On graphs containing few disjoint excluded minors. Asymptotic number and structure of graphs containing few disjoint minors K_4

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Abstract

Let $\mathrm{Ex}\,\mathcal{B}$ be a minor-closed class of graphs with a set $\mathcal{B}$ of excluded minors. Kurauskas and McDiarmid (2012) studied classes $\mathcal{A}$ of graphs that have at most k disjoint minors in $\mathcal{B}$, more precisely, at most k vertex disjoint subgraphs containing an excluded minor from $\mathcal{B}$. Denote by $\mathcal{A}_n$ the class $\mathcal{A}$ restricted to graphs on the vertex set $\{1, 2, \ldots, n\}$. In the case when all graphs in $\mathcal{B}$ are 2-connected and $\mathrm{Ex}\,\mathcal{B}$ excludes some fan (a path together with a vertex joined to each vertex on the path), they determined the asymptotics of $|\mathcal{A}_n|$ and properties of typical graphs in $\mathcal{A}_n$ as n tends to infinity. In particular, they showed that all but an exponentially small proportion of graphs G from $\mathcal{A}_n$ contain a set S of size k such that S is a $\mathcal{B}$-blocker, i.e., $G - S \in \mathrm{Ex}\,\mathcal{B}$. (An example is the class $\mathrm{Ex}\,(k+1)\{K_3\}$ of graphs containing at most k vertex-disjoint cycles.)

In the present work we consider the case when $\mathrm{Ex}\,\mathcal{B}$ contains all fans. Firstly, for good enough $\mathcal{B}$ we obtain results on asymptotics of $|\mathcal{A}_n|$. For example, we give a sufficient condition for the sequence $y_n = (|\mathcal{A}_n|/n!)^{1/n}$ to have a limit (a growth constant) as $n \to \infty$. A $\mathcal{B}$-blocker Q of G is redundant if for each $x \in Q$, $Q \setminus \{x\}$ is a $\mathcal{B}$-blocker of G. Let R_n be a graph drawn uniformly at random from $\mathcal{A}_n$. For large enough constant k we show that R_n has no $\mathcal{B}$-blocker smaller than $2k$ with probability $1 - e^{-\Omega(n)}$, and the upper limit of y_n is realised by the subclass of graphs $G \in \mathcal{A}$ that have a redundant $\mathcal{B}$-blocker Q of size $2k+1$.

Secondly, we explore the structure of graphs that have at most k disjoint minors K_4. For $k = 0$ this is the class of series-parallel graphs. For $k = 1, 2, \ldots$ we show that there are constants c_k, γ_k, such that $|\mathcal{A}_n| = c_k n^{-5/2} \gamma^n n!(1 + o(1))$. We prove that the random graph R_n with probability $1 - e^{-\Omega(n)}$ has a redundant $\{K_4\}$-blocker Q of size $2k+1$ and each vertex of Q has a linear degree. Additionally, we consider the case $\mathcal{B} = \{K_{2,3}, K_4\}$ related to outerplanar graphs.