On infinite spectra of first-order properties

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The random graph $G(n, p)$ obeys Zero-One Law if for each first-order property its probability tends to 0 or tends to 1. In [1], S. Shelah and J. Spencer showed that if $\alpha$ is an irrational positive number and $p(n) = n^{-\alpha+\alpha(1)}$, then $G(n, n^{-\alpha})$ obeys Zero-One Law.

We let $L_k$ and $L$ denote the set of properties which are expressed by first-order formulae with quantifier depths at most $k$ and the set of all first-order properties respectively. For any $L \in L$, we define two notions of its spectra, $S^1(L)$ and $S^2(L)$. The first considers $p = n^{-\alpha}$. $S^1(L)$ is the set of $\alpha \in (0, 1)$ which does not satisfy the following property: $\lim_{n \to \infty} \mathbb{P}(G(n, n^{-\alpha}) \models L)$ exists and is either zero or one. The second considers $p = n^{-\alpha+\alpha(1)}$. $S^2(L)$ is the set of $\alpha \in (0, 1)$ which does not satisfy the following property: there exists $\delta \in \{0, 1\}$ and $\epsilon > 0$ so that when $n^{-\alpha-\epsilon} < p(n) < n^{-\alpha+\epsilon}$, $\lim_{n \to \infty} \mathbb{P}(G(n, p(n)) \models L) = \delta$. It can be shown that for any rational $\alpha \in (0, 1)$ there is a first-order property $L$ such that $\alpha \in S^1(L)$. So, the Zero-One Law of Shelah and Spencer implies $\bigcup_{L \in L} S^1(L) = \mathbb{Q} \cap (0, 1)$. It is easy to show that letting $L$ be the property of every two vertices to have a common neighbor, $S^2(L) = \{\frac{1}{2}\}$ while $S^1(L) = \emptyset$. Thus, for any $L \in L$, we have $S^1(L) \subset S^2(L)$ but there may not be the equality. However, in [1] Shelah and Spencer proved that every $S^2(L)$ consists only of rational numbers as well.

Let $S^1_k$ be the union of all $S^1(L)$ where $L \in L_k$, $S^2_k$ be the union of all $S^2(L)$ where $L \in L_k$. In [2, 3], it was shown that the minimal and the maximal numbers in $S^1_k$ equal $\frac{1}{k-2}$ and $1 - \frac{1}{2^{k-2}}$ respectively. In [4], it was proved that the sets $S^1_k$ and $S^2_k$ are infinite when $k$ is large enough. It is also known [5] that all limit points of $S^1_k$ and $S^2_k$ are approached only from above. For any $j \in \{1, 2\}$, denote the set of limit points of $S^1_k$ by $(S^1_k)^j$. In our joint work with Spencer, we prove that if $k \geq 10$, then $\min(S^1_k)^j \leq \frac{1}{k-11}$. If $k \geq 16$, then $\max(S^1_k)^j \geq 1 - \frac{1}{2^{k-11}}$. Moreover, we prove that the minimal $k_1$ and $k_2$ such that $S^1_{k_1}$ and $S^2_{k_2}$ are infinite are from $\{4, \ldots, 12\}$ and $\{4, \ldots, 10\}$ respectively. Recently, we significantly improve this result.

**Theorem 1** If $k \geq 5$, then $\frac{1}{k-2} \in (S^1_k)^j$. If $k \geq 8$, then $\max(S^2_k)^j \geq \max(S^1_k)^j \geq 1 - \frac{1}{2^{k-4}}$.

Consequently, the minimal $k$ such that the set $S^1_k$ ($S^2_k$) is infinite is either 4 or 5.

**References**


