 4 sub-squares have colorings which are rotations of the coloring in square 1.

MONDAY, NOVEMBER 17, 2025.

Polya's Theorem

We now extend the above analysis to answer questions like: How many *distinct* ways are there to color an 8×8 chessboard with 32 white squares and 32 black squares?
The scenario now consists of a set D (*Domain*, a set C (colors) and $X = \{x : \ D \to C\}$ is the set of colorings of D with the color set C. G is now a group of permutations of D.

We see first how to extend each permutation of D to a permutation of X. Suppose that $x \in X$ and $g \in G$ then we define $g * x$ by

$$g * x(d) = x(g^{-1}(d)) \qquad \text{for all } d \in D.$$

Explanation: The color of d is the color of the element $g^{-1}(d)$ which is mapped to it by g.
Consider Example 1 with $n = 4$. Suppose that $g = e_1$ i.e. rotate clockwise by $\pi/2$ and $x(1) = b, x(2) = b, x(3) = r, x(4) = r$.
Then for example

$$g * x(1) = x(g^{-1}(1)) = x(4) = r, \text{ as before.}$$

Now associate a **weight** w_c with each $c \in C$.
If $x \in X$ then

$$W(x) = \prod_{d \in D} w_{x(d)}.$$

Thus, if in Example 1 we let $w(r) = R$ and $w(b) = B$ and take $x(1) = b, x(2) = b, x(3) = r, x(4) = r$ then we will write $W(x) = B^2 R^2$.

For $S \subseteq X$ we define the **inventory** of S to be

$$W(S) = \sum_{x \in S} W(x).$$

The problem we discuss now is to compute the **pattern inventory** $PI = W(S^*)$ where S^* contains one member of each orbit of X under G.

For example, in the case of Example 2, with $n = 2$, we get

$$PI = R^4 + R^3B + 2R^2B^2 + RB^3 + B^4.$$

To see that the definition of PI makes sense we need to prove

Lemma 3 If x, y are in the same orbit of X then $W(x) = W(y)$.

Proof Suppose that $g * x = y$. Then

$$\begin{aligned} W(y) &= \prod_{d \in D} w_{y(d)} \\ &= \prod_{d \in D} w_{g*x(d)} \\ &= \prod_{d \in D} w_{x(g^{-1}(d))} \qquad (21) \\ &= \prod_{d \in D} w_{x(d))} \qquad (22) \\ &= W(x) \end{aligned}$$

Note, that we can go from (21) to (22) because as d runs over D, $g^{-1}(d)$ also runs over d.

Let $\Delta = |D|$. If $g \in G$ has k_i cycles of length i then we define

$$ct(g) = x_1^{k_1} x_2^{k_2} \cdots x_\Delta^{k_\Delta}.$$

The **Cycle Index Polynomial** of G, C_G is then defined to be

$$C_G(x_1, x_2, \ldots, x_\Delta) = \frac{1}{|G|} \sum_{g \in G} ct(g).$$

In Example 2 with $n = 2$ we have

g	e	a	b	c	p	q	r	s
$ct(g)$	x_1^4	x_4	x_2^2	x_4	x_2^2	x_2^2	$x_1^2 x_2$	$x_1^2 x_2$

and so

$$C_G(x_1, x_2, x_3, x_4) = \frac{1}{8}(x_1^4 + 3x_2^2 + 2x_1^2 x_2 + 2x_4).$$

In Example 2 with $n = 3$ we have

g	e	a	b	c	p	q	r	s
$ct(g)$	x_1^9	$x_1x_4^2$	$x_1x_2^4$	$x_1x_4^2$	$x_1^3x_2^3$	$x_1^3x_2^3$	$x_1^3x_2^3$	$x_1^3x_2^3$

and so

$$C_G(x_1, x_2, x_3, x_4) = \frac{1}{8}(x_1^9 + x_1x_2^4 + 4x_1^3x_2^3 + 2x_1x_4^2).$$

Theorem (Polya)

$$PI = C_G\left(\sum_{c\in C} w_c, \sum_{c\in C} w_c^2, \ldots, \sum_{c\in C} w_c^{\Delta}\right).$$

Proof In Example 2, we replace x_1 by $R+B$, x_2 by R^2+B^2 and so on. When $n=2$ this gives

$$\begin{aligned} PI &= \frac{1}{8}((R+B)^4 + 3(R^2+B^2)^2 + \\ &\qquad 2(R+B)^2(R^2+B^2) + 2(R^4+B^4)) \\ &= R^4 + R^3B + 2R^2B^2 + RB^3 + B^4. \end{aligned}$$

Putting $R=B=1$ gives the number of distinct colorings. Note also the formula for PI tells us that there are 2 distinct colorings using 2 reds and 2 Blues.

Proof of Polya's Theorem

Let $X = X_1 \cup X_2 \cup \cdots \cup X_m$ be the equivalence clases of X under the relation

$$x \sim y \text{ iff } W(x) = W(y).$$

By Lemma 2, $g * x \sim x$ for all $x \in X, g \in G$ and so we can think of G acting on each X_i individually i.e. we use the fact that $x \in X_i$ implies $g * x \in X_i$ for all $i \in [m], g \in G$. We use the notation $g^{(i)} \in G^{(i)}$ when we restrict attention to X_i.

Let m_i denote the number of orbits $\nu_{X_i,G^{(i)}}$ and W_i denote the common PI of $G^{(i)}$ acting on X_i. Then

$$\begin{aligned} PI &= \sum_{i=1}^{m} m_i W_i \\ &= \sum_{i=1}^{m} W_i \left(\frac{1}{|G|} \sum_{g \in G} |Fix(g^{(i)})| \right) \qquad \text{by Theorem 2} \\ &= \frac{1}{|G|} \sum_{g \in G} \sum_{i=1}^{m} |Fix(g^{(i)})| W_i \\ &= \frac{1}{|G|} \sum_{g \in G} W(Fix(g)) \qquad (23) \end{aligned}$$

Note that (23) follows from $Fix(g) = \bigcup_{i=1}^{m} Fix(g^{(i)})$ since $x \in Fix(g^{(i)})$ iff $x \in X_i$ and $g * x = x$.

Suppose now that $ct(g) = x_1^{k_1} x_2^{k_2} \cdots x_\Delta^{k_\Delta}$ as above. Then we claim that

$$W(Fix(g)) = \left(\sum_{c \in C} w_c\right)^{k_1} \left(\sum_{c \in C} w_c^2\right)^{k_2} \cdots \left(\sum_{c \in C} w_c^\Delta\right)^{k_\Delta}. \quad (24)$$

Substituting (24) into (23) yields the theorem.

To verify (24) we use the fact that if $x \in Fix(g)$, then the elements of a cycle of g must be given the same color. A cycle of length i will then contribute a factor $\sum_{c \in C} w_c^i$ where the term w_c^i comes from the choice of color c for every element of the cycle. □

WEDNESDAY, NOVEMBER 19, 2025.

Hereditary Families

Given a Ground Set E, a Hereditary Family $\mathcal{A}$ on E is collection of subsets $\mathcal{I} = \{\mathcal{I}_\infty, \mathcal{I}_\in, \ldots, \mathcal{I}_\Updownarrow\}$ (the independent sets) such that

$$I \in \mathcal{I} \text{ and } \mathcal{J} \subseteq \mathcal{I} \text{ implies that } \mathcal{J} \in \mathcal{I}.$$

1. The set $\mathcal{M}$ of matchings of a graph $G = (V, E)$.
2. The set of (edge-sets of) forests of a graph $G = (V, E)$.
3. The set of stable sets of a graph $G = (V, E)$. We say that S is stable if it contains no edges.
4. If $G = (A, B, E)$ is a bipartite graph and $\mathcal{I} = \{\mathcal{S} \subseteq \mathcal{B} : \exists \text{ a matching } M \text{ that covers } S\}$.
5. Let $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n$ be the columns of an $m \times n$ matrix $\overline{A}$. Then $E = [n]$ and $\mathcal{C} = \{\mathcal{S} \subseteq [\backslash] : \{\mathbf{c}_\rangle, \rangle \in \mathcal{S}\} \text{ are linearly independent}\}$.

Matroids

An independence system is a matroid if whenever $I, J \in \mathcal{C}$ with $|J| = |I| + 1$ there exists $e \in J \setminus I$ such that $I \cup \{e\} \in \mathcal{I}$. We call this the Independent Augmentation Axiom – IAA.

Matroid independence is a generalisation of linear independence in vector spaces. Only Examples 2,4 and 5 above are matroids.

To check Example 5, let $\overline{A}_I$ be the $m \times |I|$ sub-matrix of $\overline{A}$ consisting of the columns in I. If there is no $e \in J \setminus I$ such that $I \cup \{e\} \in \mathcal{I}$ then $\overline{A}_J = \overline{A}_I \mathbf{M}$ for some $|I| \times |J|$ matrix $\mathbf{M}$.

Matrix $\mathbf{M}$ has more columns than rows and so there exists $\mathbf{x} \neq 0$ such that $\mathbf{Mx} = \mathbf{0}$. But then $\overline{A}_J \mathbf{x} = \mathbf{0}$, implying that the columns of $\overline{A}_J$ are linearly dependent. Contradiction.

These are called Representable Matroids.

Cycle Matroids/Graphic Matroids

To check Example 2 we define the vertex-edge incidence matrix $\overline{A}_G$ of graph $G = (V, E)$ over GF_2.

$\overline{A}_G$ has a row for each vertex $v \in V$ and a column for each edge $e \in E$. There is a 1 in row v, column e iff $v \in e$.

We verify that a set of columns $\mathbf{c}_i, i \in I$ are linearly dependent iff the corresponding edges contain a cycle.

If the edges contain a cycle $(v_1, v_2, \ldots, v_k, v_1)$ then the sum of the columns corresponding to the vertices of the cycle is $\mathbf{0}$.

To show that a forest F defines a linearly independent set of columns I_F, we use induction on the number of edges in the forest. This is trivial if $|E(F)| = 1$.

Cycle Matroids/Graphic Matroids

Let $\overline{A}_F$ denote the submatrix of $\overline{A}$ made up of the columns corresponding F.

Now a forest F must contain a vertex v of degree one. This means that the row corresponding to v in $\overline{A}_F$ has a single one, in column e say.

Consider the forest $F' = F \setminus \{e\}$. Its corresponding columns $I_{F'}$ are linearly independent, by induction. Adding back e adds a row with a single one and preserves independence. Let $\mathbf{B}$ denote $\overline{A}_{F'}$ minus row e.

$$\overline{A}_F = \begin{bmatrix} 1 & \mathbf{0} \\ & \mathbf{B} \end{bmatrix}.$$

Transversal Matroids

We now check Example 4. These are called Transversal Matroids. If M_1, M_2 are two matchings in a graph G then $M_1 \oplus M_2 = (M_1 \setminus M_2) \cup (M_2 \setminus M_1)$ consists of alternating paths and cycles.



Suppose now that we have two matchings M_1, M_2 in bipartite graph $G = (A, B, E)$. Let $I_j, j = 1, 2$ be the vertices in B covered by M_j. Suppose that $|I_1| > |I_2|$.

Then $M_1 \oplus M_2$ must contain an alternating path P with end points $b \in I_1 \setminus I_2, a \in A$. Let E_1 be the M_1 edges in P and let E_2 be the M_2 edges of P. Then $(M_1 \cup E_1) \setminus E_2$ is a matching that covers $I_1 \cup \{b\}$.

Representable Matroids

A matroid is binary if is representable by a matrix over GF_2.

So a graphic matroid is binary.

A matroid is regular if it can be represented by a matrix of elements in $\{0, \pm 1\}$ for which every square sub-matrix has determinant $0, \pm 1$. These are called totally unimodular matrices

A matrix with 2 non-zeros in each column, one equal to +1 and the other equal to -1 is totally unimodular. This implies that graphic matroids are regular. (Take the vertex-edge incidence matrix and replace one of the ones in each column by a -1.)

Partition Matroids

Given a partition $E_1, E_2, \ldots, E_m$ of E and non-negative integers $k_1, k_2, \ldots, k_m$ we define the associated partition matroid as follows:

$I \in \mathcal{I}$ iff $|I \cap E_i| \leq k_i, i = 1, 2, \ldots, m$.

Partition matroids are representable.

FRIDAY, NOVEMBER 21, 2025.

Bases

A matroid basis is a maximal independent set i.e. B is a basis if there does **not** exist an independent set $I \neq B$ such that $I \supset B$.

So the bases of the cycle matroid of a graph G consist of the spanning trees of G.

Lemma

If B_1, B_2 are bases of a matroid $\mathcal{M}$, then $|B_1| = |B_2|$.

Proof: If $|B_1| > |B_2|$ then there exists $e \in B_1 \setminus B_2$ such that $B_2 \cup \{e\}$ is independent. Contradicting the fact that B_2 is maximal. □

Theorem

A collection $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$ of subsets of E form the bases of a matroid on E iff for all i, j and $e \in B_i \setminus B_j$ there exists $f \in B_j \setminus B_i$ such that $(B_i \cup \{f\}) \setminus \{e\} \in \mathcal{B}$.

Proof: Suppose first that $\mathcal{B}$ are the bases of a matroid with independent sets $\mathcal{I}$ and that $e \in B_i$ and $e \notin B_j$. Then $B_i' = B_i \setminus \{e\} \in \mathcal{I}$ and $|B_i'| < |B_j|$. So there exists $f \in B_j \setminus B_i'$ such that $B_i'' = B_i' \cup \{f\} \in \mathcal{I}$. Now $f \neq e$ since $e \notin B_j$ and $|B_i''| = |B_i|$. So B_i'' must be a basis.

Conversely, suppose that $\mathcal{B}$ satisfies the conditions of the theorem and that $\mathcal{I} = \{\mathcal{S} : \exists \rangle \int . \sqcup . \mathcal{S} \subseteq \mathcal{B}_\rangle\}$. Clearly $\mathcal{I}$ is hereditary.

Bases

We first argue that all the sets in $\mathcal{B}$ are of the same size.

Suppose that $A = \{i : |B_i| = \max\{|B| : B \in \mathcal{B}\}\}$ and suppose that $A \neq [m]$. Suppose that

$$\min\{|B_i - B_j| : i \in A, j \notin A\} = |B_1 \setminus B_2|.$$

Let $x \in B_1 \setminus B_2$ and let $y \in B_2 \setminus B_1$ be such that $B' = ((B_1 \cup y) \setminus \{x\}) \in \mathcal{B}$.

Then we have $B' \in A$ and $|B' \setminus B_2| < |B_1 \setminus B_2|$, contradiction.

Bases

Suppose now that $I_1, I_2 \in \mathcal{I}$ with $|I_2| > |I_1|$ and there does not exist $e \in I_2 \setminus I_1$ for which $I_1 \cup \{e\} \in \mathcal{I}$.

Choose $B_j \supseteq I_j, j = 1, 2$ such that $|B_2 \setminus (I_2 \cup B_1)|$ is minimal.

We must have $I_2 \setminus B_1 = I_2 \setminus I_1$. If $x \in I_2 \cap B_1$ and $x \notin I_1$ then $I_1 \cup \{x\} \subseteq B_1$ and so $I_1 \cup \{x\} \in \mathcal{I}$.

Suppose there exists $x \in B_2 \setminus (I_2 \cup B_1)$. Then by assumption there is $y \in B_1 \setminus B_2$ such that $B' = (B_2 \cup \{y\}) \setminus \{x\} \in \mathcal{B}$. But then $B' \setminus (I_2 \cup B_1) = (B_2 \setminus (I_2 \cup B_1)) \setminus \{x\}$, contradicting the definition of B_2.

So $B_2 \subseteq (I_2 \cup B_1) = (I_2 \setminus B_1) \cup (B_1 \setminus I_2) = (I_2 \setminus I_1) \cup (B_1 \setminus I_2)$ and so

$$B_2 \setminus B_1 = I_2 \setminus I_1. \tag{25}$$

We show next that $B_1 \subseteq (I_1 \cup B_2)$. If there exists $x \in B_1 \setminus (I_1 \cup B_2)$ then there exists $y \in B_2 \setminus B_1$ such that $B' = (B_1 \cup \{y\}) \setminus \{x\} \in \mathcal{B}$. But $(I_1 \cup \{x\}) \subseteq B'$, contradiction.

So, $B_1 \setminus B_2 = I_1 \setminus B_2 \subseteq I_1 \setminus I_2$. Since $|B_1 \setminus B_2| = |B_2 \setminus B_1|$ we see from this and (25) that $|I_1 \setminus I_2| \geq |I_2 \setminus I_1|$ and so $|I_1| \geq |I_2|$, contradiction.

If $S \subseteq E$ then its rank

$$r(S) = \max |\{I \in \mathcal{I} : \mathcal{I} \subseteq S\}| .$$

So $S \in \mathcal{I}$ iff $r(S) = |S|$. We show next that r is submodular.

Theorem

If $S, T \subseteq E$ then $r(S \cup T) + r(S \cap T) \leq r(S) + r(T)$.

Proof: Let I_1 be a maximal independent subset of $S \cap T$ and let I_2 be a maximal independent subset of $S \cup T$ that contains I_1. (Such a set exists because of the IAA.)
But then

$$\begin{aligned} r(S \cap T) + r(S \cup T) &= |I_1| + |I_2| = |I_1 \cap S| + |I_2 \cap S \cup T| \\ &= |I_1| + |I_2 \cap S| + |I_2 \cap T| - |I_2 \cap S \cap T| \\ &= |I_2 \cap S| + |I_2 \cap T| \\ &\leq r(S) + r(T). \end{aligned}$$

Rank

For representable matroids this coresponds to the usual definition of rank.

For the cycle matroid of graph $G = (V, E)$, if $S \subseteq E$ is a set of edges and G_S is the graph (V, S) then $r(S) = |V| - \kappa(G_S)$, where $\kappa(G_S)$ is the number of components of G_S.

This clearly true for connected graphs and so if $C_1, C_2, \ldots, C_s$ are the components of G_S then $r(S) = \sum_{i=1}^{s} |C_i| - 1 = |V| - s$.

For a partition matroid as defined above,

$$r(S) = \sum_{i=1}^{m} \min\{k_i, |S \cap E_i|\}.$$

MONDAY, NOVEMBER 24, 2025.

Circuits

A circuit of a matroid $\mathcal{M}$ is a minimal dependent set. If a set $S \subseteq E, S \notin \mathcal{I}$ then S contains a circuit.

So the circuits of the cycle matroid of a graph G are the cycles.

Theorem

If C_1, C_2 are circuits of $\mathcal{M}$ and $e \in C_1 \cap C_2$ then there is a circuit $C \subseteq (C_1 \cup C_2) \setminus \{e\}$.

Proof: We have $r(C_i) = |C_i| - 1, i = 1, 2$. Also, $r(C_1 \cap C_2) = |C_1 \cap C_2|$ since $C_1 \cap C_2$ is a proper subgraph of C_1.

If $C' = (C_1 \cup C_2) \setminus \{e\}$ contains no circuit then $r(C_1 \cup C_2) \geq r(C') = |C_1 \cup C_2| - 1$. But then

$$|C_1 \cup C_2| - 1 \leq r(C_1 \cup C_2) \leq r(C_1) + r(C_2) - r(C_1 \cap C_2)$$
$$= (|C_1| - 1) + (|C_2| - 1) - |C_1 \cap C_2|.$$

Contradiction.

Theorem

If B is a basis of $\mathcal{M}$ and $e \in E \setminus B$ then $B' = B \cup \{e\}$ contains a unique circuit $C(e, B)$. Furthermore, if $f \in C(e, B)$ then $(B \cup \{e\}) \setminus \{f\}$ is also a basis of $\mathcal{M}$.

Proof: $B' \notin \mathcal{I}$ because B is maximal. So B' must contain at least one circuit.

Suppose it contains distinct circuits C_1, C_2. Then $e \in C_1 \cap C_2$ and so B' contains a circuit $C_3 \subseteq (C_1 \cup C_2) \setminus \{e\}$.

But then $C_3 \subseteq B$, contradiction. □

Dual Matroid

Theorem

If $\mathcal{B}$ denotes the set of bases of a matroid $\mathcal{M}$ on ground set E then $\mathcal{B}^* = \{E \setminus B : B \in \mathcal{B}\}$ is the set of bases of a matroid $\mathcal{M}^*$, the dual matroid.

Proof: Suppose that $B_1^*, B_2^* \in \mathcal{B}^*$ and $e \in B_1^* \setminus B_2^*$.

Let $B_i = E \setminus B_i^*, i = 1, 2$. Then $e \in B_2 \setminus B_1$.

So there exists $f \in B_1 \setminus B_2 = B_2^* m^\star B_1^*$ such that $B' = (B_2 \cup \{e\}) \setminus \{f\} \in \mathcal{B}$.

But then $E \setminus B' \in \mathcal{B}^*$ and $E \setminus B' = (B_1^* \cup \{f\}) \setminus \{e\}$. □

Greedy Algorithm

Suppose that each $e \in E$ is given a weight w_e and that the weight $w(I)$ of an independent set I is given by $w(I) = \sum_{e \in I} c_e$. The problem we discuss is

$$\text{Maximize } w(I) \text{ subject to } I \in \mathcal{I}.$$

Greedy Algorithm:
begin
    Sort $E = \{e_1, e_2, \ldots, e_m\}$ so $w(e_i) \geq w(e_{i+1})$ for $1 \leq i < m$;
    $S \leftarrow \emptyset$;
    for $i = 1, 2, \ldots, m$;
    begin
        if $S \cup \{e_i\} \in \mathcal{I}$ **then**;
        begin;
            $S \leftarrow S \cup \{e_i\}$;
        end;
    end;
end

Theorem

The greedy algorithm finds a maximum weight independent set **for all choices of** w *if and only if it is a matroid.*

Suppose first that the Greedy Algorithm always finds a maximum weight independent set. Suppose that $\emptyset \neq I, J \in \mathcal{I}$ with $|J| = |I| + 1$. Define

$$w(e) = \begin{cases} 1 + \frac{1}{2|I|} & e \in I. \\ 1 & e \in J \setminus I. \\ 0 & e \notin I \cup J. \end{cases}$$

If there does not exist $e \in J \setminus I$ such that $I \cup \{e\} \in \mathcal{I}$ then the Greedy Algorithm will choose the elements of I and stop. But I does not have maximum weight. Its weight is $|I| + 1/2 < |J|$. So if Greedy succeeds, then the IAA holds.

Greedy Algorithm

Conversely, suppose that our independence system is a matroid. We can assume that $w(e) > 0$ for all $e \in E$. Otherwise we can restrict ourselves to the matroid defined by $\mathcal{I}' = \{\mathcal{I} \subseteq \mathcal{E}^+\}$ where $E^+ = \{e \in E : w(e) > 0\}$.

Suppose now that Greedy chooses $I_G = e_{i_1}, e_{i_2}, \ldots, e_{i_k}$ where $i_t < i_{t+1}$ for $1 \leq t < k$. Let $I = e_{j_1}, e_{j_2}, \ldots, e_{j_\ell}$ be any other independent set and assume that $j_t < j_{t+1}$ for $1 \leq t < \ell$. We can assume that $\ell \geq k$, for otherwise we can add something from I_G to I to give it larger weight.

We show next that $k = \ell$ and that $i_t \leq j_t$ for $1 \leq t \leq k$. This implies that $w(I_G) \geq w(I)$.

Greedy Algorithm

Suppose then that there exists t such that $i_t > j_t$ and let t be as small as possible for this to be true.

Now consider $I = \{e_{i_s} : s = 1, 2, \ldots, t-1\}$ and $J = \{e_{j_s} : s = 1, 2, \ldots, t\}$. Now there exists $e_{j_s} \in J \setminus I$ such that $I \cup \{e_{j_s}\} \in \mathcal{I}$.

But $j_s \leq j_t < i_t$ and Greedy should have chosen e_{j_s} before choosing $e_{i_{t+1}}$.

Also, $i_k \leq j_k$ implies that $k = \ell$. Otherwise Greedy can find another element from $I \setminus I_G$ to add.

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Minors

Given a graph $G = (V, E)$ and an edge e we can get new graphs by deleting e or contracting e.

We describe a corresponding notion for matroids. Suppose that $F \subseteq E$ then we define the matroid $\mathcal{M}_{\Updownarrow^\star \mathcal{F}}$ with independent sets $\mathcal{I}_{\Updownarrow^\star \mathcal{F}}$ obtained by deleting F: $I \in \mathcal{I}_{\Updownarrow^\star \mathcal{F}}$ if $I \in \mathcal{I}$, $\mathcal{I} \cap \mathcal{F} = \emptyset$.

It is clear that the IAA holds for $\mathcal{M}_{\Updownarrow^\star \mathcal{F}}$ and so it is a matroid.

For contraction we will assume that $F \in \mathcal{I}$. Then contracting F defines $\mathcal{M}.\mathcal{F}$with independent sets
$\mathcal{I}.\mathcal{F} = \{\mathcal{I} \in \mathcal{I} : \mathcal{I} \cap \mathcal{F} = \emptyset, \mathcal{I} \cup \mathcal{F} \in \mathcal{I}\}$.

We argue next that $\mathcal{M}.\mathcal{F}$ is also a matroid.

Lemma

$\mathcal{M}.\mathcal{F} = (\mathcal{M}^*_{\Updownarrow^\star \mathcal{F}})^*$ *and* $\mathcal{M}_{\Updownarrow^\star \mathcal{F}} = (\mathcal{M}^*.\mathcal{F})^*$.

Proof:

$$\begin{aligned} I \in \mathcal{I}.\mathcal{F} &\leftrightarrow \exists B \in \mathcal{B}_{m^\star F}, I \subseteq B \\ &\leftrightarrow \exists B^* \in \mathcal{B}^*_{m^\star F}, I \cap B^* = \emptyset \\ &\leftrightarrow I \in (\mathcal{I}^*_{\Updownarrow^\star \mathcal{F}})^*. \end{aligned}$$

For the second claim we use

$$\mathcal{M}^*.\mathcal{F} = (\mathcal{M}^{**}_{\Updownarrow^\star \mathcal{F}})^* = (\mathcal{M}_{\Updownarrow^\star \mathcal{F}})^*.$$

□

Matroid Intersection

Suppose we are given two matroids $\mathcal{M}_\infty, \mathcal{M}_\in$ on the same ground set E with $\mathcal{I}_\infty, \mathcal{I}_\in$ and r_1, r_2 etc. having there obvious meaning.

An intersection is a set $I \in \mathcal{I}_\infty \cap \mathcal{I}_\in$. We give a min-max relation for the size of the largest independent intersection. Let $\mathcal{J}$ denote the set of intersections.

Theorem (Edmonds)

$$\max\{J \in \mathcal{J}\} = \min\{\nabla_\infty(\mathcal{A}) + \nabla_\in(\mathcal{E} \Updownarrow^\star \mathcal{A}) : \mathcal{A} \subseteq \mathcal{E}\}.$$

Matroid Intersection

Before proving the theorem let us see a couple of applications:

Hall's Theorem: suppose we are given a bipartite graph $G = (A, B, E)$. Let $\mathcal{M}_A, \mathcal{M}_B$ be the following two partition matroids.

For $\mathcal{M}_A$ we define the partition $E_a = \{e \in x : a \in e\}$, $a \in A$. We let $k_a = 1$ for $a \in A$. We define $\mathcal{M}_B$ similarly.

Intersections correspond to matchings and $r_1(A)$ is the number of vertices in A that are incident with an edge of A. Similarly $r_2(E \setminus A)$ is the number of vertices in B that are incident with an edge not in A.

Matroid Intersection

For $X \subseteq A$, let

$$A_X = \{v \in A : v \in e \text{ for some } e \in X\}.$$

Define B_X similarly.

So

$$\max\{|M|\} = \min\{|A_X| + |B_{E\setminus X}| : \ X \subseteq E\}.$$

Now we can assume that if $e \in E \setminus X$ then $e \cap A_X = \emptyset$, otherwise moving e to X does not increase the RHS of the above.

Let $S = A \setminus A_X$. Then $|B_{E\setminus X}| = |N(A)|$ and so

$$\max\{|M|\} = \min\{|A| - |S| + |N(S)| : S \subseteq A\}.$$