ON JAKOVLEV SPACES

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ABSTRACT. A topological space X is called weakly first countable, if for every point x there is a countable family $\{C^x_n \mid n \in \omega\}$ such that $x \in C^x_{n+1} \subseteq C^x_n$ and such that $U \subset X$ is open iff for each $x \in U$ some C^x_n is contained in U. This weakening of first countability is due to A. V. Arhangelskii from 1966, who asked whether compact weakly first countable spaces are first countable. In 1976, N.N. Jakovlev gave a negative answer under the assumption of continuum hypothesis. His result was strengthened by V.I.Malykhin in 1982, again under CH. In the present paper we construct various Jakovlev type spaces under the weaker assumption $\mathfrak{b} = \mathfrak{c}$, and also by forcing.

INTRODUCTION

Motivated by a range of topological considerations and problems, Alexander Arhangelskii introduced the notion of weakly first-countable topological space in his 1966 paper "Mappings and spaces" ([2], p.129 ff.). He posed there two fundamental questions:

Question 0.1. *Do the two notions, weakly first-countable and first-countable, coincide in the class of compact Hausdorff spaces?*

Question 0.2. *Is there a weakly first-countable compact space of cardinality $>\ \mathfrak{c}$?*

Both of these questions are still open. They acquired especial poignancy after Arhangelskii proved in 1969 [3] the celebrated conjecture of Alexandroff and Urysohn that the size of compact Hausdorff first-countable space is at most the continuum. In view of his theorem, *yes* to the second question implies *no* to the first one.

Date: November 7, 2002.

1991 *Mathematics Subject Classification.* Primary 54D70, 54A35; Secondary 54A25, 03E17.

Key words and phrases. Weakly first-countable, compact, scattered, small cardinals, forcing.

The second author was supported by the Ben-Gurion University Center for Advanced Studies in Mathematics, Be'er Sheva.

Definition 0.3. *A topological space* $(X, \mathcal{T})$ *is called* **weakly first-countable**, *if for each point* $x \in X$ *there is a countable family* σ^x, $\sigma^x = \{C_n^x \mid n \in \omega\}$ *with* $x \in C_{n+1}^x \subseteq C_n^x \subseteq X$, *such that*

$$V \in \mathcal{T} \iff (\forall x \in V)\ (\exists n \in \omega)\ \ C_n^x \subseteq V.$$

Although, as it stands, the definition of *weakly first-countable* relies on the total *system* $\{\sigma^x \mid x \in X\}$, we will say that σ^x is a *local weak base* at x, and C_n^x are *weak neighborhoods* of x. In the following proposition we collect some basic facts about weakly first-countable spaces.

Proposition 0.4.

(1) *Weakly first-countable spaces are sequential.*
(2) *Weakly first-countable Hausdorff spaces are first-countable iff Frechet-Urysohn.*
(3) *Compact sequential spaces of size* $< \mathfrak{b}$ *are Frechet-Urysohn.*

Proof. A space is sequential if any non-closed set A has a sequence that converges to a point not in A. A space is Frechet-Urysohn if any set has sequences converging to any point in its closure.

(1) The key is to observe that any set meeting all members of a local weak base of a point in a weakly first-countable space has a countable subset converging to this point. At this point we remark for the interested reader that weakly first-countable spaces are k-spaces, and that they are α_1 on the α_i $(i = 1, 2, 3, 4)$ scale of A. Arhangelskii (defined in [4]).

(2) Suppose a space X is Frechet-Urysohn and weakly first-countable. We will show that $x \in \mathrm{Int}(C_n^x)$. If not, by Frechet-Urysohn, there is a sequence from $X \setminus C_n^x$ converging to x. This contradicts the fact that in a weakly first-countable Hausdorff space a countable set converges to a point iff it is almost included in all weak neighborhoods of that point.

(3) A good straight-forward proof is in [6]. (See also [7].) □

The first consistent example of a compact Hausdorff weakly first-countable not first-countable space - and so a consistent answer *no* to the first question of Arhangelskii - was given in 1976 by N. Jakovlev [8]. He constructed under CH such a space of cardinality $\aleph_1 = \mathfrak{c}$. (Non-compact examples are abundant, perhaps the simplest being Arens space of the "sequence of sequences" [1]).

After that, V. Malykhin ([9], Theorem 1) realized that the CH construction of Jakovlev can be pushed up to any cardinal above the continuum $\mathfrak{c} = \aleph_1$ (we believe that the argument in [9] contains a gap, and we were able to reconstruct the proof only for $\aleph_2$). This gives a consistent *yes* to the second question of Arhangelskii.

The same paper [9], also contains a sketch (which we did not completely understand) of a forcing argument designed to add a Jakovlev type space of size $\aleph_1$, consistently with the negation of CH.

In the first section we define Jakovlev spaces (by Lemma 1.3, they are all counterexamples to Question 0.1, and the example of Jakovlev [8] is a "Jakovlev space"), and in Theorem 1.4 we prove the existence of Jakovlev spaces of cardinality $\mathfrak{c}$ under the assumption $\mathfrak{b} = \mathfrak{c}$.

Under the same set-theoretic assumption, we construct in Theorem 1.5 a Jakovlev space of cardinality $\mathfrak{c}^+$.

By Proposition 0.4, under $\aleph_1 < \mathfrak{b}$ (just add $\aleph_2$ Hechler reals), there are no Jakovlev spaces of cardinality $\aleph_1$. However, we prove in the last section, Theorem 2.1, that Jakovlev spaces of size $\aleph_1$ always exist in the extension of any universe by adding $\aleph_1$ Cohen reals (note that $\mathfrak{b} = \aleph_1$ there). this shows striking dissimilarity between weak first countability and first countability, because the size of a first countable compact space can be only countable or continuum.

Do Jakovlev spaces of size $\mathfrak{b}$ always exist? Not necessarily: Alan Dow has a model ([5], Theorem 4.6) in which $\aleph_1 = \mathfrak{b} < \aleph_2 = \mathfrak{s} = \mathfrak{a} = \mathfrak{c}$ and all compact sequential spaces of weight $\mathfrak{b}$ are Frechet-Urysohn, hence no Jakovlev spaces of size $\mathfrak{b}$. Together with our result, this gives consistency and independence of the existence of Jakovlev spaces of size $\mathfrak{b}$.

In the last section we prove (for example) that if CH holds in the ground model then there are generic extensions that add no new reals, in which $2^{\aleph_1}$ is arbitrarily high and there exists a Jakovlev space of size $2^{\aleph_1}$.

Still, many tantalizing questions remain unanswered:

Question 0.5. *Is yes to Question 0.1 consistent? At least, is it consistent that there are no Jakovlev spaces?*

Question 0.6. *Is there a ZFC bound on the sizes of Jakovlev spaces? Is $2^{\mathfrak{b}}$ such a bound?*

In connection with this question, the referee points out that the stepping up techniques of [10] may prove to be relevant.

Acknowledgments

The authors are very grateful to Alexander Arhangelskii whose encouragement and interest were crucial in bringing this work to life.

The second author is especially grateful to Menachem Kojman for inviting him to Ben-Gurion University for the duration of writing of this paper, and wants to express his appreciation of extraordinary friendliness of the BGU staff.

1. Jakovlev spaces

We begin with definitions of pre-Jakovlev and Jakovlev spaces. As a preparation we define spaces of a spcific form that will be useful here—J spaces.

Let κ be some ordinal, and L_n for $n \in \omega$ be disjoint copies of κ. We call L_n the n-th "line". For every $n \in \omega$, $\alpha_n : L_n \to \kappa$ is a bijection which sets the correspondence between the n-th line and κ. We write $\alpha(p)$ rather than $\alpha_n(p)$ for $p \in L_n$ when the identity of n is obvious. So, for $p \in L_n$, $\alpha(p) \in \kappa$ is the "ordinal of p".

We say (in this paper) that a topological space with universe $L = \bigcup_{n\in\omega} L_n$ as above is a "J-space over L" if each point $p \in L$ has a countable neighborhood base of compact, clopen, countable sets that "look down and to the left". By this we mean that there are clopen (closed and open) sets B^p_n for $n \in \omega$ such that

(1) $B^p_0 \supseteq \cdots B^p_n \supseteq B^p_{n+1} \cdots$ form a neighborhood base of p, and
(2) if $p \in L_m$ and $x \in B^p_0$ then either $x = p$ or else $x \in L_k$ for some $k < m$ and $\alpha(x) < \alpha(p)$.
(3) Each B^p_n is compact and countable.

Thus a J-space is scattered, Hausdorff, first-countable, zero-dimensional, locally compact, and locally countable.

Now we define pre-Jakovelv spaces.

Definition 1.1. *Let κ be some ordinal, we say that a topological space K is pre-Jakovlev built over κ ("of cardinality κ" when κ is a cardinal) iff the following hold: For some disjoint union $L = \bigcup_{n\in\omega} L_n$ of copies of κ K is a J-space over L so that every infinite vertical set has an accumulation point. (A subset of L is said to be "vertical" iff it is a subset of some L_n.)*

Definition 1.2. *Let K be a pre-Jakovlev space built over some uncountable cardinal κ. The Jakovlev space $K^* = K \cup \{*\}$ is defined by the addition of a special point $*$ and the declaration that each open neighborhood of $*$ has the form $\{*\} \cup B$ where $B \subseteq K$ is open and includes for some $k \in \omega$ the set $\bigcup_{n\geq k} L_n$.*

Lemma 1.3. *If K is a pre-Jakovlev space built over an uncountable cardinal κ, and K^* is the Jakovlev space as defined above, then K^* is a compact, Hausdorff, zero-dimensional, weakly first-countable, not first-countable, scattered space of scattered height $\omega + 1$, and of sequential order 2.*

Proof. Since the pre-space K is Hausdorff, and as points in K have clopen neighborhoods, it is evident that K^* is Hausdorff. Indeed, the separation of any two points in K is assumed and the separation of $*$ from a point in K is a consequence of the assumption that points in K have clopen neighborhoods that lie entirely in the union of finitely many lines of K.

We next prove that every open neighborhood of the special point $*$ is co-countable. Let $G \subseteq K^*$ be an open neighborhood of $*$. Then by definition $G \cap K$ is open. We assume that the closed set $C = K \setminus G$ is uncountable and obtain a contradiction. Necessarily there is some $n \in \omega$ such that $C \cap L_n$ is uncountable, but there are only finitely such indices. Let n be the maximal one. Let A be the collection of accumulation points of $C \cap L_n$ in K. ($a \in A$ iff $a \in K$ and every neighborhood of a has an infinite intersection with $C \cap L_n$.) Since C is closed, A is a subset of C, and by property 2(b) every point of A lies in L_k for some $k > n$. It follows from the maximality of n that A is countable. For every $a \in A$, B^a_0 is an open and countable neighborhood of a. It follows that the uncountable set $(C \cap L_n) \setminus \bigcup_{a \in A} B^a_0$ has no accumulation point in K, which is a contradiction to property 3.

We now argue for compactness of K^*. Since every open neighborhood of $*$ is co-countable, the space K^* is Lindelof. But every infinite set has an accumulation point, because if $X \subset K^*$ is infinite then either $X \cap L_m \neq \emptyset$ for infinitely many indices m (and then $*$ is an accumulation point of X), or else $X \cap L_n$ is infinite for some n (and then, by property 3, X has an accumulation point in K). A Lindelof space in which every infinite set has an accumulation point is compact. So K^* is compact.

An uncountable Jakovlev space is not first countable at the special point $*$. This follows directly from our observation that the complement of every open neighborhood of $*$ is countable, and so the intersection of countably many neighborhoods of $*$ is always infinite. Moreover, a moment's consideration convinces that the point $*$ has a local base of a particularly simple form

$$\mathcal{B} = \{K^* \setminus \bigcup_{p \in F} B^p_0 \mid F \in [K]^{<\omega}\}.$$

Since $\mathcal{B}$ consists of clopen sets, K^* is zero-dimensional. $\square$

1.1. Jakovlev spaces of cardinality $\mathfrak{c}$, assuming $\mathfrak{b} = \mathfrak{c}$.

Theorem 1.4. *Suppose that $\mathfrak{b} = \mathfrak{c}$. Then there exists a Jakovlev space $J^*_\mathfrak{c}$ of cardinality $\mathfrak{c}$.*

Proof. Assume $\mathfrak{b} = \mathfrak{c}$. By the previous lemma, it is enough to construct a pre-Jakovlev space J of cardinality $\mathfrak{c}$. Let L_n, for $n \in \omega$, be pairwise disjoint copies of $\mathfrak{c}$, and for $\xi < \mathfrak{c}$ let $L_n \restriction \xi$ be the set of first ξ members of L_n. That is, $L_n \restriction \xi = \{a \in L_n \mid \alpha(a) < \xi\}$. Define $J \restriction \xi = \bigcup_{n\in\omega} L_n \restriction \xi$. Fix an enumeration $\{F_\beta \mid \beta < \mathfrak{c}\}$ of all vertical sets of order type ω (in the ordering of each L_n induced by the correspondence α with $\mathfrak{c}$). We may assume that F_β is always a subset of $J \restriction \beta$. We shall define by induction on $\xi < \mathfrak{c}$ a topology on $J \restriction \xi$ by defining, for every point $p \in J \restriction \xi$, a neighborhood base $\langle B_i^p \mid i \in \omega\rangle$ consisting of compact, clopen, countable sets that form a J-space over $J \restriction \xi$. The final topology on J is determined by the local neighborhood bases thus defined. Observe that the neighborhood base of a point p in $J \restriction \xi_0$ remains the base of p in any of the spaces $J \restriction \xi$ for $\xi \geq \xi_0$. So for $\xi' < \xi$, $J \restriction \xi'$ is an open subspace of $J \restriction \xi$. Hence every compact subset of $J \restriction \xi'$ remains compact in $J \restriction \xi$, and in particular the neighborhood base of a point in $J \restriction \xi'$ consists of compact, clopen sets that remain compact, clopen in any later $J \restriction \xi$.

At limit stages ξ there is thus nothing to define, and the topological space $J \restriction \xi$ is the union of all spaces $J \restriction \xi'$ for $\xi' < \xi$.

Suppose that the bases for points in $J \restriction \xi$ are defined. We shall define the topological space $J \restriction \xi + 1$ by defining the neighborhoods of the points in $J \restriction \xi + 1 \setminus J \restriction \xi$. There are ω points in this level, formed by the ξ-th point taken from each L_n. Say $q_n \in L_n$ is the ξ-th point in L_n. With the possible exception of one point, all q_k shall be defined to be isolated (a point q is isolated when $\{q\}$ is open). The exception is determined by considering $F = F_\xi$. In case there is some n such that $F \subset L_n$ and F has no accumulation point in the space $J \restriction \xi$ so far defined, we define q_{n+1} as a limit of F in the following fashion.

Let $\langle f_i \mid i \in \omega\rangle$ be an increasing enumeration of F. Our intension is to find for each $i \in \omega$ a basic neighborhood $B_{k_i}^{f_i}$ of f_i such that the family $\{B_{k_i}^{f_i} \mid i \in \omega\}$ is discrete in $J \restriction \xi$. (That is, every $p \in J \restriction \xi$ has a neighborhood that intersects at most one $B_{k_i}^{f_i}$.) When we achieve this, we will make $q = q_{n+1}$ the one point compactification of this family by setting

$$B_m^q = \{q\} \cup \bigcup_{i\geq m} B_{k_i}^{f_i}.$$

The Hausdorff property of $J \restriction \xi + 1$ will follow from the discreetness of our family.

There are basic neighborhoods of each f_i that are pairwise disjoint (these can be defined inductively). Consider an arbitrary point $p \in (J \restriction \xi) \setminus F$. Since F has no accumulation points, we can fix some clopen neighborhood G_p of p such that $G_p \cap F = \emptyset$. We define a function

$g_p : \omega \longrightarrow \omega$ by the requirement that for every $i \in \omega$

$$G_p \cap B^{f_i}_{g_p(i)} = \emptyset.$$

The number $g_p(i)$ exists because f_i is not in G_p, and hence some set in the neighborhood basis of f_i is included in the complement of the closed set G_p. The collection $\{g_p \mid p \in (J \restriction \xi) \setminus F\}$ has cardinality $|\xi| < \mathfrak{c}$, and hence, by assumption $\mathfrak{b} = \mathfrak{c}$, there exists a function $g \in \omega^\omega$ such that $g_p <^* g$ for every $p \in (J \restriction \xi) \setminus F$. We may assume that the neighborhoods $\{B^{f_i}_{g(i)} \mid i \in \omega\}$ are pairwise disjoint. (Recall that the points f_i have pairwise disjoint clopen neighborhoods; take $g(i)$ sufficiently big to enter these neighborhoods.) Now define the neighborhoods of $q = q_{n+1} \in L_{n+1}$ by the formula

$$B^q_m = \{q\} \cup \bigcup_{i \geq m} B^{f_i}_{g(i)}.$$

Each B^q_m is countable and open in $J \restriction \xi + 1$ (a union of open and countable sets), but it is also closed. Indeed, for every $p \in (J \restriction \xi) \setminus B^q_m$ there are two possibilities. If $p \in F$, then p is one of $f_0, \ldots, f_{m-1}$ (say $p = f_j$) and then $B^{f_j}_{g(j)}$ is disjoint from B^q_m. If $p \notin F$, then G_p meets only finitely many $B^{f_i}_{g(i)}$ (since $g_p <^* g$) and hence $(G_p - B^q_m)$ is a neighborhood of p disjoint from B^q_m.

This ends the definition of $J = \bigcup_{\xi < \mathfrak{c}} J_\xi$. We argue that every infinite, vertical set has an accumulation point. It is enough to consider vertical sets of order-type ω. If $F \subset L_n$ is such a vertical set then $F = F_\xi$ for some ξ that is not smaller than the supremum of F. At stage ξ, F was considered. If it had an accumulation point at that stage, then that point remained an accumulation point. Otherwise, a limit point q was added in that stage.

Finally, the special point $*$ is added, completing the construction of the Jakovlev space $J^\star_\mathfrak{c}$. □

1.2. **Jakovlev spaces of cardinality $\mathfrak{c}^+$.**

Theorem 1.5. *Assume $\mathfrak{b} = \mathfrak{c}$. Then there exists a compact, Hausdorff, weakly first-countable space of cardinality $\mathfrak{c}^+$. In fact there is a Jakovlev space of cardinality $\mathfrak{c}^+$.*

Proof. By Lemma 1.3, we only need to construct a pre-Jakovlev space of cardinality $\mathfrak{c}^+$. Let $\kappa = \mathfrak{c}^+$ be the successor of the continuum; the construction is carried out in κ stages, each stage consisting of $\mathfrak{c}$ steps. Let L_n be pairwise disjoint copies of κ for every $n \in \omega$. Fix a bijection $\alpha : L_n \to \kappa$, and let q^ζ_n be the ζ-th member of L_n, that is such that $\alpha(q^\zeta_n) = \zeta$. As before, the space J is defined by assigning inductively for every q_n^ξ $(n \in \omega)$ clopen neighborhoods. The construction, however, is carried out somewhat differently.

(1) For every $\epsilon \in \kappa$ define $\gamma_\epsilon = \mathfrak{c} \cdot \epsilon$. Here $\mathfrak{c} \cdot \epsilon$ is the ordinal product, so that γ_ϵ is that ordinal consisting of ϵ copies of $\mathfrak{c}$.
(2) For every $\epsilon \in \kappa$ define the set of the ϵ stage:
$$K(\epsilon) = \{q_n^{\gamma_\epsilon+\xi} \mid \xi < \mathfrak{c} \text{ and } n \in \omega\}.$$
And, for $\zeta < \mathfrak{c}$, define $K(\epsilon) \restriction \zeta = \{q_n^{\gamma_\epsilon+\xi} \mid \xi < \zeta \text{ and } n \in \omega\}$. So $K(\epsilon)$ contains $\mathfrak{c}$ elements which we intend to be the ϵ-th copy of some pre-Jakovlev space of cardinality $\mathfrak{c}$.
(3) For every $\xi < \kappa$ define $L_n \restriction \xi = \{a \in L_n \mid \alpha(a) < \xi\}$, and $J \restriction \xi = \bigcup_{n\in\omega} L_n \restriction \xi$.
So $J \restriction \gamma_{\epsilon+1} = (J \restriction \gamma_\epsilon) \cup K(\epsilon)$ and, for limit ϵ,
$$J \restriction \gamma_\epsilon = \bigcup_{\epsilon'<\epsilon} J \restriction \gamma_{\epsilon'}.$$

The pre-Jakovlev space that we construct is $J = J \restriction \kappa = \bigcup_{\epsilon<\kappa} J \restriction \gamma_\epsilon$. The construction is by stages. In each stage $\epsilon < \kappa$ we define $K(\epsilon)$ in $\mathfrak{c}$ steps. At the ξ-th step, the neighborhoods of the points $q_n^{\gamma_\epsilon+\xi}$, $n \in \omega$ are defined. This construction is similar to the one in Theorem 1.4 (for cardinality $\mathfrak{c}$), and so we mainly describe the needed changes. We continue to denote with B_n^p the n-th neighborhood of p. It is, again, compact, clopen, and countable.

Since for limit ordinals ϵ, the topological space $J \restriction \gamma_\epsilon$ is defined as the union of the previous spaces $J \restriction \gamma_{\epsilon'}$ for $\epsilon' < \epsilon$, the inductive mission is to define the topology on $J \restriction \gamma_{\epsilon+1}$ from that on $J \restriction \gamma_\epsilon$.

We make two induction assumptions:

(1) The first inductive assumption is that for every *successor* ϵ the space $J \restriction \gamma_\epsilon$ is a pre-Jakovlev space.
It follows then that if ϵ is a limit ordinal in κ of uncountable cofinality, then $J \restriction \gamma_\epsilon$ is also a pre-Jakovlev space. The main point is that $J \restriction \gamma_\epsilon$ is the union of previously constructed $J \restriction \gamma_{\epsilon'}$ for $\epsilon' < \epsilon$, and so every countably infinite vertical set is included in some $J \restriction \gamma_{\epsilon'}$ for ϵ' a successor ordinal.
In case ϵ is a limit ordinal of countable cofinality, $J \restriction \gamma_\epsilon$ is a J-space, but obviously not a pre-Jakovlev space, since cofinal sequences have no accumulation points. This will be taken care of at the next stage of the construction (which endures $\mathfrak{c}$ steps, and is of the same character as the construction for Theorem 1.4).
(2) The second inductive assumption is that if $p \in K(\epsilon)$, then for every $\epsilon' < \epsilon$ there is $n \in \omega$ such that $B_n^p \cap J \restriction \gamma_{\epsilon'} = \emptyset$

If $\epsilon < \kappa$ is a successor ordinal or a limit ordinal of uncountable cofinality (or zero), then the topology of $K(\epsilon)$ is defined as in the previous section as a Jakovlev space of cardinality $\mathfrak{c}$, and then $J \restriction \gamma_{\epsilon+1} = J \restriction \gamma_\epsilon \cup K(\epsilon)$.

So suppose that $\epsilon < \kappa$ is a limit ordinal of countable cofinality and $J \restriction \gamma_\epsilon = \bigcup_{\epsilon' < \epsilon} J \restriction \gamma_{\epsilon'}$ has been defined. We must define the topology on $J \restriction \gamma_{\epsilon+1}$ by defining the clopen neighborhoods B^p_n for points $p \in K(\epsilon) = J \restriction \gamma_{\epsilon+1} \setminus J \restriction \gamma_\epsilon$. We fix an enumeration $\{F_\beta \mid \beta < \mathfrak{c}\}$ of all countable, vertical subsets of $J \restriction \gamma_{\epsilon+1}$. But actually we are only interested in two types of such sets.

(1) A countable, infinite vertical set $F \subset L_n$ of order-type ω is "of the first type" iff F is an unbounded subset of $L_n \restriction \gamma_\epsilon$ of order-type ω in the well-ordering of L_n.
(2) F is "of the second type" iff it is included in $K(\epsilon)$.

We are not interested in infinite vertical sets that are bounded in γ_ϵ because they are already in some $J \restriction \epsilon'$ for a successor $\epsilon' < \epsilon$ and thus have an accumulation point by the inductive assumption.

Every $q \in K(\epsilon)$ has the form $q^{\gamma_\epsilon+\xi}_n$, for some $\xi < \mathfrak{c}$, and the definition of the neighborhoods of q is by induction on ξ. Suppose that $J \restriction \gamma_\epsilon \cup K(\epsilon) \restriction \xi$ has been defined. At the step ξ we define the neighborhoods of the ω points at the level ξ, namely $q^{\gamma_\epsilon+\xi}_n$ for $n \in \omega$. For this we consider the set F_ξ enumerated at this step. In case F_ξ is of the second type, the construction proceeds just as in the Jakovlev construction of the previous section.

A slightly more complex case is when F_ξ is of the first type. The problem is that $J \restriction \gamma_\epsilon$ may already have size $\mathfrak{c}$, so that it is not possible to dominate all the functions that its points define. Fortunately there is no need for this, by the second inductive property.

How does this help? Suppose that $F = F_\xi$ is of the first type, and $F \subset L_n$ has no accumulation point in the space $J \restriction \gamma_\epsilon + \xi$ so far constructed. Enumerate $F = \{f_i \mid i \in \omega\}$ in increasing order (i.e., $\alpha(f_i) < \alpha(f_{i+1})$). By diluting F we may assume (for simplicity of expression) that each f_i lies in a separate $K(\epsilon')$. That is, $f_i \in K(\epsilon_i)$ where $\epsilon_i < \epsilon_{i+1}$. Choose $j(i) \in \omega$ such that the neighborhoods $B^{f_{i+1}}_{j(i+1)} \cap J \restriction \gamma_{\epsilon_i} = \emptyset$. It is here that the inductive hypothesis is used. Since the sequence F is unbounded in γ_ϵ, we are sure that the family $\{B^{f_i}_{j(i)} \mid i \in \omega\}$ is discrete with respect to the points in $J \restriction \gamma_\epsilon$. But the collection of remaining points in $K(\epsilon) \restriction \xi$ is a set of cardinality $< \mathfrak{c}$, and hence the assumption that $\mathfrak{b} = \mathfrak{c}$ can be used as in the proof of Theorem 1.4 to refine the neighborhoods $\{B^{f_i}_{j(i)} \mid i \in \omega\}$ and to obtain a family discrete in $J \restriction \gamma_\epsilon \cup K(\epsilon) \restriction \xi$, as required. □

2. A Jakovlev space of cardinality $\aleph_1 < 2^{\aleph_0}$

In this section we show the consistency of $2^{\aleph_0} > \aleph_1$ with the existence of a Jakovlev space of cardinality $\aleph_1$. We prove that such a space exists in any extension of the universe obtained by adding (at least) ω_1 Cohen reals.

Theorem 2.1. *If $\overline{r} = \langle r_\alpha \mid \alpha < \omega_1 \rangle$ is a generic sequence of ω_1 Cohen reals over the ground model V, then in $V[\overline{r}]$ there exists a Jakovlev space of cardinality $\aleph_1$.*

Proof. We work in $V[\overline{r}]$. The idea is this. The space J is constructed as in section 1, and we use the same terminology used there with ω_1 replacing $\mathfrak{c}$. In particular, we intend to construct a pre-Jakovlev space by defining for every point $p \in J_{\omega_1}$ a countable base B^p_n, $n \in \omega$, consisting of compact, clopen, countable sets that satisfy properties 2(a)–2(c). But now there is no enumeration of all possible countable vertical sets, since the continuum may be larger than ω_1. Instead, we let the generic real r_α choose the omega sequence at the α-th step of the construction. Since r_α is generic, we will be able to prove that every countable vertical set has an accumulation point.

Fix two functions, $\beta : \omega_1 \to \omega_1$ (with $\beta(\xi) \leq \xi$) and $n : \omega_1 \to \omega$, such that for every $\beta_0 \in \omega_1$ and $n_0 \in \omega$ there are uncountably many ordinals $\xi \in \omega_1$ with $\beta(\xi) = \beta_0$ and $n(\xi) = n_0$. At step $\xi < \omega_1$ of the construction, we assume that $J \restriction \xi = \bigcup_{n\in\omega} L_n \restriction \xi$ is defined. That is, we assume that for every p with $\alpha(p) < \xi$ neighborhoods B^p_n that satisfy properties (1)–(3) of J-spaces have been defined. In particular, B^p_n is a clopen, compact, and countable neighborhood of p. Consider $\beta_0 = \beta(\xi)$ and $n_0 = n(\xi)$. We shall try to define an omega sequence $F \subset L_{n_0} \restriction \beta_0$ that has no accumulation points in J_ξ.

For this we define a forcing poset P as follows. A condition $p \in P$ has the form $p = (\langle b_0, \dots, b_{k-1} \rangle, X)$ where $b^p = \langle b_0, \dots, b_{k-1} \rangle$ is a finite sequence such that its members b_i all lie in $L_{n_0} \restriction \beta_0$ and are enumerated in the increasing order (i.e., $\alpha(b_0) < \alpha(b_1) < \dots < \alpha(b_{k-1})$, and $X^p = X$ is a finite subset of $J \restriction \xi$. As for the ordering, we define $p_1 \leq p_2$ (read: p_2 extends p_1) iff b^{p_2} is an end-extension of b^{p_1} (say $b^{p_2} = b^{p_1} \frown \langle b_k, \dots, b_{m-1} \rangle$) and for every $x \in X^{p_1}$, $\{b_k, \dots, b_{m-1}\} \cap B^x_0 = \emptyset$. That is, a condition p_1 ensures that the continuation of its ω sequence is disjoint from the first basic neighborhood of each point in X^{p_1}.

It may well be the case that some condition p has no extensions that continue the b^p sequence. If this happens then $X = X^p$ has the property that $\bigcup\{B^x_0 \mid x \in X\}$ covers a final segment of $L_{n_0} \restriction \beta_0$. In this case we should not worry. Since each B^x_0 is compact, every infinite sequence from B_0^x has an accumulation point and hence every sequence unbounded in $L_{n_0} \restriction \beta_0$ has an accumulation point.

So assume that for every finite set $X \subset J \restriction \xi$, $L_{n_0} \restriction \beta_0 \setminus \bigcup\{B_0^x \mid x \in X\}$ is unbounded in $L_{n_0} \restriction \beta_0$. In this case the generic filter G produces an increasing ω sequence, $F_G = \langle b_0, b_1, \ldots\rangle$, of members of $L_{n_0} \restriction \beta_0$ that has finite intersection with any B_0^p for $p \in J \restriction \xi$. So F_G has no accumulation points in $J \restriction \xi$. Obviously, the poset P is countable, and is thence isomorphic to the Cohen forcing for adding a real. Thus r_ξ can be viewed as $V[\langle r_\alpha \mid \alpha < \xi\rangle]$ generic, yielding a generic sequence R_ξ $(= F_G$ above). The construction continues as in Section 1 by defining a discrete family of compact neighborhoods of the points of R_ξ, and using them to define a neighborhood base for $q^\xi_{n_0+1}$. (The fact that $J \restriction \xi$ is countable is used here to find a function that dominates countably many functions.)

Now we must prove property (3), namely that every infinite vertical subset of J has an accumulation point. Suppose that $A \subset L_{n_0}$ is such a set (of order-type ω) and let $\beta_0 = \sup A$. There is some ordinal $\xi < \omega_1$ such that $A \in V[\langle r_{\xi'} \mid \xi' < \xi\rangle]$, and such that $\beta(\xi) = \beta_0$ and $n(\xi) = n_0$. If a final segment of $L_{n_0} \restriction \beta_0$ is covered by some finite union of basic neighborhoods B_0^p $(p \in J \restriction \xi)$, then A has an accumulation point in $J \restriction \xi$, being almost included in a finite union of compact sets. Otherwise, a density argument shows that the generic ω sequence constructed at the ξ-th step intersects A infinitely often. Thus $q^\xi_{n_0+1}$ is an accumulation point of A. $\square$

3. Jakovlev spaces obtained by countably closed forcing

We prove in this section that if, for example, κ is a cardinal with $\mathrm{cf}(\kappa) \geq \aleph_1$. Then there is a countably closed, $\mathfrak{c}^+$-c.c. forcing poset P such that in any forcing extension there exists a Jakovlev space of size κ. More generally we have the following.

Theorem 3.1. *Suppose that $\mathfrak{b} = \mathfrak{c}$ and $\mathfrak{c}^{<\mathfrak{c}} = \mathfrak{c}$. Let $\kappa \geq \mathfrak{c}$ be a cardinal such that $\kappa^{<\mathfrak{c}} = \kappa$ and $\mathrm{cf}(\kappa) > \mathfrak{c}$. There is then a $\mathfrak{b}$-closed, $\mathfrak{b}^+$-c.c. forcing poset P such that in any forcing extension made via P there exists a Jakovlev space of size κ (cardinals are not collapsed and $2^\mathfrak{c} \geq \kappa$ in the extension).*

Proof. Let L_n for $n \in \omega$ be some fixed, pairwise disjoint copies of κ, and $L = \bigcup_{n\in\omega} L_n$. Let $\alpha_n : \kappa \to L_n$ be a one-to-one correspondence. For every $S \subset \kappa$ with $|S| < \mathfrak{c}$ let $S_n = \alpha_n[S]$ be the corresponding copy of S in L_n. We shall write $\alpha[S] = \langle S_n \mid n \in \omega\rangle$. So $\alpha[S]$ is an ω sequence of copies of some ordinal strictly smaller than $\mathfrak{c}$, and hence the notion of a J-space over $\alpha[S]$ is applicable.

A condition $p \in P$ is a topological space $p = \langle J^p, \mathcal{T}^p \rangle$ where $J^p \subset L$ is of the form $\alpha[S]$ for some $S \in [\kappa]^{<\mathfrak{c}}$ and $\mathcal{T}^p$ is a neighborhood system for points of J^p, so that p is a J-space over J^p.

The extension relation is: $p_1 \leq p_2$ (p_2 extends p_1) if $J^{p_1} \subseteq J^{p_2}$ and $\mathcal{T}^{p_2}$ retains the neighborhood systems of $\mathcal{T}^{p_1}$ for points in J^{p_1}.

It is obvious that P is $\mathfrak{c}$-closed. If $\langle p_i \mid i \in \gamma \rangle$ is an increasing sequence of conditions in P of length $\gamma < \mathfrak{c}$, then the union can be defined and it is an upper bound of the sequence.

Since $\mathfrak{c}^{<\mathfrak{c}} = \mathfrak{c}$, P satisfies the $\mathfrak{c}^+$-c.c., namely every antichain is of cardinality $\leq \mathfrak{c}$. Thus forcing with P collapses no cardinals. The size of P is $k^{<\mathfrak{c}}$; so if $\mathrm{cf}(\kappa) > \mathfrak{c}$ and $\lambda^{<\mathfrak{c}} < \kappa$ for all $\lambda < \kappa$ then $2^{\mathfrak{c}} = \kappa$ in the extension, and, in V^P, $2^{\mathfrak{c}} \geq \kappa$.

It is easy to see that for any condition $p \in P$ and set $X \in [L]^{<\mathfrak{c}}$ there exists an extension $p' \geq p$ such that X is a subset of the universe of p'. (Define the new points to be isolated.)

Therefore, if G is a generic filter over P, the topology defined by $\bigcup G$ is a J-space over L. We shall prove that it is a pre-Jakovlev space, and hence there exists in $V[G]$ a Jakovlev space of cardinality κ. We must prove that every countable subset C of L_n has an accumulation point in $\bigcup G$. This is a density argument. Let $p \in P$ be any condition and let C be any countable subset of L_n (we may assume that C is in V). Extending p if necessary, we may further assume that C is a subset of the universe of p. Say $C = \{c_i \mid i \in \omega\}$ is an increasing enumeration. The argument now is exactly as in the proof of Theorem 1.4. Consider the topological space p. If C has already an accumulation point in p then we are done. Otherwise choose some new point (not in J^p) in L_{n+1} above all members of C. Now use the fact that J^p has size $< \mathfrak{b}$ to define clopen neighborhoods B_i of c_i such that for every point $x \in J^p$ there exists a neighborhood H of x that has non-empty intersection with only a finite number of B_i's.

$\square$

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