SET THEORY BASIC EXAM: SEPTEMBER 2016

Attempt four of the following six questions. All questions carry equal weight.

(1) Prove that ♦ω1 implies the following guessing principle (diamond for functions): there is a sequence ⟨fα: α < ω1⟩ such that fα: α → α, and for every f: ω1 → ω1 there are stationarily many α such that f↾α = fα. Outline some construction of a Souslin tree assuming ♦ω1.

(2) Give the definition of the class HOD, and prove that it is a model of the Axiom of Replacement (you may assume the Reflection Theorem, but you must state it correctly).

(3) State Shoenfield’s absoluteness theorem and outline the main steps in the proof. Prove that the assertion “there is a countable transitive model of ZFC” is Σ12. We call this assertion φ. Prove that if M is a countable transitive model of ZFC with On ∩ M minimal then φ is false in M but true in V, and explain why this does not contradict Shoenfield’s theorem.

(4) Prove that:
(a) If κ is a singular cardinal and the function λ ↦ 2λ is eventually constant for λ < κ with eventual value µ, then 2κ = µ.
(b) The generalised continuum hypothesis (GCH) states that 2κ = κ+ for all infinite cardinals κ. Assuming GCH prove that if κ and λ are infinite cardinals with λ < cf(κ) then κλ = κ. What happens when λ ≥ cf(κ)?

(5) Let θ > ω2 be regular and let <θ be a wellordering of Hθ. Prove that:
(a) If M ⊨ (Hθ, ∈, <θ) and M is countable then M ∩ ω1 ∈ ω1, and furthermore M ∩ ω1 ∈ C for all C ∈ M with C club in ω1.
(b) The following are equivalent for S ⊆ ω1:
   • S is stationary.
   • There is a countable M ⊨ (Hθ, ∈, <θ) with M ∩ ω1 ∈ S and S ∈ M.

(6) Assume that V = L. For each α < ω1, let f(α) be the least β > α (if it exists) such that α is countable in Lβ. Prove that:
(a) f(α) exists.
(b) f(α) is a successor ordinal.
(c) f(α) < ω1.