1. Let \((X_n)\) be IID random variables taking values in \([-1, 1]\) and having the common mean \(\mu = \mathbb{E}[X_n] = 0\) and the variance \(\sigma^2 = \mathbb{E}[X_n^2] > 0\). Let \((a_n)\) be a sequence in \((-1, 1)\). Define
\[
Y_n = \prod_{k=1}^{n}(1 + a_k X_k), \quad n \geq 1.
\]
Will \((Y_n)\) converge (i) almost surely, (ii) in probability, (iii) in \(L_1\), (iv) weakly to some random variable \(Y_\infty\)? Will \(Y_n = \mathbb{E}[Y_\infty | \sigma(X_1, \ldots, X_n)]\)? If needed, formulate additional (ideally, necessary and sufficient) conditions on the sequence \((a_n)\) under which the answers are affirmative.

2. Let \(X\) and \(Y\) be random variables in \(L_2(\Omega, \mathcal{F}, \mathbb{P})\) such that
\[
\mathbb{E}[X] = \mathbb{E}[Y] = \mathbb{E}[XY] = 0,
\]
\[
\mathbb{E}[X^2] = \mathbb{E}[Y^2] = 1,
\]
and \(\mathcal{A}\) be a sub-\(\sigma\)-algebra of \(\mathcal{F}\).

(a) Show that
\[
\mathbb{E}[\mathbb{E}[X | \mathcal{A}] \mathbb{E}[Y | \mathcal{A}]] \leq \frac{1}{2}.
\]

(b) Assume in addition that \(X\) and \(Y\) are IID RVs. Find a sub-\(\sigma\)-algebra \(\mathcal{A}\) of \(\mathcal{F}\) such that
\[
\mathbb{E}[\mathbb{E}[X | \mathcal{A}] \mathbb{E}[Y | \mathcal{A}]] = \frac{1}{2}.
\]

3. Let \((X_n)\) be IID RVs in \(L_1\) and denote by \(\mu = \mathbb{E}[X_1]\) their common first moment. Let \(\tau\) be a stopping time with \(\mathbb{E}[\tau] < \infty\) and \(S_n = \sum_{k=1}^{n} X_k\). Show that \(S_\tau \in L_1\) and compute \(\mathbb{E}[S_\tau]\).
4. Let \((X_n)\) be IID RVs in \(L_2\) with common zero mean. Denote

\[ Y_n = \frac{1}{n} \sum_{k=1}^{n} X_k, \quad Y^* = \sup_n Y_n. \]

Show that

\[ \mathbb{E} \left[ (Y^*)^2 \right] \leq 4 \mathbb{E} \left[ X_1^2 \right]. \]

5. A dice is thrown repeatedly. Compute the expected time of getting the sequence “6,6, . . . ,6” of length \(N\).