Integration

Notes by Walter Noll, 1986, rev. 1989

1 Function spaces and algebras

Let S be any non-empty set. The set

$$\operatorname{Fun} S := \operatorname{Map}(S, \mathbb{R})$$

of all real-valued functions on S acquires the structure of a linear space when addition and scalar multiplication are defined value-wise. Fun$S$ has also the structure of a ring whose multiplication is defined by value-wise multiplications of functions.

Fun$S$ has also a natural order $\leq$, which is defined by

$$f \leq g \;:\Leftrightarrow f(x) \;\leq g(x) \quad \text{for all } x \in S. \tag{1.1}$$

If S contains more than one element, this order on Fun$S$ is only partial, i.e. not total. If $f, g \in \operatorname{Fun} S$, then $\sup\{f, g\} \in \operatorname{Fun} S$ and $\inf\{f, g\} \in \operatorname{Fun} S$ exist, i.e. the order $\leq$ in Fun$S$ is a lattice-order. We have

$$\left.\begin{array}{rcl} (\sup\{f,g\})(x) & = & \max\{f(x), g(x)\} \\ (\inf\{f,g\})(x) & = & \min\{f(x), g(x)\} \end{array}\right\} \text{ for all } x \in S. \tag{1.2}$$

For every $f \in \operatorname{Fun} S$ we put

$$f^+ := \sup\{f, 0\}, \quad f^- := -\inf\{f, 0\}. \tag{1.3}$$

We then have

$$f = f^+ - f^-, \quad |f| = f^+ + f^-, \tag{1.4}$$

where $|f| \in \operatorname{Fun} S$ is defined by $|f|(x) := |f(x)|$ for all $x \in S$.

The *support* $\operatorname{Supt} f$ of a function $f \in \operatorname{Fun} S$ is the subset of S given by

$$\operatorname{Supt} f := \{x \in S \mid f(x) \neq 0\}. \tag{1.5}$$

We note that $\operatorname{Supt}(f+g) \subset \operatorname{Supt} f \cup \operatorname{Supt} g$, $\operatorname{Supt} fg = \operatorname{Supt} f \cap \operatorname{Supt} g$, and $\operatorname{Supt} |f| = \operatorname{Supt} f$ for all $f, g \in \operatorname{Fun} S$.

A subspace of Fun$S$ is called a **function space**. To decide whether a non-empty subset of Fun$S$ is a function space, one has to ascertain whether it is stable under addition and scalar multiplication. A subspace of Fun$S$ that is stable under multiplication is called a **function algebra**. Examples of function algebras as the set of all continuous functions in Fun$S$ and the set of all functions in Fun$S$ that have a bounded range. The set of all constants is a function algebra that is naturally isomorphic to the real field $\mathbb{R}$. The isomorphism associates with each constant its value. Ordinarily, we use the same symbol for a number in $\mathbb{R}$ and the corresponding constant in Fun$S$.

Let I be a genuine interval. If $f \in \text{Fun}\, I$, we denote the set of discontinuities of f by

$$\text{Dis} f := \{t \in I \mid f \text{ is not continuous at } t\} \tag{1.6}$$

We note that $\text{Dis}(f+g) \subset \text{Dis} f \cup \text{Dis}\, g$, $\text{Dis}(fg) \subset \text{Dis} f \cup \text{Dis}\, g$ and $\text{Dis}|f| \subset \text{Dis} f$.We say that $f \in \text{Fun}\, I$ is **nearly continuous** if $\text{Dis} f$ is finite.

The set of all functions in Fun$I$ with bounded range, the set of all functions in Fun$I$ with bounded support, and the set of all nearly continuous functions in Fun$I$ each are function algebras. The intersection of these three function algebras is again a function algebra, which we denote by Bbnc$I$, so that

$$\text{Bbnc}\, I = \text{le}\{f \in \text{Fun}\, I \mid \text{Rng}\, f \text{ is bounded, } \text{Supt}\, f \text{ is bounded, } \text{Dis} f \text{ is finite}\} \tag{1.7}$$

It is easily seen that if $f, g \in \text{Bbnc}\, I$ then also $\sup\{f,g\},\ \inf\{f,g\} \in \text{Bbnc}\, I$. In particular, $f \in \text{Bbnc}\, I$ implies $f^+, f^-, |f| \in \text{Bbnc}\, I$. We note that the non-zero constants do *not* belong to Bbnc$I$ if I is not bounded.

Given $f \in \text{Fun}\, \mathbb{R}$ and $s \in \mathbb{R}$, we call $f \circ (\iota - s)$ the **translation** of f by the amount s. Intuitively, if $s > 0$, the graph of $f \circ (\iota - s)$ is obtained by shifting the graph of f to the right by the distance s. Let A be a subset of $\mathbb{R}$. Recall the the characteristic function $\text{ch}_A \in \text{Fun}\, \mathbb{R}$ of A is given by

$$\text{ch}_A(t) := \left.\begin{array}{ll} 1 & \text{if} \quad t \in A \\ 0 & \text{if} \quad t \in \mathbb{R}\backslash A \end{array}\right\} \tag{1.8}$$

Given $s \in \mathbb{R}$ we have

$$\text{ch}_A \circ (\iota - s) = \text{ch}_{(A+s)}, \text{ where } A + s := \{t + s \mid t \in A\}. \tag{1.9}$$

We now deal with Bbnc$\mathbb{R}$. The characteristic function ch_I of an interval I belongs to Bbnc$\mathbb{R}$ if and only if I is bounded. If $f \in$ Bbnc $\mathbb{R}$, it is obvious that then also $f \circ (\iota - s) \in$ Bbnc $\mathbb{R}$ for every $s \in \mathbb{R}$. Hence we can define, for each $s \in \mathbb{R}$

$$T_s : \text{ Bbnc}\,\mathbb{R} \rightarrow \text{ Bbnc}\,\mathbb{R} \quad \text{by} \quad T_s f := f \circ (\iota - s). \tag{1.10}$$

It is easily seen that T_s is linear, i.e. that $T_s \in$ Lin Bbnc $\mathbb{R}$ for all $s \in \mathbb{R}$ and isotone, i.e. that

$$f \leq g \;\Rightarrow\; T_s f \leq T_s g \quad \text{for all } s \in \mathbb{R}. \tag{1.11}$$

2 Definition of the integral

We wish to associate with every function $f \in$ Bbnc $\mathbb{R}$ a number Igl f, the *integral* of f. Thus, Igl should be a *functional* on Bbnc$\mathbb{R}$ i.e., a mapping from Bbnc $\mathbb{R}$ to $\mathbb{R}$. Intuitively, if $f \geq 0$, the Igl f should be the area measure of the region between the graph of f and the horizontal axis. If $f \in$ Bbnc $\mathbb{R}$ is arbitrary, we should have Igl $f =$ Igl $f^+ -$ Igl f^-.

We will give an indirect definition of the integral. First, we will say that **an** integral is a functional with certain properties, properties which reflect some of our intuitive notions about area measures. Then we will prove that there exists exactly one integral, which makes it then legitimate to refer to **the** integral.

Defintiion 1: *A functional* Igl : Bbnc$\mathbb{R} \rightarrow \mathbb{R}$ *is called an* **integral** *if*

(a) Igl *is linear*

(b) Igl *is isotone, i.e.* $f \geq g \implies$ Igl $f \geq$ Igl g.

(c) Igl *is translation invariant, i.e.*

$$\text{Igl} \circ T_s = \text{Igl} \quad \text{for all} \quad s \in \mathbb{R}$$

and

(d) Igl $\mathrm{ch}_{[0,1[} = 1$.

We note that if (a) is valid, then (b) is equivalent to (b′) $f \geq 0 \implies \text{Igl } f \geq 0$ for all $f \in$ Bbnc$\mathbb{R}$.

Theorem 2.1. *There is exactly one integral* Igl. *It has the property that*

(d$_1$) Igl ch$_I$ $= \sup I - \inf I$ *for all bounded non-empty intervals I.*

We defer the proof to the last section.

If $I := \{a\}$ is a singleton, (d$_1$) yields Igl $\text{ch}_{\{a\}} = 0$. If $h \in \text{Fun}\,\mathbb{R}$ has finite support, we have

$$h = \sum_{t \in \text{ Supt } h} h(t)\ \text{ch}_{\{t\}}.$$

Using the linearity of Igl it follows that Igl $h = 0$. Applying this result to the difference $h := f - g$ of two functions and using the linearity again, we obtain the following result:

Proposition 2.1. *If $f, g \in$* Bbnc $\mathbb{R}$ *agree on all but a finite number of points in $\mathbb{R}$, then Igl $f =$ Igl g.*

Definition 2: *Let $S \in$* Sub $\mathbb{R}$ *and $f \in$* Fun $\mathbb{R}$ *be given: if I is an interval included in S, we define $f|_I^0 \in$* Fun $\mathbb{R}$ *by*

$$f|_I^0(t) := \left\{ \begin{array}{lll} f(t) & \text{if} & t \in I \\ 0 & \text{if} & t \in \mathbb{R} \backslash I \end{array} \right\} \tag{2.1}$$

If $f|_I^0 \in$ Bbnc $\mathbb{R}$ *we define the* **integral of f over I** *by*

$$\int_I f := \text{ Igl } f|_I^0. \tag{2.2}$$

Remark 1: Assume that $a, b \in \mathbb{R}, a < b$, that f is a function whose domain includes $[a, b]$, and that $f|_{[a,b]}$ is continuous. By the Theorem on Attainment of Extrema, Rng $f|_{[a,b]} = f_>([a, b])$ is then a bounded set. Hence $f|_{[a,b]}^0 \in$ Bbnc $\mathbb{R}$ and $\displaystyle\int_{[a,b]} f$ is meaningful. ■

Remark 2: If K is an interval and if $f \in$ Bbnc K then, for every interval I included in K we have $f|_I^0 \in$ Bbnc $\mathbb{R}$ and hence $\displaystyle\int_I f$ is meaningful. ■

Proposition 2.2. *Let f be a function whose domain includes the interval I and assume that $f|_I^0 \in$ Bbnc$\,\mathbb{R}$. If $\mathcal{P}$ is a finite interval-partition of I, we then have $f|_J^0 \in$ Bbnc$\,\mathbb{R}$ for all $J \in \mathcal{P}$ and*

$$\int_I = \sum_{J\in\mathcal{P}} \int_J f. \tag{2.3}$$

Definition 3: *Given an interval I and $f \in$ Bbnc$\,I$, we define:*

$$\int_a^b f := \left\{ \begin{array}{ll} \displaystyle\int_{[a,b]} f & \text{if} \quad a \le b \\ -\displaystyle\int_{[b,a]} f & \text{if} \quad b < a \end{array} \right\} \text{ for all } a, b \in I. \tag{2.4}$$

It is clear that

$$\int_a^b f = -\int_b^a f \quad \text{for all} \quad a, b \in I, \tag{2.5}$$

and it follows from Prop. 2 that

$$\int_a^b f + \int_b^c f = \int_a^c f \quad \text{for all } a, b, c \in I. \tag{2.6}$$

3 Some theorems of integral calculus

In the previous section we already listed the basic properties (a), (b), (c), (b′), and (d_1) of the integral and we derived a few easy consequences. Here we derive some other important consequences. We assume that an Interval I and a function $f \in \operatorname{Fun} I$ are given.

Proposition 3.1. *If $f \in$ Bbnc$\,I$ then $|f| \in$ Bbnc$\,I$ and*

$$\left| \int_I f \right| \le \int_I |f|. \tag{3.1}$$

Proof The fact that $f \in$ Bbnc$\,I$ implies $|f| \in$ Bbnc$\,I$ was already mentioned in Sect. 1. Clearly we have $-|f| \le f \le |f|$. Since Igl is linear we have $\int_I (-|f|) = -\int_I |f|)$, and hence, since Igl is isotone, we obtain

$$-\int_I |f| \leq \int_I f \leq \int_I |f|$$

which is equivalent to the assertion. ■

Proposition 3.2. *If* $f \in$ Bbnc I *and* $a, b \in I$ *with* $a < b$, *then*

$$\inf f_>([a,b]) \leq \frac{1}{b-a} \int_a^b f \leq \sup f_>([a,b]). \tag{3.2}$$

Proof We have $\inf f_>([a,b]) \leq f|_{[a,b]} \leq \sup f_>([a,b])$. Hence, by the isotonicity and linearity of the itnegral, we get

$$\inf f_>([a,b]) \int_a^b 1 \leq \int_a^b f \leq \sup f_>([a,b]) \int_a^b 1. \tag{3.3}$$

Since $\displaystyle\int_a^b 1 =$ Igl $\mathrm{ch}_{[a,b]} = b - a$ by property (d_1), the assertion follows. ■

Proposition 3.3. *If* $f \in$ Bbnc I *then*

$$\int_I |f| = 0 \quad \Leftrightarrow \quad f \text{ has finite support.}$$

Proof If f has finite support, so has $|f|$, and $\displaystyle\int_| |f| = 0$ follows from the remarks after the Theorem in Sect. 2.

Assume now that f does not have finite support. Since Dis$f$ is finite and since I has at most two endpoints, we can choose an interior point $c \in I$ such that $|f(c)| > 0$ and such that f and hence $|f|$ is continuous at c. We choose $\alpha \in]0, |f(c)|$ (for example ($\alpha := \frac{1}{2}|f(c)|$) and determine $\delta \in \mathbb{P}^\times$ such that $]c-\delta, c+\delta[\subset I$ and

$$|f(c)| - |f||_{]c-\delta,c+\delta[} \leq |f(c)| - \alpha$$

so that

$$|f| \geq \alpha \text{ ch }_{]c-\delta,c+\delta[} |_I \ .$$

The isotonicity of the integral and (d_1) of Sect. 2 give

$$\int_I |f| \geq \int_I \alpha \ \mathrm{ch}_{]c-\delta,c+\delta[} = \alpha 2\delta > 0.$$

■

Proposition 3.4. *Let $c \in I$ be given. If $f|_{[a,b]} \in \mathrm{Bbnc}[a,b]$ for every $a, b \in I$ with $a < b$ then the function $F : I \to \mathbb{R}$ defined by*

$$F(t) := \int_c^t f \text{ for all } t \in I \tag{3.4}$$

is continuous.

Proof Let $a, b \in I$ with $a < b$ be given. It follows from (3.4), from (2.5) and (2.6) of Sect. 2, and from Props. 3 and 4 that

$$|F(t) - F(s)| = \left| \int_s^t f \right| \leq \left| \int_x^t |f| \right| \leq |t - s| \sup |f|_> ([a, b]) \tag{3.5}$$

holds for all $s, t \in [a, b]$. If $\varepsilon \in \mathbb{P}^\times$ is given, we may choose δ such that $\delta \sup |f|_>([a, b]) \leq q\varepsilon$ and conclude from (3.5) that

$$|s - t| < \delta \Rightarrow |F(t) - F(s)| < \varepsilon$$

for all $s, t \in [a, b]$. We conclude that $F|_{[a,b]}$ is uniformly continuous for all $a, b \in I$ with $a < b$ and hence that F is continuous. ■

Theorem 3.1. Fundamental Theorem of Calculus: *Assume that $f \in$* Bbnc *I. Let $c \in I$ be given and let $F : I \Leftrightarrow \mathbb{R}$ be defined by (3.4). If f is continuous at $t \in I$, then F is differentiable at t and $\partial_t F = f(t)$.*

Proof Using the definition (3.4) of F as well as (2.5) and (2.6) of Sect. 2, we find that

$$\frac{1}{s}(F(t + s) - F(t)) = \frac{1}{s} \int_t^{t+s} f \tag{3.6}$$

for all $s \in \mathbb{R}^\times$ such that $t + s \in I$. Now let $\varepsilon \in \mathbb{P}^\times$ be given. Since f is continuous at t, we can determine $\delta \in \mathbb{P}^\times$ such that

$$f_>(]t - \delta,\ t + \delta[\ \cap I) \ \subset \ ]f(t) - \varepsilon,\ f(t) + \varepsilon[,$$

and hence

$$\sup f_>(]t-\delta,\ t+\delta[\ \cap I) \leq f(t)+\varepsilon$$

and

$$f(t)-\varepsilon \leq \inf f_>(]t-\delta,\ t+\delta[\ \cap I).$$

Using Prop. 4 we easily conclude that

$$f(t)-\varepsilon \leq \frac{1}{s}\int_t^{t+s} f \leq f(t)+\varepsilon \quad \text{for all} \ \ s \in \,]-\delta,\delta[\, \cap (I-t). \tag{3.7}$$

Since $\varepsilon \in \mathbb{P}^\times$ was arbitrary, it follows from (3.6) and (3.7) that

$$\lim_{s\to 0} \frac{1}{s}\,(F(t+s)-F(t)) = f(t),$$

which is the assertion. ■

Corollary 3.1. *If $f : I \to \mathbb{R}$ is continuous then it has antiderivatives. If G is an antiderivative of f and $s,t \in I$ then*

$$\int_s^t f = G(t)-G(s). \tag{3.8}$$

Proof Let $s \in I$ be given. Since f is continuous, we have $f|_{[a,b]} \in$ Bbnc $[a,b]$ for all $a,b \in I$ with $a<b$ (see Remark 1 of Sect. 2), and hence we may apply the Fundamental Theorem of Cale ulus with $c := s$ in (3.4) to f and obtain an antiderivative F that satisifes $F(s)=0$. If G is any antiderivative of f, we have $F^\bullet = G^\bullet = f$ and hence $(F-G)^\bullet = 0$. Using a theorem of differential calculus, we conclude that $F-G$ is a constant, i.e. that $F-G = F(s)-G(s) = -G(s)$. We conclude that $F(t) = G(t) = G(s)$ for all $t \in I$, which is the assertion. ■

The Formula (3.8) is particularly useful for finding integrals of those functions for which antiderivatives can be obtained by methods other than integration. The formula (3.4) is also often useful for defining interesting new functions from known functions. For example, the function $\log : \mathbb{P}^\times \to \mathbb{R}$ can be defined by

$$\log(t) := \int_1^t \frac{1}{\iota} \quad \text{for all} \ \ t \in \mathbb{P}^\times.$$

Other examples are the *integral sine function* $S_i : \mathbb{R} \to \mathbb{R}$ defined by

$$S_i(t) := \int_0^t \frac{\sin}{\iota} \quad \text{for all } t \in \mathbb{R}$$

and the *Gauss error function* $\Phi : \mathbb{R} \to \mathbb{R}$ defined by

$$\Phi(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-\iota^2} \quad \text{for all } t \in \mathbb{R}.$$

4 Uniqueness and existence of the integral

We now give the proof of the existence and uniqueness theorem of Sect. 2. First we introduce some notation and then we prove several preliminary lemmas.

Let a bounded non-empty interval I be given. The *left endpoint* of I is $\inf I$ and the *right endpoint* is $\sup I$. The *closure* $\overline{I}$ of I is defined by

$$\overline{I} := [\inf I, \sup I], \tag{4.1}$$

i.e. $\overline{I}$ is obtained from I by joining the endpoints of I to I. The *length* of I is defined by

$$\mathrm{le}(I) := \sup I - \inf I. \tag{4.2}$$

It is clear that I and its closure $\overline{I}$ have the same lengths: $\mathrm{le}(\overline{I}) = \mathrm{le}(I)$. If I is a non-empty bounded interval and if $\mathcal{P}$ is a finite interval-partition of I, then

$$\mathrm{le}(I) = \sum_{J \in \mathcal{P}} \mathrm{le}(J). \tag{4.3}$$

For every $p \in \mathbb{N}$, we consider the partition

$$\mathcal{P}_p := \left\{ [\frac{n}{2p}, \frac{n+1}{2p}[\ | \ n \in \mathbb{Z} \right\} = \left\{ [0, \frac{1}{2p}[+ \frac{n}{2p} \ | \ n \in \mathbb{Z} \right\} \tag{4.4}$$

of $\mathbb{R}$ into half-open intervals of length $\frac{1}{2p}$. If $q \geq p$, then $\mathcal{P}_q$ is a refinement of $\mathcal{P}_p$ and hence, by (4.3), we have

$$\frac{1}{2p} = \mathrm{le}(I) = \sum_{J \in \mathcal{P}_q, J \subset I} \mathrm{le}(J) \quad \text{for all} \quad I \in \mathcal{P}_p. \tag{4.5}$$

Let $f \in$ Bbnc$\,\mathbb{R}$ be given. Since f is bounded, we have $\inf f_>(I) \in \mathbb{R}$ and $\sup f_>(I) \in \mathbb{R}$ for all intervals I. Let $p \in \mathbb{N}$ be given. Since f has bounded support, we have $f|_I = 0$ and hence $\sup f_>(I) = \inf f_>(I) = 0$ for all but a finite number of intervals I in $\mathcal{P}_p$. Hence

$$\overline{S}_p(f) := \frac{1}{2p} \sum_{I \in \mathcal{P}_p} \sup f_>(I), \quad \underline{S}_p(f) := \frac{1}{2p} \sum_{I \in \mathcal{P}_p} \inf f_>(I) \tag{4.6}$$

are well defined, because only a finite number of terms in the sums can be non-zero.

Lemma 4.1. *The sequence $\big(\overline{S}_p(f) \mid p \in \mathbb{N}\big)$ is antitone and bounded below by every $\underline{S}_q(f), q \in \mathbb{N}$. The sequence $\big(\underline{S}_p(f) \mid p \in \mathbb{N}\big)$ is isotone and bounded above by every $\overline{S}_q(f), q \in \mathbb{N}$.*

Proof: Since $\sup f_>(I) \geq \sup f_>(J)$ whenever $J \subset I$, we see, with the help of (4.5), that

$$\begin{aligned} \frac{1}{2p} \sup f_>(I) = \sup f_>(I) \sum_{J \in \mathcal{P}_q, J \subset I} \mathrm{le}(J) \quad &\geq \quad \sum_{J \in \mathcal{P}_q, J \subset I} \sup f_>(J) \mathrm{le}(J) \\ &= \quad \frac{1}{2q} \sum_{J \in \mathcal{P}_q, J \subset I} \sup f_>(J) \end{aligned}$$

is valid for all $I \in \mathcal{P}_p$ such that $q \geq p$. Summing this inequality over all $I \in \mathcal{P}_q$ and observing (4.6), we obtain $\overline{S}_p(t) \geq \overline{S}_q(f)$. Hence $\big(\overline{S}_p(t) \mid p \in \mathbb{N}\big)$ is antitone. A similar argument shows that $\big(\underline{S}_p(f) \mid p \in \mathbb{N}\big)$ is isotone.

Since $\inf f_>(I) \leq \sup f(I)$ for all non-empty intervals I, we immediately conclude that $\underline{S}(f) \leq \overline{S}_p(f)$ for all $p \in \mathbb{N}$. Using the already proved face that $(\overline{S}_p(t) \mid p \in \mathbb{N})$ is antitone and $(\underline{S}_p(f) \mid p \in \mathbb{N})$ isotone, it follows that $\underline{S}_p(f) \leq \overline{S}_q(t)$ for all $p, q \in \mathbb{N}$, which proves the boundedness assertions. ■

Lemma 4.2. *The sequences $(\underline{S}(f) \mid p \in \mathbb{N})$ and $(\overline{S}(f) \mid p \in \mathbb{N})$ both converge and have the same limit*

$$S(f) := \lim_{p \to \infty} \underline{S}_p(f) = \lim_{p \to \infty} \overline{S}_p(t). \tag{4.7}$$

Proof: The convergence of the sequences follows from Lemma 1. To prove that their limits are the same, we must show that

$$\lim_{p\to\infty}\left(\overline{S}_p(f)-\underline{S}_p(f)\right)=0.$$

Since the sequence $\left(\overline{S}_p(f)-\underline{S}_p(f)\mid p\in\mathbb{N}\right)$ has positive terms and is antitone by Lemma 1, it is sufficient if we show that for every $\varepsilon\in\mathbb{P}^\times$ one can find a $q\in\mathbb{N}$ such that

$$\sum_{J\in\mathcal{P}_q}(\sup f_>(J)-\inf f_>(J))\mathrm{le}(J)=\overline{S}_q(f)-\underline{S}_q(t)<\varepsilon \tag{4.8}$$

We use the abbreviations

$$m:=\#\mathrm{Dis}f,\quad b:=\sup\ \mathrm{Rng}f-\inf Rngf. \tag{4.9}$$

We may assume, without loss, that $f\neq 0$ and hence $b>0$.

Now let $\varepsilon\in\mathbb{P}^\times$ be given. We determine $p\in\mathbb{N}$ such that

$$\frac{2mb}{2^p}\leq\frac{\varepsilon}{2}. \tag{4.10}$$

For every $I\in\mathcal{P}_p$, one of the following three mutually exclusive situations must obtain:

(i) $f|_{\overline{I}}$ fails to be continuous,

(ii) $f|_{\overline{I}}$ is continuous and not zero,

(iii) $f|_{\overline{I}}=0$.

Hence, if we define

$$D:=\bigcup\{I\in\mathcal{P}_p\mid\ \mathrm{Dis}\ f|_{\overline{I}}\ \neq\emptyset\}, \tag{4.11}$$

$$C:=\bigcup\{I\in\mathcal{P}_p\mid\ \mathrm{Dis}\ f\||_{\overline{I}}\ =\emptyset\text{ and }f\ |_{\overline{I}}\neq 0\}, \tag{4.12}$$

we have

$$D\cap C=\emptyset\quad\text{and}\quad\mathrm{Supt}f\subset D\cup C. \tag{4.13}$$

Now let $q \in p + \mathbb{N}$ be given. Since $\mathcal{P}_q$ is a refinement of $\mathcal{P}_p$ and since $\operatorname{Supt} f$ is bounded, it follows from (4.13) that $\{J \in \mathcal{P}_q \mid J \subset D\}$ and $\{J \in \mathcal{P}_q \mid J \subset C\}$ are finite and, by (4.5) that

$$\sum_{J \in \mathcal{P}_q, J \subset D} \operatorname{le}(J) = \delta := \sum_{I \in \mathcal{P}_p, I \subset D} \operatorname{le}(I), \tag{4.14}$$

$$\sum_{J \in \mathcal{P}_q, J \subset C} \operatorname{le}(J) = \gamma := \sum_{I \in \mathcal{P}_p, I \subset C} \operatorname{le}(I). \tag{4.15}$$

Since a given $t \in I$ can belong to at most two closures of intervals belonging to $\mathcal{P}_p$, we conclude, using (4.14), (4.11), (4.9_1) $(5)_1$ and (4.10), that

$$\delta \leq \frac{2m}{2p} \leq \frac{\varepsilon}{2b}. \tag{4.16}$$

If follows from $(4.9)_2$, (4.14), and (4.16) that

$$\sum_{J \in \mathcal{P}_q, J \subset D} (\sup f, (J) - \inf f_{>}(J)) \operatorname{le}(J) \leq \frac{\varepsilon}{2}. \tag{4.17}$$

We now assume that $C \neq \emptyset$ and hence $\gamma > 0$ in (4.15). Let $I \in \mathcal{P}_p$ with $I \subset C$ be given. Since $f|_{\overline{I}}$ is continuous by (4.12) and since $\overline{I}$ is closed and bounded, the Uniform Continuity Theorem ensures that $f|_{\overline{I}}$ is uniformly continuous. Hence, we can choose $\sigma_I \in \mathbb{P}^{\times}$ such that

$$|s - t| < \sigma_I \quad \Longrightarrow \quad |f(s) - f(t)| < \frac{\varepsilon}{2\gamma} \quad \text{for all } s, t \in I \ ,$$

and hence

$$\sup f_{>}(J) - \inf f_{>}(J) = \sup\{|f(s) - f(t)| \mid s, t \in J\} < \frac{\varepsilon}{2\gamma} \tag{4.18}$$

for every interval J included in I such that $\operatorname{le}(J) < \sigma_I$. Since $\{I \in \mathcal{P}_p \mid I \subset C\}$ is finite, we can determine $q \in p + \mathbb{N}$ such that $\frac{1}{2^q} < \sigma_I$ for all $I \in \mathcal{P}_p$ such that $I \subset C$. Hence, since $\operatorname{le}(J) = \frac{1}{2^q}$ for all $J \in \mathcal{P}_q$, (4.18) is valid for all $J \in \mathcal{P}_q$ such that $J \subset C$. Summing (4.18) over all such J and using (4.15), we obtain

$$\sum_{J\in\mathcal{P}_q, J\subset C} (\sup f_>(J) - \inf f_>(J))\mathrm{le}(J) \leq \frac{\varepsilon}{2\gamma}\gamma = \frac{\varepsilon}{2}. \qquad (4.19)$$

If $C = \emptyset$ then (4.19) is trivially valid. We now add the inequalities (4.17) and (4.19). Observing (4.13), we conclude that (4.8) holds. ■

Lemma 4.3. *The functional* $S : \mathrm{Bbnc}\,\mathbb{R} \to \mathbb{R}$ *defined by (4.7) has the property that* $S(ch_I) = \mathrm{le}(I)$ *for all non-empty bounded intervals* I.

Proof Let I be a non-empty bound interval. We observe that for every $J \in$ Sub $\mathbb{R}$, we have

$$\sup ch_{I^>}(J) = \left\{ \begin{array}{lll} 1 & \text{if} & I\cap J \neq \emptyset \\ 0 & \text{if} & I \cap J = \emptyset \end{array} \right\},$$

$$\inf ch_{I^>}(J) = \left\{ \begin{array}{lll} 1 & \text{if} & J \subset I \\ 0 & \text{if} & J \not\subset I \end{array} \right\}.$$

Now, let $p \in \mathbb{N}$ be given. Then

$$K_p := \bigcup \{J \in \mathcal{P}_p \mid I \cap J \neq \emptyset\}$$

is a bounded interval that includes I and

$$L_p := \bigcup \{J \in \mathcal{P}_p \mid J \subset I\}$$

is a bounded interval included in I. Hence, in view of (4.3), the definitions (4.6) give

$$\begin{aligned} \overline{S}_p(ch_I) &= \sum_{J\in\mathcal{P}_p} (\sup ch_{I^>}(J))\mathrm{le}(J) = \sum_{J\in\mathcal{P}_p, I\cap J\neq\emptyset} \mathrm{le}(J) = \mathrm{le}(K_p), \\ \underline{S}_p(ch_I) &= \sum_{J\in\mathcal{P}_p} (\inf ch_{I^>}(J))\mathrm{le}(J) = \sum_{J\in\mathcal{P}_p, J\subset I} \mathrm{le}(J) = \mathrm{le}(L_p). \end{aligned}$$

Since $L_p \subset I \subset K_p$ we conclude that

$$\underline{S}_p(ch_I) \leq \mathrm{le}(I) \leq \overline{S}_p(ch_I) \quad \text{for all } p \in \mathbb{N}.$$

Taking the limit $p \to \infty$, we obtain the desired result from Lemma 2. ■

Lemma 4.4. *There is exactly one functional* $S' :$ Bbnc $\mathbb{R} \to \mathbb{R}$ *that is linear and isotone and has the property*

$$S'(ch_I) = \mathrm{le}(I) = \frac{1}{2^p} \text{ when } I \in \mathcal{P}_p, p \in \mathbb{N}. \tag{4.20}$$

Moreover we have $S' = S$*, where* S *is defined by (4.7).*

Proof: Let S' be a functional with the desired properties. Let $f \in$ Bbnc $\mathbb{R}$ and $p \in \mathbb{N}$ be given. It is easily seen that

$$\sum_{I \in \mathcal{P}_p} (\inf f_>(I)) ch_I \leq f \leq \sum_{I \in \mathcal{P}_q} (\sup f_>(I)) S'(ch_I).$$

Since S' is assumed to be isotone and linear, it follows that

$$\sum_{I \in \mathcal{P}_p} (\inf f_>(I)) S'(ch_I) \leq S'(f) \leq \sum_{I \in \mathcal{P}_p} (\sup f_>(I)) S'(ch_I).$$

Since (4.20) is assumed to hold, the definitions (4.6) yield

$$\underline{S}_p(f) \leq S'(f) \leq \overline{S}_p(f).$$

Taking the limit $p \to \infty$ and using Lemma 2, we obtain $S(f) = S'(f)$. Since $f \in$ Bbnc $\mathbb{R}$ was arbitrary, we see that S is the only possibility for a functional S' with the desired properties.

It is a fairly easy exercise to show that $S' := S$ does indeed have the desired properties. ∎

Proof of the Theorem Let Igl be an integral, i.e., a functional with the properties (a) - (d) of Definition 1. Let $p \in \mathbb{N}$ be given. We observe that

$$ch_{[0,1[} = \sum_{n \in (2^p)[} ch_{[0,\frac{1}{2^p}[+\frac{n}{2^p}}$$

Since $ch_{[0,\frac{1}{2^p}[+\frac{n}{2^p}} = T_{\frac{n}{2^p}} ch_{[0,\frac{1}{2^p}[}$ by (1.9) of Sect. 1, it follows from the assumed linearity and translation invariance of Igl that

$$\mathrm{Igl}(ch_{[0,1[}) = 2^p \ \mathrm{Igl}\ (ch_{[0,\frac{1}{2^p}[}).$$

The assumption Igl $(ch_{[0,1[}) = 1$ gives $\mathrm{Igl}(ch_{[0,\frac{1}{2^p}[}) = \frac{1}{2^p}$. For every $I \in \mathcal{P}_p$ we have $ch_I = T_s ch_{[0,\frac{1}{2^p}[}$ for a suitable $s \in \mathbb{R}$. Hence, by the translation invariance of Igl, we have

$$\text{Igl}(ch_I) = \frac{1}{2^p} = \text{le}(I) \text{ for all } I \in \mathcal{P}_p,$$

i.e., (4.20) is satisfied. It follows from Lemma 4 that S is the only possibility for an integral.

If we define Igl $:= S$, it follows from Lemma 4 that Igl has the properties, (a), (b), and (d). In order to prove that it also has the property (c), we let $s \in \mathbb{R}$ be given and define

$$\text{Igl}' := \text{ Igl } \circ T_s = S \circ T_s.$$

Since both Igl and T_s are linear and isotone, so is Igl$'$. Let I be any non-empty bounded interval. It follows from (1.9) of Sect. 1 and Lemma 3 that

$$\text{Igl}'(ch_I) = \text{ Igl } (T_s ch_I) = \text{Igl } (ch_{I+s}) = \text{le}(I+s) = \text{le}(I).$$

Hence Igl$'$ has the property (4.20). By the uniqueness assertion of Lemma 4, it follows that Igl $=$ Igl$'$ $=$ Igl$\circ T_s$, which proves the translation invariance of Igl.

The property (d_1) of the integral follows from Lemma 3.

5 Extended Integrals

Let a genuine interval I be given. We denote the set of all bounded intervals included in I by Bint I. We say that a function $f \in$ Fun I is **nearly continuous** if $f|_J$ has only a finite number of discontinuities for every $J \in$ Bint I and we use the notation

$$\text{Nc } I := \{f \in \text{ Fun } I \mid \text{ Dis } (f\ |_J) \text{ if finite for every } J \in \text{ Bint } I\} \tag{5.1}$$

for the set of all nearly continuous functions on I. It is clear the Nc I is a subspace of the linear space Fun I. The of all *positive* nearly continuous functions on I will be denoted by

$$\text{Pnc } I := \{f \in \text{ Nc } I \mid f \geqq 0\}. \tag{5.2}$$

For every $f \in$ Pnc I and every $\gamma \in \mathbb{P}$ we define $C_\gamma f \in$ Pnc I by $C_\gamma f := \inf\{f, \gamma\}$, so that

$$(C_\gamma f)(t) := \min\{f(t), \gamma\} \quad \text{for all} \quad t \in I \tag{5.3}$$

and we say that $C_\gamma f$ is obtained from f by **truncation** at γ.

Let $f \in \text{Pnc}\, I$ be given. Using the definitions of Sect. 1, it is easily seen that

$$C_\gamma f|_J^0 = C_\gamma(f|_J^0) \in \text{ Bbnc}\, \mathbb{R}$$

and hence that

$$\int_J C_\gamma f = \text{ Igl}(C_\gamma f)|_J^0 \in \mathbb{P} \tag{5.4}$$

is meaningful for every $\gamma \in \mathbb{P}$ and every $J \in$ Bint I.

We note that

$$(\gamma' \geq \gamma \quad \text{and} \quad J' \subset J) \implies \int_{J'} C_{\gamma'} f \geq \int_J C_\gamma f \tag{5.5}$$

for all $\gamma, \gamma' \in \mathbb{P}$, and $J, J' \in$ Bint I.

Definition 1: For every $f \in$ Pnc I, we define the **integral** $\int_I f \in \overline{\mathbb{P}}$ of f by

$$\int_I f = \sup\{\text{Igl}(C_\gamma f|_J^0) \mid \gamma \in \mathbb{P},\ J \in \text{ Bint}\, I\} \tag{5.6}$$

We say that $f \in$ Pnc I is **integrable** if $\int_I f < \infty$.

It is clear that if $f \in$ Bbnc I and $f \geq 0$, then $C_\gamma f|_J^0 = f$ when $\gamma \geq$ sup Rng f and Supt $f \subset J$ and hence that the integral $\int_I f$ in the sense of (5.6) gives the same result as the integral defined by (1.2) as it should. Hence the concept of integral of Def.1 is an extension of the concept of the integral given by (1.2).

If I is a bounded interval to begin with, then (5.6) reduces to

$$\int_I f = \sup\{\text{Igl}(C_\gamma f|^0) \mid \gamma \in \mathbb{P}\}. \tag{5.7}$$

If f is a bounded function, i.e., if Rng f is bounded, then (5.6) reduces to

$$\int_I f = \sup\{\mathrm{Igl}(f|_J^0) \mid J \in \ \mathrm{Bint}\ I\}. \tag{5.8}$$

Proposition 5.1. Pnc I *is an additive submonoid of Fun I and the mapping* $(f \mapsto \int_I f) : \mathrm{Pnc}\ I \to \overline{\mathbb{P}}$ *is an additive-monoid homomorphism, and it is isotone. In particular, given* $f, g \in$ Pnc$I$ *such that* $f \leq g$*, if* g *is summable, so is* f*, and if* f *fails to be summable, so does* g*.*

Proof. It is clear that Pnc I contains the zero-function and that is is stable under value-wise addition. It is also evident that $\int_I 0 = 0$. To show that the integral preserves additions, let $f, g \in$ Pnc I be given. It is clear that $f + g \in$ Pnc I. Let $\gamma, \gamma' \in \mathbb{P}^\times$ and $J, J' \in$ Bint$I$ be given. Using the results of Sect. 2 and (5.6), we see that

$$\int_J C_\gamma f + \int_{J'} C_{\gamma'} g \leq \int_{J \cup J'} C_{\max\{\gamma,\gamma'\}}(f+g) \leq \int_I f + g.$$

Hence, since $\gamma, \gamma' \in \mathbb{P}$ and $J, J' \in$ Bint I were arbitrary, it follows from (5.6) that

$$\int_I f + \int_I g \leq \int_I (f+g). \tag{5.9}$$

On the other hand, let $\gamma \in \mathbb{P}$ and $J \in$ Bint I be given. It is easily seen that $C_\gamma(f+g) \leq C_\gamma f + C_\gamma g$ and hence, by the linearity and isotonicity of Igl and by(5.6), that

$$\int_I (f+g) \leq \int_I f + \int_I g) \tag{5.10}$$

It follows from (5.9) and (5.10) that $\int_I (f+g) = \int_I f + \int_I g$.

The isotonicity of $(f \mapsto \int_I f) :$ Pnc $I \to \overline{\mathbb{P}}$ is an immediate consequence of the isotonicity of Igl . ☐

Definition 2: *We say that a given* $f \in$ *Nc I is* **integrable** *if both* $f^+ \in$ Pnc I *and* $f^- \in$ Pnc I *are integrable in the sense of Def. 1. If this is the case, we put*

$$\int_I f := \int_I f^+ - \int_I f^- \in \mathbb{R} \tag{5.11}$$

and call it the **integral** *of f. We use the notation*

$$\text{Inc } I := \{f \in \text{ Nc } I \mid f \text{ is integrable}\}. \tag{5.12}$$

The following results are analogous to Props. 4-6 of Sect. 4 of **Limits and Continuity** concerning summability. Their proofs are also analogous to the corresponding results on summability.

Proposition 5.2. *A given $f \in$ Nc I is integrable if and only if $|f| \in$ Nc I is integrable.*

Proposition 5.3. *The set* Inc I *is a subspace of the linear space* Nc I *and the mapping*

$$(f \mapsto \int_I f) : \text{ Inc } \to \mathbb{R} \tag{5.13}$$

is linear and isotone.

Proposition 5.4. *A given $f \in$ Nc I is integrable with $\int_I f = s$ if and only if, for every $\varepsilon \in \mathbb{P}^\times$, there is $J \in$ Bint I and $\eta \in \mathbb{P}$ such that*

$$\left| \int_K \sup\{\inf\{f, \gamma\}, -\gamma\} - s \right| < \varepsilon \quad \text{for all} \quad K \in \text{ Bint} J \quad \text{and } \gamma, \gamma' \in \eta + \mathbb{P}. \tag{5.14}$$

We now consider the case when $I := \mathbb{R}$ and we define $\overline{\text{Igl}}$: Inc $\mathbb{R} \to \mathbb{R}$ by

$$\overline{\text{Igl}}(f) := \int_{\mathbb{R}} f \quad \text{for all} \quad f \in \text{Inc}\,\mathbb{R}. \tag{5.16}$$

Proposition 5.5. Inc *is a translation-stable subspace of* Nc, *and* Inc $\mathbb{R} \to \mathbb{R}$ *is linear, isotone, and translation-invariant. Moreover, Bbnc* $\mathbb{R}$ *is a subspace of* Inc $\mathbb{R}$ *and* $\text{Igl} = \overline{\text{Igl}}|_{\text{Bbnc } \mathbb{R}}$.