Martingales and Local Martingales

Philip Protter, Columbia University
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What is a Martingale?

- We begin with a complete probability space $L^2(\Omega, \mathcal{F}, P)$, with P a probability measure on $\mathcal{F}$
- Suppose there is a sequence of nested σ algebras: $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots$ with every $\mathcal{F}_i \subset \mathcal{F}$
- Then we have a sequence of nested Hilbert spaces:

$$L^2(\Omega, \mathcal{F}_1, P) \subset L^2(\Omega, \mathcal{F}_1, P) \subset \cdots \subset L^2(\Omega, \mathcal{F}, P)$$

- Let $X \in L^2(\Omega, \mathcal{F}, P)$, and define M_n by

$$M_n = \pi_n(X)$$

where π_n is the Hilbert space projection onto $L^2(\Omega, \mathcal{F}_n, P)$

- The Hilbert space projection $\pi_n(X)$ is the same thing as the **conditional expectation** of X given $\mathcal{F}_n$, written $E(X|\mathcal{F}_n)$
- We have then $E(M_n = \pi_n(X) = X|\mathcal{F}_n)$ and by properties of Hilbert space projection we have $E(M_{n+1}|\mathcal{F}_n) = M_n$, for each n
- A stochastic process $M = (M_n)_{n \geq 0}$ with the relation that $E(M_m|\mathcal{F}_n) = M_n$ a.s. for any $m \geq n$ is called a **martingale**
- Typically we go beyond L^2 and the analogy to Hilbert spaces: conditional expectation makes sense in L^1
- So we can extend, in some sense, Hilbert space projection to the Banach space L^1
- We also do not need the existence of some random variable X in the über σ algebra $\mathcal{F}$

- In Probability Theory, martingales are often cited as the mathematical model of a fair game: if M_n represents your fortunes at time n, then your conditional expectation of your future fortune M_m (with $m \geq n$) is $E(M_m|\mathcal{F}_n)$
- In other words, $E(M_m|\mathcal{F}_n)$ is your best guess of your future fortune, given all observable events up to and including the present time n
- In stochastic process theory we usually use continuous time: $\mathbb{R}_+$ replaces $\mathbb{N}$; then we have M is a martingale if for any $u \geq t$:

$$E(M_u|\mathcal{F}_t) = M_t \quad a.s.$$

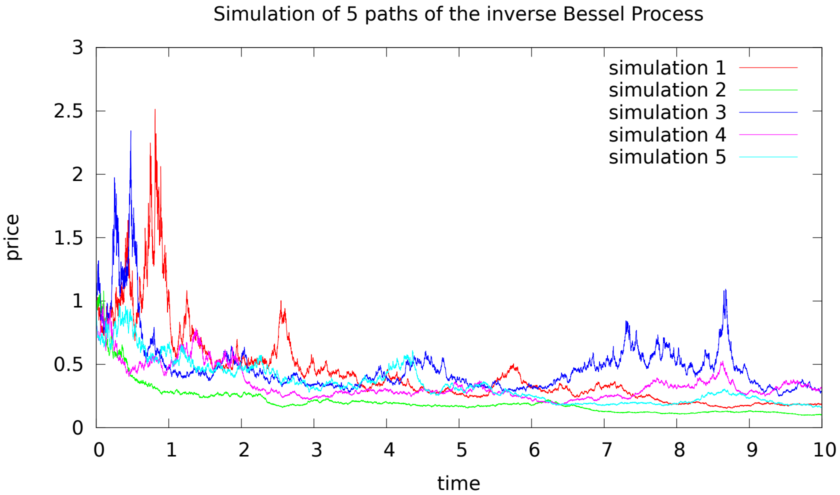
- Martingales have the elementary property that $t \mapsto E(M_t)$ is constant

- In Probability Theory, an object of interest is the random time something happens
- A random time is simply a positive valued function $T : \Omega \to \mathbb{R}_+$
- This is too general; the class of random times that are mathematically useful are the **stopping times**:
- T is a stopping time if $\{\omega : T(\omega) \leq t\} = \{T \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$
- **Theorem:** M is a martingale if and only if $E(M_T) = E(M_0)$ for all stopping times T

Local Martingales

- A **local martingale** is a stochastic processes which is locally a martingale
- A process X is a **local martingale** if there exists a sequence of stopping times T_n with $T_n \nearrow \infty$ a.s., $T_n < T$ a.s. on $\{T > 0\}$, and $\lim_{n\to\infty} T_n = T$ a.s. and moreover $X_{t\wedge T_n}$ is a martingale for each n
- **P. A. Meyer (1973)** showed that there are no local martingales in discrete time; they are a continuous time phenomenon
- The original example of a local martingale (**G. Johnson & L.L. Helms, 1963**) is the inverse Bessel Process: Let W be a 3D Brownian motion not starting at $(0, 0, 0)$, and let $Y_t = \| W_t \|$ and $X_t = \frac{1}{Y_t}$

Simulations of the Inverse Bessel Process



Examples of Continuous Local Martingales

- We now have more examples, thanks to **S. Kotani (2006)** and **Mijatovic & Urusov (2012)**: Let X be a solution of a stochastic differential equation (SDE) of the form

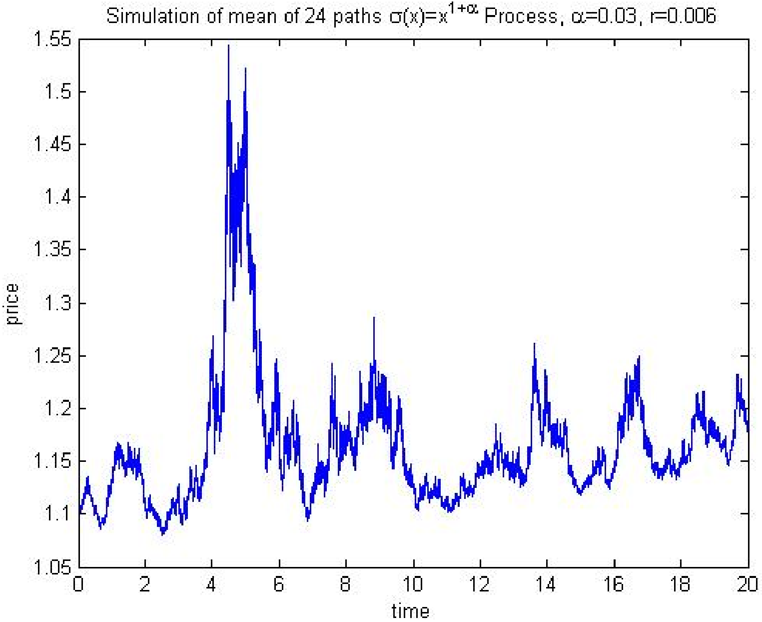
$$dX_t = \sigma(X_t)dB_t; \quad X_0 = 1$$

  where B is standard Brownian motion

- X is a positive **Strict Local Martingale** (a local martingale that is not a martingale) if for any $\varepsilon > 0$:

$$\int_0^{\varepsilon} \frac{x}{\sigma(x)^2} dx = \infty \text{ and } \int_{\varepsilon}^{\infty} \frac{x}{\sigma(x)^2} dx < \infty$$

Simulation of the solution of an SDE of M-U type



A Little History

- **The Doob-Meyer Decomposition**, **P.A. Meyer (1963)**: X a submartingale of Class D, then

$$X_t = M_t + A_t \quad \text{(uniquely)}$$

  with M a martingale and A an increasing, predictably measurable process
- Local Martingales were invented by **K. Itô and S. Watanabe** in 1965 (2 years after Johnson & Helms) to obtain a general multiplicative decomposition of multiplicative functions of Markov processes

Local Martingales

- Doob-Meyer was then extended easily to no longer needing Class D if M were a local martingale and A still a predictably measurable increasing process
- Stochastic Integration (**H. Kunita and S. Watanabe, 1967; P.A. Meyer, 1967**): the integral $\int_0^t H_s dM_s$ need not be martingale even if M were one, but it is always a local martingale (in the continuous case); in the general case it's a sigma martingale, a slight generalization

Strict Local Martingales in Finance

- **Absence of Arbitrage Opportunities:** The gold standard for this is the condition **No Free Lunch with Vanishing Risk$\sim$: NFLVR**
- The goal is to show that if one has a positive price process (such as a stock price), then there exists an equivalent probability measure that turns the price process into a martingale
- Such a result, first proved in a very special case by **J.M. Harrison & S. Pliska, J.M. Harrison & D. Kreps** (1978-1981), was extended by Kreps to a more general case; but it was too complicated to be useful.
- **F. Delbaen and W. Schachermayer** extended it to its present (and useful!) form in two papers in 1994 & 1998
- Except when S is bounded, in the continuous case they needed strict local martingales for the theorem to be true

Strict Local Martingales and Stochastic Volatility

- A simple **stochastic volatility model** (sometimes called a Heston paradigm):

$$\begin{aligned} dX_t &= \sigma_t X_t dB_t; \quad X_0 = x > 0 \\ d\sigma_t &= \alpha \sigma_t dZ_t; \quad \sigma_0 = \sigma > 0 \end{aligned} \tag{1}$$

  where $Z_t = \rho W_t + \sqrt{1-\rho^2} B_t$, and (W, B) is a standard 2D Brownian motion, $\alpha > 0$, and $\rho \in [-1, +1]$ is the correlation parameter

- **P.L. Lions & M. Musiela (2007)** show that, depending primarily on the values of ρ, the solution of the system (1) gives rise to strict local martingales, and therefore is no longer "meaningful" for its usual applications to finance
- A similar result was obtained by **L. Andersen and V. Piterbarg (2007)**

Models of Financial Bubbles

- Let $S = (S_t)_{t \geq 0}$ be a nonnegative price process of (to be concrete) a stock
- The **fundamental price** of S under a risk neutral measure Q is
$$S_t^* = E_Q\{ \text{ the future cash flow of the stock } |\mathcal{F}_t\}$$
- Typically, $S_t = S_t^*$ as should be the case if markets are "rational"
- In a bubble situation, $\beta_t \equiv S_t^* - S_t; \quad \beta_t \geq 0$
- The most interesting case is the finite horizon case: working on the time interval $[0, T]$

- **Theorem:** In the finite horizon case, there is a bubble if and only if S is a strict local martingale, and S^* is a martingale (**Jarrow-P^2-Shimbo, 2007, 2010**)
- We do not need to know what S^* is, if we can determine that S is a strict local martingale under the chosen risk neutral measure Q
- This is not well defined, since there is a choice of risk neutral measures (incomplete markets), and so the fundamental price is not well defined if we consider all risk neutral measures simultaneously
- But if the stock price follows an equation of the form

$$dS_t = \sigma(S_t)dB_t + b(t, S_t, Y_t)dt; \quad S_0 = s_0,$$

then $dS_t = \sigma(S_t)dB_t$ under all the risk neutral measures (reasonable hypotheses on σ, b, Y)

Strict Local Martingales and Filtration Shrinkage

- We assume we have two filtrations $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ and $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ with $\mathbb{F} \subset \mathbb{G}$
- The **optional projection** of a stochastic process $X = (X_t)_{t \geq 0}$ onto a filtration $\mathbb{F}$ to which it is not adapted, is a process $({}^oX_t)_{t \geq 0}$ where ${}^oX_t = E\{X_t | \mathcal{F}_t\}$ a.s., each $t \geq 0$ (**P.A. Meyer, 1968; C. Dellacherie, 1972**)
- **Theorem:** Let M be a $\mathbb{G}$ martingale. Then ${}^oM$ is an $\mathbb{F}$ martingale
- The above theorem is no longer true in general for $\mathbb{G}$ local martingales

- **Theorem:** Let X be a local martingale for a filtration $\mathbb{G}$ and let $\mathbb{F}$ be a subfiltration of $\mathbb{G}$. Then the optional projection of X onto $\mathbb{F}$, ${}^{o}X$, is an $\mathbb{F}$ local martingale if there exists a sequence of reducing stopping times $(T_n)_{n\geq 1}$ for X in $\mathbb{G}$ which are also stopping times in $\mathbb{F}$.
- Conversely, if X is positive, then a reducing sequence of stopping times for ${}^{o}X$ is also a reducing sequence for X in $\mathbb{G}$. (**H. Föllmer, P^2, 2010**)

The Inverse Bessel Process and Filtration Shrinkage

- Let $(B_t)_{t\geq 0} = (B_t^1, B_t^2, B_t^3)_{t\geq 0}$ denote a standard three dimensional Brownian motion starting at 0, and with natural completed filtration $\mathbb{H}$
- Let $x_0 = (1, 0, 0)$, so that $U_t = \| B_t - x_0 \|, t \geq 0$ is a Bes(3) process, with $U_0 = 1$
- Let $M_t = 1/U_t$ be the inverse Bessel process which is a strict local martingale
- We consider the subfiltration $\mathbb{F}$ of $\mathbb{H}$ defined as

$$\mathcal{F}_t = \sigma(B_s^1; s \leq t); \quad t \geq 0,$$

- The process

$$N_t = E(M_t | \mathcal{F}_t); \quad t \geq 0$$

  is the optional projection of M onto the smaller ("shrunken") filtration $\mathbb{F}$, a filtration to which M is not adapted

- $$\begin{aligned} N_t &= E(M_t|\mathcal{F}_t) \\ &= E^{(2,3)}\{\left((B_t^1-1)^2+(B_t^2)^2+(B_t^3)^2\right)^{-\frac{1}{2}}\} \\ &= u(B_t^1, t) \end{aligned}$$

  where $E^{(2,3)}$ denotes expectation with respect to the second and third coordinates
- The second line above is justified by the independence of B^1 with (B^2, B^3)
- $$u(x,t) = \int_0^\infty \left((x-1)^2 + tr^2\right)^{-\frac{1}{2}} re^{-r^2/2} dr$$

- A change of variables yields a "closed form" expression for the function u:

$$u(x,t) = \sqrt{\frac{2\pi}{t}} \exp\left(\frac{(x-1)^2}{2t}\right)\left(1 - \Phi\left(\sqrt{\frac{(x-1)^2}{t}}\right)\right)$$

where Φ is the distribution function of a $N(0,1)$ random variable

- **The $\mathbb{F}$ decomposition of N is**

$$N_t = \text{a local martingale } - \int_0^t \frac{1}{s} dL_s^1. \tag{2}$$

- From equation (2) we see that the optional projection of the strict local martingale M onto the subfiltration generated by the first Brownian component yields a supermartingale which is not a local martingale, since the increasing term in its remarkable Doob-Meyer decomposition is $\int_0^t \frac{1}{s} dL_s^1$

- It is interesting to note that N is in fact a strict local martingale up to the hitting time T_1 of 1 for the Brownian motion B^1, which starts at 0
- The increasing term in (2) has paths which are singular with respect to Lebesgue measure, while the local martingale term has a quadratic variation process which is absolutely continuous. Thus from a **Mathematical Finance perspective**, M does not yield arbitrage, but its projection onto this smaller filtration does in fact yield arbitrage opportunities
- This observation has been used to describe **illusory arbitrage** and to give an explanation of how hedge funds, through ignorance, sell arbitrage generating strategies **("positive alpha")** that they think contain arbitrage opportunities (**R. Jarrow & P^2, 2013**)

Examples of Strict Local Martingales with Jumps

- Recently **Fontana-Jeanblanc-Song (2013)** and **Kardaras-Dreher-Nikeghbali** (2013) have remarked that there is a paucity of examples of strict local martingales with jumps, other than that of **O. Chybiryakov** (2007)
- We present here a way to construct strict local martingales through filtration shrinkage (**P^2, 2013**)
- Recall the continuous result:

$$dX_t = \sigma(X_t)dB_t; \quad X_0 = 1 \tag{3}$$

  where B is standard Brownian motion

- X is a positive Strict Local Martingale if for any $\varepsilon > 0$:

$$\int_0^{\varepsilon} \frac{x}{\sigma(x)^2} dx = \infty \text{ and } \int_{\varepsilon}^{\infty} \frac{x}{\sigma(x)^2} dx < \infty \tag{4}$$

- The idea is to project a solution of (3),(4) onto a smaller filtration
- Let Z be a $\mathbb{G}$ continuous nonnegative strict local martingale
- Let U be an arbitrary continuous $\mathbb{G}$ adapted process, and let $\Lambda \subset [0, \infty)$ such that Λ has left isolated points, and the left isolated points contain a sequence tending to ∞.
- $$\tau_x = \inf\{t > 0 : U_t \geq x\}. \tag{5}$$
- We define $\mathbb{F}$ by

$$\mathcal{F}_t = \sigma(\tau_x \leq s, s \leq t, x \in \Lambda) \tag{6}$$

- **Theorem:** If the reducing stopping times of Z are also stopping times in $\mathbb{F}$, then the optional projection M of Z onto $\mathbb{F}$ is an $\mathbb{F}$ strict local martingale. Moreover it has jumps at every time T_β, where β is a left isolated point of Λ
- With more hypotheses, we can infer more structure; specifically, when are the compensators of the jump time absolutely continuous?

- **Theorem:** Assume given a Brownian motion B on the space $(\Omega, \mathcal{G}, P, \mathbb{G})$ and a set $\Lambda \subset \mathbb{R}_+$, such that there exists at least one sequence of left isolated points increasing to ∞. We define $\mathbb{F}$ by

$$\mathcal{F}_t = \sigma(\tau_x \leq s, s \leq t, x \in \Lambda) \tag{7}$$

Let X be the solution of (3) with conditions (4) holding. Moreover assume $\sigma > 0$, $\sigma \in \mathcal{C}^2$, and both of σ and $\sigma\sigma'$ are Lipschitz continuous. Let $M = {}^o X$ be its projection onto $\mathbb{F}$. Then M is a strict local martingale, and its jumps have **absolutely continuous compensators**.

Connections of the theory of Mathematical Finance

- For stocks, prices in the United States markets are quoted in pennies.
- This means that even if a price process is modeled as a continuous process it can be observed only at a grid of prices (ie, in 1¢units).
- This naturally creates a situation of filtration shrinkage, where one observes the process only at the times it crosses the grid of prices separated by penny units.
- This is in the spirit of the work of **A. Deniz Sezer (2007)** and **Jarrow-P^2-Sezer (2007)**

The Issue of Transaction Times

- A common interpretation of models in Mathematical Finance is that a price process evolves continuously, for example following a diffusion
- But one can observe only at the random times when a transaction takes place
- One observes the process at a well ordered sequence of stopping times, the times when trades occur. It is typically assumed that nevertheless one "knows" the price process at all times, especially so if the transaction times occur with high frequency, a more common event in the modern era with the presence of high frequency trading and ultra high frequency trading.

- However this is a small leap, and it is more precise to model the information one has by the filtration obtained by seeing the process only at the transaction times, a framework amenable to the ideas given today
- This allows to give a connection between continuous time models and "discrete time" models, via a filtration shrinkage corresponding to the transaction times and the pricing grid crossing times, in the spirit of **Jarrow & P^2 (2004, 2012)**

The End

Thank You for Your Attention